Journal of Mathematical Extension
Vol. 10, No. 1, (2016), 87-105
ISSN: 1735-8299
URL: http://www.ijmex.com

Interval-Valued Parametric Distribution Functions: Methodologies and Applications

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Abstract. A lot of methods and models in classical reliability theory assume that all parameters of lifetime density function are precise. But in the real world applications imprecise information is often mixed up in the lifetimes and/or parameters of systems. However, the parameters sometimes cannot be recorded precisely due to machine errors, experiment, personal judgment, estimation or some other unexpected situations. When parameters in the lifetime distribution are interval-valued, the conventional reliability system may have difficulty for handling reliability function. Therefore, estimation methods for reliability characteristics have to be adapted to the situation of interval-valued parameters of life times in order to obtain more realistic results. In this regard, the present paper will discuss the system reliability for coherent system based on a new notion of random variable with interval-valued parameters. The concepts of probability density function and cumulative distribution function of the random variable with interval-valued parameters will be stated in this paper. Using the same techniques in classical probability theory, the probability measure of the random variable can be constructed from the probability density function and cumulative distribution function. In the proceeding discussion, several numerical examples are provided in reliability systems and queueing theory to clarify our discussions.

AMS Subject Classification: 68M15; 90B25; 46N30; 65C50
Keywords and Phrases: Interval-valued parameter, random variable, reliability system, probability of an event, queueing

Received: March 2015; Accepted: September 2015
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1. Introduction

The classical probabilistic methods and statistical techniques provide a rigorous framework for modeling uncertainty due to randomness. These methods are inadequate for dealing with certain kinds of uncertainty due to human judgment. However, there are many situations where we have insufficient information regarding the underlying model. For instance, it is frequently difficult to assume that the parameters, for which the distribution of a random variable is determined, have a precise or crisp value. In such a case, closed intervals may be more effective in encoding parameters instead of crisp ones. To achieve suitable probability theory dealing with imprecise information, we need to model the imprecise information and extend the usual probability space to imprecise environments. On the other hand, the conventional reliability of a system is defined as the probability that the system performs its assigned function properly during a predefined period under the condition that the system behavior can be fully characterized in the context of probability measures [9]. Classical reliability assessment is based largely on crisp or precise information. In practice, however, some information about an underlying system might not be assumed crisp and they are represented in the form of vague quantities. In this regards, over the past decades, different approaches and theories have been proposed for treating imprecision and uncertainty generalizing the classical methods to vague environments for studying and analyzing the systems of interests (see Taheri and Zarei [18] for reviewing of fuzzy reliability and vague reliability studies). This suggests the need for a theory of defining new notions of probability density function and cumulative distribution function of random variable with interval-valued parameters. The topic of probability theory with imprecise information has been studied by some authors. Below is a brief review of some studies relevant to the present work.

Smets [14] proposed some axioms to justify the natural definition of the probability of a fuzzy event. A fuzzy event is a fuzzy set whose membership function is Borel measurable and its probability is defined by Zadeh [23] as the expected value of the membership function characterizing the fuzzy set. Stein [15] discussed the treatment of fuzzy probabilities in setting of fuzzy variables and joint possibility distributions. Yager [21] introduced a methodology for obtaining a crisp fuzzy measure of the probability of a fuzzy event in the face of probabilistic uncertainty on the base elements. Klement [8] suggested a modification of Yager's [21] definition which leads to a piecewise continuous fuzzy subset. Yager [22] provided an appropriate interpretation for Klement's [8] modification and used it to provide an alternative definition for a fuzzy probability of a fuzzy event. Plasecki [11] defined the probability of fuzzy events as a denumerable additivity measure and proposed the notions of conditional probability of fuzzy events, complete fuzzy repartition and independent fuzzy events. Cheng and Liu [3] extended the fuzzy probability of a fuzzy event from a fuzzy algebra to the fuzzy σ-algebra. Toth [19] redesigned some definitions of a probability of a fuzzy event based on the operational viewpoint of f-set theory and on some concepts of operational statistics. Heilpern [7] studied the fuzzy subsets of the space of all probability measures so that the probability of fuzzy event is obtained as a fuzzy set. Baldwin et al. [1] introduced the probability of a fuzzy event by using mass assignment theory techniques for processing uncertainty toughener with the t-norm definition of conditional probabilities. Grzegorzewski [6] generalized the notion of independence of events and the concept of conditional probability on the intuitionistic fuzzy events. Chinag and Yao [4] considered fuzzy probabilities constructed over fuzzy topological spaces. Mesiar and Komorníková [10] transformed probability measures on intuitionistic fuzzy events were axiomatically characterized by Riečan [12] as interval-valued fuzzy sets. Stojaković [16] and Stojaković and Gajić [17] defined set valued probability and fuzzy valued probability over a measurable space as a fuzzy valued set functions which were used for analyzing and modeling highly uncertain probability systems. In Bayesian reliability analysis, Viertl and Mirzaei Yeganeh [20] proposed fuzzy probability distributions in reliability analysis, fuzzy HPD-regions, and fuzzy predictive distributions and generalized the concept of highest a-posteriori density regions for parameters and the predictive densities for lifetimes based on fuzzy data. En-lin and You-ming [5], based on the interval probability, studied the second kind of fuzzy random problem and introduced definitions of fuzzy probability, random variable and its distribution function, distribution sequence, fuzzy math expectation and fuzzy variance.

The purpose of this paper is to provide a novel method of constructing a probability measure induced by a family of interval-valued parametric distribution function. In order to satisfy the purpose of this paper, we discuss the notion of probability density function and the way of constructing the probability measure of random variable from the known probability density function. From a different perspective, in this paper the problem of the evaluation of system reliability will be considered in which the lifetimes of components are described using random variables with interval-valued parameters. Furthermore, the reliability functions of series systems, parallel systems and k-out-of-n systems are discussed. Finally, some numerical examples are presented to illustrate how to calculate the reliability function.

This paper is organized as follows: In Section 2, the computational procedures and several examples are provided in order to extend the classical probability measure based on a family of distributions with interval-valued parameters. In Section 3, in order to clarify the theory discussed in this paper and to give a possible insight for applying the random variables with interval-valued parameters several examples are provided in system reliability assessment. Finally, Section 4 provides an overall conclusion.

2. Probability Measure with Interval-Valued Parameters

In statistical inference, the data set is viewed as a realization or observation of a random element defined on a probability space $(\Omega, \mathcal{A}, \mathcal{P})$ related to the random experiment. A set of probability measures $P_{\boldsymbol{\theta}}$ on $(\Omega, \mathcal{A})$ indexed by a parameter $\boldsymbol{\theta} \in \Theta$ is said to be a parametric family if and only if $\Theta \subseteq \mathbb{R}^p$ for some fixed positive integer p and each $P_{\boldsymbol{\theta}}$ is a known probability measure when $\boldsymbol{\theta}$ is known. The set Θ is called the p-dimensional parameter space. A parametric model refers to the assumption that the probability measure P is in a given parametric family.

Let $X : \Omega \to \mathbb{R}$ be a random variable, where $\mathbb{R}$ is equipped with the σ-algebra $\mathcal{B}(\mathbb{R})$, the set of all Borel subsets of $\mathbb{R}$. Then the probability measure induced by X, i.e. $P_X : \mathcal{B}(\mathbb{R}) \to [0,1]$, is defined as follows for all $A \in \mathcal{B}(\mathbb{R})$ [13]

$$\begin{aligned} P_X(A) &= P\{X \in A\} \\ &= \int_{\{\omega\in\Omega | X(\omega)\in A\}} dP(\omega). \end{aligned}$$

In particular, if $A = (-\infty, x]$, then the cumulative distribution function of X, for each $x \in \mathbb{R}$, is obtained as follows

$$\begin{aligned} F_X(x) &= P\{X \leq x\} \\ &= \int_{\{\omega\in\Omega | X(\omega)\leq x\}} dP(\omega). \end{aligned}$$

So, the probability of $A = (a, b]$ can be obtained by $P(a < X \leq b) = F_X(b) - F_X(a)$.

In the following we extend the concept of probability of an event induced by a family of distribution with interval-valued parameters.

Definition 2.1. *Suppose that a random experiment is defined on a probability space* $(\Omega, \mathcal{A}, P_{\boldsymbol{\theta}})$, $\boldsymbol{\theta} \in \Theta \subseteq \mathbb{R}^p$. *Let* $X : (\Omega, \mathcal{A}, P_{\boldsymbol{\theta}}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$ *be a random variable related to this random experiment having a distribution with interval parameter* $\overline{\boldsymbol{\theta}} = (\overline{\theta}_1, \ldots, \overline{\theta}_p) \in (\mathcal{C}(\Theta))^p$, *where* $\mathcal{C}(\Theta)$ *is the class of nonempty compact intervals on real numbers* $\mathbb{R}$. *Then the probability measure induced by* X, *i.e.* $P_X : \mathcal{B}(\mathbb{R}) \to [0,1]$, *with interval parameter* $\overline{\boldsymbol{\theta}}$ *is defined as follows*

$$\begin{aligned} P_X(B) &= P\{X \in B\} \\ &= \int_0^1 \int_{\{\omega\in\Omega | X(\omega)\in B\}} dP_{\boldsymbol{\theta}_\lambda}(\omega) d\lambda, \end{aligned} \tag{1}$$

where $B \in \mathcal{B}$, $\overline{\boldsymbol{\theta}} = (\overline{\theta}_1, \ldots, \overline{\theta}_p)$, $\boldsymbol{\theta}_\lambda = (\theta_{1\lambda}, \ldots, \theta_{p\lambda})$, *and* $\theta_{j\lambda} = \lambda\theta_j^l + (1-\lambda)\theta_j^u$ *(the convex combination of endpoints of the interval* $\overline{\theta}_j = [\theta_j^l, \theta_j^u]$, $j = 1, \ldots, p$, $\lambda \in [0,1]$*).*

Remark 2.2. *Note that an interval-valued parameter sometimes happens when an expert estimates the value of a parameter* $\boldsymbol{\theta}$ *based on the smallest possible value (or "pessimistic") and the highest possible value* *(or "optimistic"). We can solicit this estimate from the expert as is done in estimating job times in project scheduling. So we can construct/consider an interval for parameter θ in computation (for more see [2]).*

Remark 2.3. *Let $B = (-\infty, x]$ in Eq. (1), then the cumulative distribution function of X is obtained as follows*

$$\begin{aligned} F_X(x) &= P\{X \le x\} \\ &= \int_0^1 \int_{\{\omega\in\Omega | X(\omega)\le x\}} dP_{\boldsymbol{\theta}_\lambda}(\omega) d\lambda \\ &= \int_0^1 P_{\boldsymbol{\theta}_\lambda}(X \le x) d\lambda, \end{aligned}$$

We denote by $X \sim F_{\overline{\boldsymbol{\theta}}}$ the random variable X from distribution F with interval-valued parameter $\overline{\boldsymbol{\theta}}$. Differentiating both sides of the preceding equation yields the relationship between the cumulative distribution F_X and the probability density f_X which is expressed by

$$\begin{aligned} f_X(x) &= \frac{d}{dx} F_X(x) \\ &= \int_0^1 \frac{d}{dx} P_{\boldsymbol{\theta}_\lambda}(X \le x) d\lambda, \qquad x \in \mathbb{R}, \end{aligned}$$

Therefore, it is seen that the following relationship between the cumulative distribution F_X and the probability density f_X is held for any $x \in \mathbb{R}$

$$F_X(x) = \begin{cases} \int_{-\infty}^{x} f_X(t)dt & if \quad X \text{ is a continuous random variable,} \\ \sum_{t\le x} f_X(t) & if \quad X \text{ is a discrete random variable.} \end{cases}$$

Example 2.4. Assume that $X \sim F_{\overline{\theta}}$ where $F_\theta(x) = 1 - e^{-\theta x}$ and $\overline{\theta} = [\theta^l, \theta^u] \in \mathcal{C}((0,\infty))$. The cumulative distribution function of X is given by

$$\begin{aligned} F_X(x) &= \int_0^1 P_{\boldsymbol{\theta}_\lambda}(X \le x) d\lambda \\ &= \int_0^1 (1 - e^{-(\lambda\theta^l + (1-\lambda)\theta^u)x}) d\lambda \\ &= \begin{cases} 0 & x \le 0; \\ 1 + \frac{e^{-\theta^u x} - e^{-\theta^l x}}{(\theta^u - \theta^l)x} & x > 0. \end{cases} \end{aligned}$$

For example assume that $\overline{\theta} = [0.02, 0.04])$, then $P_X(45 < X < 65) = F_X(65) - F_X(45) = 0.11$.

Example 2.5. Assume that $X \sim f_{\overline{\theta}}$ where $f_\theta(x) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$, $x = 0, 1, 2, \ldots, n$ and $\overline{\theta} = [\theta^l, \theta^u] \in \mathcal{C}((0,1))$. The cumulative distribution function of X is obtained as follows

$$\begin{aligned} F_X(x) &= \int_0^1 P_{\theta_\lambda}(X \leq x) d\lambda \\ &= \sum_{i=0}^{[x]} \binom{n}{i} \int_0^1 \Big(\lambda\theta^l + (1-\lambda)\theta^u\Big)^i \Big(1 - (\lambda\theta^l \\ &+ (1-\lambda)\theta^u)\Big)^{n-i} d\lambda, \end{aligned}$$

where $[x]$ denotes the greatest integer smaller than or equal to x. Now, let $\overline{\theta} = [0.7, 0.8]$, then $P_X(4 \leq X \leq 7) = F_X(7) - F_X(3) = 0.4728 - 0.0042 = 0.469$.

Example 2.6. Assume that X is normally distributed with interval parameters $\overline{\mu} = [-1, 1]$ and $\overline{\sigma^2} = [0.5, 1.5]$. The cumulative distribution function of X is obtained as follows

$$\begin{aligned} F_X(x) &= \int_0^1 P_{\boldsymbol{\theta}_\lambda}(X \leq x) d\lambda \\ &= \int_0^1 \int_{-\infty}^x \frac{1}{\sqrt{2\pi}(1.5-\lambda)} e^{-\frac{1}{2}\left(\frac{t-(1-2\lambda)}{1.5-\lambda}\right)^2} dt\, d\lambda. \end{aligned}$$

Now the probability of any event such as $A = [-1, 2]$ can be easily calculated as $P_X(-1 \leq X \leq 2) = F_X(2) - F_X(-1) = 0.75$.

Remark 2.7. *The situation with the usual (precise) random variable with crisp parameters is a special case of the proposed procedure. If the parameter gets a crisp value, the lower and upper bounds will become equal, which means that the results and concepts of our approach coincide with the classical probability concepts.*

3. Applied Numerical Examples

3.1. Application in reliability systems analysis

The name reliability is given to the field of study that attempts to assign numbers to the probability of systems to fail. In a more restrictive sense, the term reliability is defined to be the probability that a system performs its mission successfully. Because the mission is often specified in terms of time, reliability is often defined as the probability that a system will operate satisfactorily for a given period of time. When parameters in the lifetime distribution are interval-valued, the conventional reliability system may have difficulty for handling reliability function. Therefore, in the following the estimation methods for reliability characteristics will be adapted to the situation of interval-valued parameters of lifetimes in order to obtain more realistic results.

Let X be a random variable with pdf f_X of known functional form but depending on an unknown p-dimensional interval-valued vector parameter $\overline{\boldsymbol{\theta}}$ which belongs to the family of parametric distribution $\{f_X(x) : \overline{\boldsymbol{\theta}} \in (\mathcal{C}(\Theta))^p, x \in (0, \infty)\}$. Based on the concepts of probability density function and cumulative distribution function of the random variable with interval-valued parameters the components of a lifetime distribution at time $t > 0$ are defined as follows

1. The Reliability Function (**R.F.**): $R(x) = 1 - F_X(x)$, for all $x \in \mathbb{R}$.

2. The Mean Time to Failure (**M.T.F.**): $E = \int_0^\infty (1 - F_X(x))\, dx$.

3. The Mean Residual Life Function (**M.R.L.F.**):
$$\mu(x) = E(X - x | X > x) = \frac{\int_x^\infty R(t)\, dt}{R(x)}, \quad \text{for all } x \in \mathbb{R}.$$

4. The Residual Life Function (**R.L.F.**) (or Hazard Rate Function) $h(x) = \frac{f_X(x)}{1 - F_X(x)}$, for all $x \in \mathbb{R}$.

5. The Cumulative Hazard Rate function (**C.H.R.**): $H(x) = \int_0^x h(t)\, dt$, for all $x \in \mathbb{R}$.

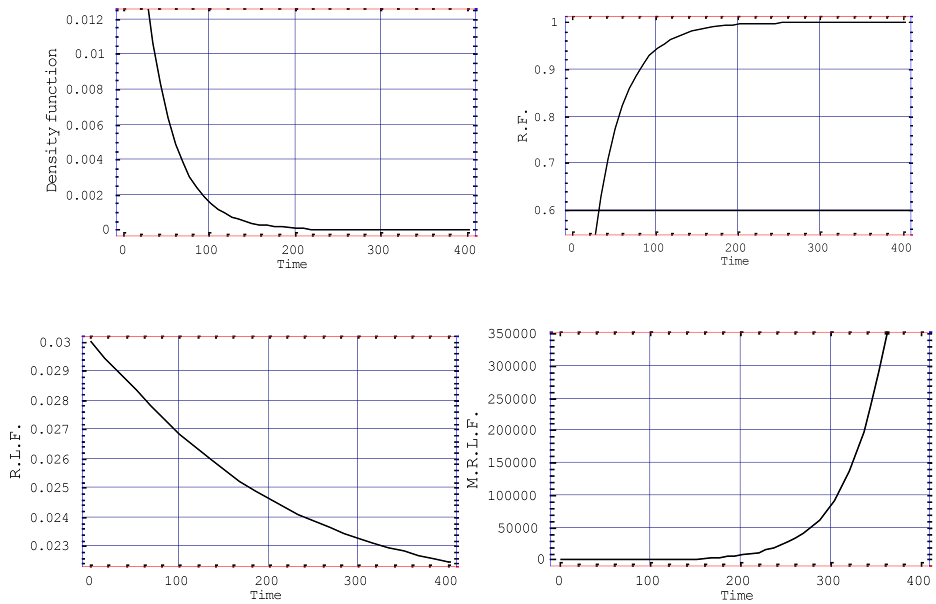

Figure 1: The components of the life time distribution at any time $t > 0$ for the system reliability in Example 3.1

6. Conditional Reliability (**C.R.**):

$$C(X,t) = P(\text{no failure in } (t, t+X)| \text{ no failure in } (0,t)) = \frac{R(t+X)}{R(t)}, \quad \text{for all } t \in \mathbb{R}.$$

Example 3.1. Assume that the random variable X has the exponential distribution with interval parameter $\overline{\lambda} = [0.02, 0.04]$. The reliability function $(R(x))$, density function $(f_X(x))$, mean residual life function $(\mu(x))$, residual life function $(h(x))$ of the random variable X at any time are shown in Figure 1.

Estimating reliability is essentially a problem in probability modeling. A system consists of a number of components. In the simplest case, each component has two states, operating or failed. When the set of operating components and the set of failed components is specified, it is possible to discern the status of the system. The problem is to compute the probability that the system is operating, i.e. the reliability of the system. Let $c_1, c_2, \ldots, c_n$ denote the n components in a reliability systems. Assume that the n components operate independently, and $P(c_j$ works until time $t) = R_j(t)$. Certain types of systems frequently arisen in practice usually are k-out-of-n systems. The reliability of the k-out-of-n system is given by

$$R_S(t) = \sum_{j=k}^{n} \binom{n}{j} (R_j(t))^j (1 - R_j(t))^{n-j}.$$

If $k = 1$, the system reduces to a parallel system with the reliability of $R_S(t) = 1 - \prod_{j=1}^{n}(1 - R_j(t))$, and if $k = n$ the system reduces to a series system with the reliability of $R_S(t) = \prod_{j=1}^{n} R_j(t)$.

Example 3.2. Consider a system that has five synchronous computers which analyze all other systems and compare the results among each other. For a launch take place, four out of five computers must agree on the system parameters. If all agree, the launch takes place. If one computer fails and four agree, the fifth computer is ignored and the launch occurs. If two computers fail to agree, the launch is scrubbed. Assume that all components are exponentially distributed (with c.d.f. $F_\gamma(x) = 1 - e^{-\gamma x}$) with interval-valued parameter $\overline{\gamma} = [0.003, 0.005]$. In this case, the probability of a successful launch is

$$\begin{aligned}
R_S(1) &= \sum_{j=4}^{5} \binom{5}{j} (R_j(1))^j (1 - R_j(1))^{5-j} \\
&= \sum_{j=4}^{5} \binom{5}{j} \left(\int_0^1 e^{-(0.003\lambda + 0.005(1-\lambda))} d\lambda \right)^j \\
&\quad \left(1 - \int_0^1 e^{-(0.003\lambda + 0.005(1-\lambda))} d\lambda \right)^{5-j} \\
&= 0.99.
\end{aligned}$$

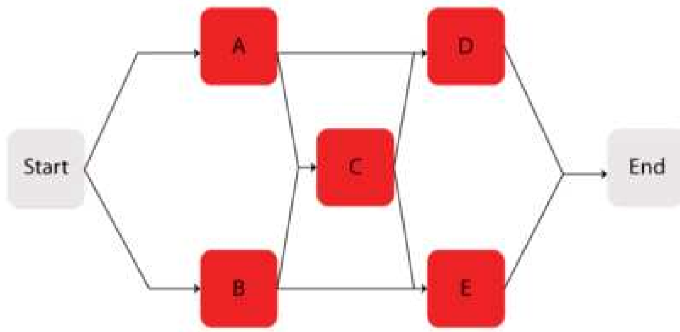

Figure 2: The complex system in Example 3.3

Example 3.3. Consider the complex system as shown in Figure 2 where the starting and ending blocks cannot fail. Assume that the components A through E are Weibull distributed (with c.d.f. $F_{\beta\eta}(x) = 1 - e^{-(\frac{x}{\eta})^\beta}$) where $\overline{\beta} = [1.1, 1.3]$ and $\overline{\eta} = [1225, 1235]$ are interval-valued parameters. The first step is to obtain the reliability function for the system. Since the components in this example are identical, the system reliability equation is obtained as follows

$$R_S(t) = 2(R(t))^2 + 2(R(t))^3 - 5(R(t))^4 + 2(R(t))^5,$$

where

$$\begin{aligned} R(t) &= \int_0^1 e^{-(\frac{t}{\overline{\eta}_\lambda})^{\overline{\beta}_\lambda}} d\lambda \\ &= \int_0^1 e^{-(\frac{t}{1225+10\lambda})^{1.1+0.2\lambda}} d\lambda. \end{aligned}$$

The reliability function, probability density function of the system, residual lifetime function, and cumulative hazard rate function are shown in Figures 3-6, respectively. In addition $E = \int_0^\infty (1 - F_{\overline{\boldsymbol{\theta}}_\lambda}(x))dx = 1006.87$.

Sometimes it is desirable to know the time value associated with a certain reliability. Warranty periods are often calculated by determining what percentage of the failure population can be covered financially and

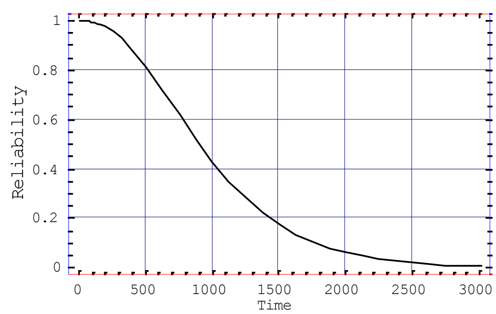

Figure 3: Reliability function of the system reliability in Example 3.3

estimating the time at which this portion of the population will fail. So, the warranty time can be obtained by solving $R_S(t) = 0.9$ with respect to time for the system reliability which leads to a time of $t = 372.7$ hours. Lastly, the conditional reliability can be obtained using

$$C(X,t) = \frac{R(t+X)}{R(t)}.$$

For instance

$$C(200,400) = \frac{R(400)}{R(200)} = 0.906.$$

3.2. Application in queueing theory

In this section the application of the proposed method will be considered in Queueing Theory. An $M/M/1$ queueing system is the simplest non-trivial queue where the requests arrive according to a Poisson process with rate θ. The service times are also assumed to be independent and exponentially distributed with parameter μ. Furthermore, all the involved random variables are supposed to be independent of each other. The model is considered stable only if $\theta < \mu$. If, on average, arrivals happen faster than service completions the queue will grow indefinitely long and the system will not have a stationary distribution. Various performance measures can be computed explicitly for the $M/M/1$ queue. Below we recall some common measures:

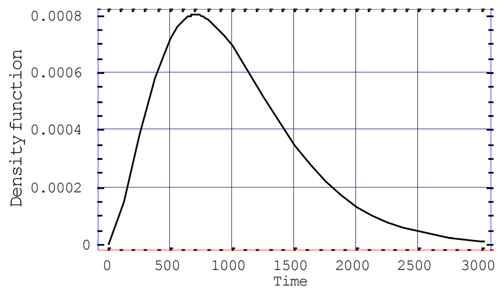

Figure 4: Probability density function of the system reliability in Example 3.3

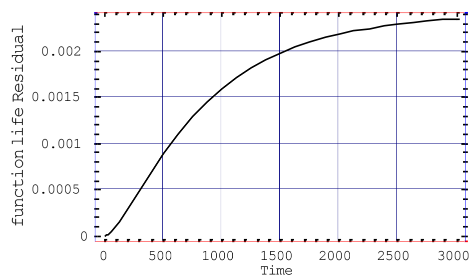

Figure 5: Residual life function of the system reliability in Example 3.3

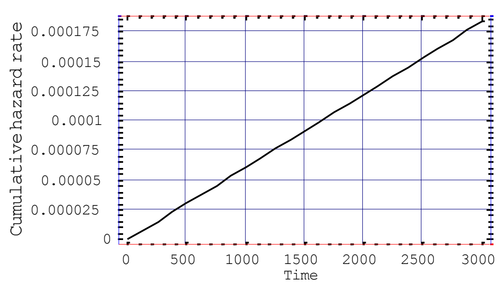

Figure 6: Cumulative hazard rate function of the system reliability in Example 3.3

1. The probability that the stationary process is in state n is

$$P(N=n)=(1-\rho)\rho^n, \quad n=0,1,2,\ldots, \quad \rho=\frac{\theta}{\mu}.$$

The average number of the customers in the system is $E(N)=\frac{\rho}{(1-\rho)}$.

2. Let T_q be the time that a customer spends in waiting line waiting for service, then

$$\begin{aligned} P(T_q=0) &= P(N=0)=1-\rho, \\ F_{T_q}(x) &= P(T_q \leq x)=1-\rho e^{-\mu(1-\rho)x}, \quad x>0, \\ E(T_q) &= \frac{\rho}{(1-\rho)\mu}. \end{aligned}$$

3. Let T be the time that a customer spends in the system (in waiting line and being served). Then T is distributed as exponential distribution with parameter $\mu-\theta$, hence the average time that a customer spends in the system is $E(T)=\frac{1}{\mu-\theta}$.

4. Let U be percentage of the time that all servers are busy, then $U=\frac{\theta}{\mu}$.

However, as it is mentioned in Introduction, "service times" are often reported as vague numbers (non-crisp values). Now consider an $M/M/1$ queueing system with a Poisson process with rate θ and the interval-valued service time $\overline{\mu}=[\mu^l,\mu^u]\in\mathcal{C}((0,\infty))$. Now, by applying the proposed method, the above performance measures are extended as follows:

1. The probability that the stationary process is in state n is

$$P(N=n)=\int_0^1 (1-\rho_\lambda)(\rho_\lambda)^n d\lambda, \quad n=0,1,\ldots, \quad \rho_\lambda=\frac{\theta}{\mu_\lambda}, \theta<\overline{\mu}^l.$$

The average number of the customers in the system is $E(N)=\int_0^1 \frac{\rho_\lambda}{(1-\rho_\lambda)}d\lambda$.

2. Let T_q be the time that a customer spends in the waiting line waiting for service. Then

$$\begin{aligned} P(T_q = 0) &= 1 - \int_0^1 \rho_\lambda d\lambda, \\ F_{T_q}(x) &= 1 - \int_0^1 \rho_\lambda e^{-\mu_\lambda(1-\rho_\lambda)x} d\lambda, \quad x > 0, \\ E(T_q) &= \int_0^1 \frac{\rho_\lambda}{(1-\rho_\lambda)\mu_\lambda} d\lambda. \end{aligned}$$

3. Let T be the time that a customer spends in the system (in waiting line and being served). Then $E(T) = \int_0^1 \frac{1}{\mu_\lambda - \theta} d\lambda$.

4. Let U be the percentage of the time that all servers are busy, then $U = \int_0^1 \frac{\theta}{\mu_\lambda} d\lambda$.

Example 3.4. Our observations of one cashier in a supermarket have shown that the arrival distribution of customers follows Poisson distribution with arrival rate of $\theta = 0.75$ customers per every 5 minutes. The distribution of service time follows exponential distribution with service time about 1 minute per customer described as the interval $\overline{\mu} = [0.5, 1.5]$. The measurements of effectiveness of the queuing system are calculated as $E(N) = 1.37$, $P(T_q = 0) = 0.50$, $W_q = 1.94$, and $E(T) = 3.04$.

Example 3.5. The Erlang distribution can be used to model the time to complete n operations in series, where each operation requires an exponential period of time to complete. In Queueing Theory, it is usually used to model inter-arrival time and service time with a low coefficient of variation. An Erlang random variable X with rate θ and n stages has the following probability density function

$$f_X(x) = \frac{1}{(n-1)!\,\theta^n} x^{n-1} e^{-\frac{x}{\theta}}, \qquad x > 0.$$

Now, consider a queue with three people ahead of you. Assume one is being served and two are waiting. Their service times S_1, S_2, and S_3 are independent exponential random variables with mean about 2 minutes per each person given by $\overline{\theta} = [1, 3]$. Your conditional time in the queue given the system state $N = 3$ upon your arrival is $T = S_1 + S_2 + S_3$. Now, using Remark 2.3, one can obtain that T is distributed according to extended Erlang distributed with the following density function

$$f_T(t) = \int_0^1 \frac{1}{2(1+2\lambda)^3} t^2 e^{-\frac{t}{1+2\lambda}} d\lambda, \quad t > 0.$$

Hence, the probability that you wait, for instance, more than 7 minutes in the queue is

$$\begin{aligned} P(T > 7) &= \int_0^1 \int_7^\infty \frac{1}{2(1+2\lambda)^3} t^2 e^{-\frac{t}{1+2\lambda}} dt d\lambda \\ &= 0.313. \end{aligned}$$

Moreover, the expected time that you wait in the queue is evaluate by

$$\begin{aligned} E(T) &= \int_0^1 \int_0^\infty \frac{1}{2(1+2\lambda)^3} t^3 e^{-\frac{t}{1+2\lambda}} dt d\lambda \\ &= 6. \end{aligned}$$

4. Conclusions

There are many situations in real world applications that we deal with imprecise (non-crisp) information in applied statistics. For instance, the parameters of lifetimes of components are usually not crisp and therefore statistical inferences such as the Reliability Systems or Queueing Theory are not feasible for such cases. In this regard, the subject of probability theory induced by the new notion of random variable with interval-valued parameters has been successfully introduced and discussed in this paper to overcome this difficulty. The numerical results show that the induced probability distribution is a solution to overcome such uncertainty in such applications of statistics. The proposed probability theory is a generalization of conventional probability theory since if parameters are crisp, it turns in to the conventional reliability system.

The proposed methodology may be used/extended for a more general problem when parameters are recorded as fuzzy quantities rather than crisp values or interval values. This is a potential subject for future study.

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