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Original Research Paper

On the Cofinite Properties of Generalized Local Cohomology Modules

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Abstract. Let A be a commutative Noetherian ring with non-zero identity and let I be an ideal of A. Consider F as a finitely generated A-module and C as an arbitrary A-module. In this context, for a non-negative integer n, we establish that if the local cohomology modules $\mathrm{H}^i_I(C)$ are $(\mathrm{FD}_{<n}, I)$-cofinite over A for all integers i and if the collection of such $(\mathrm{FD}_{<n}, I)$-cofinite A-modules constitutes an Abelian category, then the generalized local cohomology modules $\mathrm{H}^i_I(F, C)$ are likewise $(\mathrm{FD}_{<n}, I)$-cofinite over A for all integers i.

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1 Introduction

Throughout this document, the symbol A denotes a commutative Noetherian ring with non-zero identity. The notation I refers to an ideal within A (since A is Noetherian, I is finitely generated). The modules C and D are arbitrary A-modules, which may not necessarily be finitely generated. Additionally, F and G are finitely generated A-modules and n represents a non-negative integer. For any A-module M, $\operatorname{Ext}^i_A(M,-)$ denotes the ith right derived functor of $\operatorname{Hom}_A(M,-)$ and $\mathrm{H}^i_I(-)$ indicates the ith right derived functor of the I-torsion functor $\Gamma_I(-)$. In particular, $\operatorname{Ext}^i_A(M,N)=0$ and $\mathrm{H}^i_I(N)=0$ for all integers $i<0$ and any A-module N. For essential concepts, notation, and terminology that are not explicitly defined herein, readers are advised to refer to the following references: [10, 9, 24].

In [17], Hartshorne introduced the concept of cofinite modules, providing a notable counterexample to a conjecture originally proposed by Grothendieck [16, Expose XIII, Conjecture 1.1]. He characterized an I-torsion A-module C as being *I-cofinite* (or *cofinite with respect to I*) if the modules $\operatorname{Ext}^i_A(A/I,C)$ are finitely generated over A for all integers i. Building upon this framework, Hartshorne posed a significant research question (see [17, First Question]). This question has been the subject of extensive investigation, with positive results established under various specific conditions (for instance, refer to [17, 20, 12, 21, 22, 7, 23, 25]).

Question 1.1. *Under what conditions does the local cohomology module $\mathrm{H}^i_I(G)$ possess the property of I-cofiniteness as an A-module for every integer i?*

Extensions of classical local cohomology theory have been investigated, with foundational contributions beginning with Herzog [19]. He introduced the concept of the *ith generalized local cohomology module of D and C with respect to I*, defined as follows:

$$\mathrm{H}^i_I(D,C)\cong\varinjlim_{j\in\mathbb{N}}\operatorname{Ext}^i_A(D/I^jD,C)$$

(in particular, $\mathrm{H}^i_I(D,C)=0$ for all integers $i<0$). It is worth noting that $\mathrm{H}^i_I(R,C)$ aligns with the ordinary local cohomology module $\mathrm{H}^i_I(C)$

of C with respect to I. In continuation of aforementioned question, we consider the following question related to generalized local cohomology (see [32, Question 2.7]):

Question 1.2. *Under what conditions does the generalized local cohomology module* $\mathrm{H}^i_I(F,G)$ *possess the property of* I*-cofiniteness as an* A*-module for every integer* i*?*

This question has been extensively investigated, with positive findings demonstrated in numerous studies (for example, refer to [32, 14, 18, 8, 13, 11, 3]). In particular, Hassanzadeh and the first author established in [18, Theorem 3.7] that if F is finitely generated with finite projective dimension and C is finitely generated such that $\mathrm{H}^i_I(C)$ is I-cofinite for all i, assuming the category of I-cofinite A-modules is Abelian, then $\mathrm{H}^i_I(F,C)$ is I-cofinite for all i.

The A-module C is considered to be $\mathrm{FD}_{<n}$ (or *in dimension* $< n$) if there exists a finitely generated submodule $C' \subseteq C$ such that $\dim_A(C/C') < n$ [2, 5]. According to [33, Theorem 2.3], the collection of all $\mathrm{FD}_{<n}$ A-modules forms a Serre subcategory within the category of A-modules; that is, this class is closed under taking submodules, quotients, and extensions. An A-module C that is I-torsion is said to be $(\mathrm{FD}_{<n}, I)$-cofinite if the modules $\mathrm{Ext}^i_A(A/I,C)$ are $\mathrm{FD}_{<n}$ over A for all integers i [4, Definition 4.1]. It is important to note that C is finitely generated over A precisely when it is $\mathrm{FD}_{<0}$, and C is I-cofinite if and only if it is $(\mathrm{FD}_{<0}, I)$-cofinite. This conceptual framework naturally raises broader research questions that extend previous investigations. Specifically, the research investigates the following questions (see [1, 27, 29, 28, 30, 26, 31]):

Question 1.3. *Under what conditions does the local cohomology module* $\mathrm{H}^i_I(G)$ *possess the property of* $(\mathrm{FD}_{<n}, I)$*-cofiniteness as an* A*-module for every integer* i*?*

Question 1.4. *Under what conditions does the generalized local cohomology module* $\mathrm{H}^i_I(F,G)$ *possess the property of* $(\mathrm{FD}_{<n}, I)$*-cofiniteness as an* A*-module for every integer* i*?*

In this study, we examine Question 1.4. Utilizing Grothendieck spectral sequences, we extend and refine [18, Theorem 3.7] to broaden their

scope to include modules that are $(\mathrm{FD}_{<n}, I)$-cofinite, without imposing restrictions on the finiteness of the projective dimension of F or the requirement that C be finitely generated. Specifically, assuming that $\dim(A) \leq n+2$ (respectively, $\dim(A/I) \leq n+1$) and that $\mathrm{Ext}^i_A(A/I, C)$ is $\mathrm{FD}_{<n}$ for all $i \leq n+2$ (respectively, for all i), we conclude that $\mathrm{H}^i_I(F, C)$ is $(\mathrm{FD}_{<n}, I)$-cofinite for all i. Furthermore, we present additional results concerning the classical cofiniteness of the generalized local cohomology modules. In particular, we observe that if the ideal I meets the criteria outlined in Question 1.1, then it also satisfies the criteria specified in Question 1.2.

2 Main Results

Roughly speaking, an A-module is said to be $\mathrm{FD}_{<n}$ if it is "almost finitely generated" in the sense that, up to a finitely generated submodule, the quotient has Krull dimension strictly less than n. For $n = 0$, the condition $\dim < 0$ implies that the quotient must be zero; therefore, $\mathrm{FD}_{<0}$ simply means finitely generated. Note that every finitely generated module, as well as every module with Krull dimension less than n, is $\mathrm{FD}_{<n}$ (however, the converse of this statement is not true; see Example 2.1(a)). Now, an I-torsion module C is called $(\mathrm{FD}_{<n}, I)$-cofinite if the modules $\mathrm{Ext}^i_A(A/I, C)$ are $\mathrm{FD}_{<n}$ for all integers i. This generalizes the classical notion of I-cofiniteness, which corresponds to the case $n = 0$; in that case, $\mathrm{Ext}^i_A(A/I, C)$ are finitely generated for all integers i. The concept of $(\mathrm{FD}_{<n}, I)$-cofiniteness thus extends the traditional framework by allowing the Ext modules to be "almost finitely generated" with Krull dimension less than n, rather than requiring strict finite generation.

Example 2.1. Suppose that $A = k[x_1, \ldots, x_{n+2}]$, where k is a field and $x_1, \ldots, x_{n+2}$ are independent indeterminates. Additionally, assume that $P = (x_1, x_2, x_3)$, $C = \oplus_{i\in\mathbb{N}} A/P^i$, and $D = A \oplus C$.

(a) The module D is an $\mathrm{FD}_{<n}$ A-module that is not finitely generated and does not have Krull dimension less than n.

(b) The module C is a P-torsion A-module. Since $\dim_A(C) < n$, it follows that $\dim_A(\mathrm{Ext}^i_A(A/P, C)) < n$ and so $\mathrm{Ext}^i_A(A/P, C)$ is an

$\mathrm{FD}_{<n}$ A-module for all integers i. Therefore, C is an $(\mathrm{FD}_{<n}, P)$-cofinite A-module that is not finitely generated.

(c) Given that $\dim_A(C) < n$, for any ideal I of A, the modules $\mathrm{Ext}^i_A(A/I, C)$ satisfy $\dim_A(\mathrm{Ext}^i_A(A/I, C)) < n$. Consequently, the modules $\mathrm{Ext}^i_A(A/I, C)$ are $\mathrm{FD}_{<n}$ A-modules for all integers i. According to [29, Theorem 3.1], the local cohomology modules $\mathrm{H}^i_I(C)$ are $(\mathrm{FD}_{<n}, I)$-cofinite A-modules for all integers i.

The classical result by Hassanzadeh and the first author [18, Theorem 3.7] states that if F is finitely generated with finite projective dimension, C is finitely generated, and $\mathrm{H}^i_I(C)$ is I-cofinite for all i (where the category of I-cofinite modules is Abelian), then $\mathrm{H}^i_I(F, C)$ is I-cofinite for all i. This result has two primary limitations: first, it requires that C be finitely generated; second, it assumes that F has finite projective dimension. Furthermore, it addresses only classical I-cofiniteness (corresponding to the case $n = 0$ in our terminology). The objective of the present work is to eliminate both of these restrictions simultaneously and to extend the result to the more general setting of $(\mathrm{FD}_{<n}, I)$-cofiniteness. Specifically, we permit C to be an arbitrary (not necessarily finitely generated) A-module, and we do not impose any finiteness condition on the projective dimension of F. Under the assumptions that $\mathrm{H}^i_I(C)$ is $(\mathrm{FD}_{<n}, I)$-cofinite for all i, and that the category of such modules is Abelian, we establish in the following theorem that $\mathrm{H}^i_I(F, C)$ is also $(\mathrm{FD}_{<n}, I)$-cofinite for all i. This generalization not only encompasses the classical result (by setting $n = 0$) but also offers new insights for finitely generated modules F with infinite projective dimension and for modules C that are not finitely generated.

Theorem 2.2. *Let F be a finitely generated A-module and let C be an arbitrary A-module such that the local cohomology modules $\mathrm{H}^i_I(C)$ (computed over A) are $(\mathrm{FD}_{<n}, I)$-cofinite A-modules for all integers i. Additionally, assume that the category of $(\mathrm{FD}_{<n}, I)$-cofinite A-modules forms an Abelian category. Then the generalized local cohomology modules $\mathrm{H}^i_I(F, C)$ are $(\mathrm{FD}_{<n}, I)$-cofinite A-modules for all integers i.*

Proof. According to [24, Theorem 10.47] and [15, Lemma 2.1(i)], a

third quadrant spectral sequence exists with

$$\mathrm{I}_2^{u,v} := \operatorname{Ext}_A^u(F, \mathrm{H}_I^v(C)) \underset{u}{\Longrightarrow} \mathrm{H}_I^{u+v}(F, C).$$

We proceed by induction on $k \geq 2$ to demonstrate that each term $\mathrm{I}_k^{j,i}$ in the spectral sequence is an $(\mathrm{FD}_{<n}, I)$-cofinite A-module for all integers j, i.

Base case ($k = 2$). Fix $i \geq 0$. Select a free resolution of F,

$$F'_\bullet = \cdots \longrightarrow F'_j \longrightarrow \cdots \longrightarrow F'_0 \longrightarrow F \longrightarrow 0,$$

where each F'_j is a finitely generated free A-module. Applying the functor $\operatorname{Hom}_A(-, \mathrm{H}_I^i(C))$ to its deletion, denoted as $F'_{\bullet F}$, yields a complex

$$\operatorname{Hom}_A(F'_{\bullet F}, \mathrm{H}_I^i(C)) =$$

$$0 \longrightarrow \operatorname{Hom}_A(F'_0, \mathrm{H}_I^i(C)) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_A(F'_j, \mathrm{H}_I^i(C)) \longrightarrow \cdots .$$

Each $\operatorname{Hom}_A(F'_j, \mathrm{H}_I^i(C))$ is $(\mathrm{FD}_{<n}, I)$-cofinite because F'_j is finitely generated and free, and $\mathrm{H}_I^i(C)$ is $(\mathrm{FD}_{<n}, I)$-cofinite from assumption. Consequently, $\mathrm{I}_2^{j,i} = \operatorname{Ext}_A^j(F, \mathrm{H}_I^i(C))$ is $(\mathrm{FD}_{<n}, I)$-cofinite for all j, by the assumption that the category of $(\mathrm{FD}_{<n}, I)$-cofinite A-modules is Abelian.

Inductive step ($k > 2$). Assume that $\mathrm{I}_{k-1}^{j,i}$ is an $(\mathrm{FD}_{<n}, I)$-cofinite A-module for all integers j, i. Fix $j, i \geq 0$. Based on the induction hypothesis, the modules $\mathrm{I}_{k-1}^{j-k+1,i+k-2}$, $\mathrm{I}_{k-1}^{j,i}$, and $\mathrm{I}_{k-1}^{j+k-1,i-k+2}$ are known to be $(\mathrm{FD}_{<n}, I)$-cofinite modules over A. Consequently, $\operatorname{Im}(\mathrm{I}_{k-1}^{j-k+1,i+k-2} \to \mathrm{I}_{k-1}^{j,i})$ and $\operatorname{Ker}(\mathrm{I}_{k-1}^{j,i} \to \mathrm{I}_{k-1}^{j+k-1,i-k+2})$ are also $(\mathrm{FD}_{<n}, I)$-cofinite modules over A, given that the category of such modules is Abelian. Therefore, the module $\mathrm{I}_k^{j,i}$, defined as the quotient

$$\operatorname{Ker}(\mathrm{I}_{k-1}^{j,i} \to \mathrm{I}_{k-1}^{j+k-1,i-k+2}) / \operatorname{Im}(\mathrm{I}_{k-1}^{j-k+1,i+k-2} \to \mathrm{I}_{k-1}^{j,i})$$

is itself an $(\mathrm{FD}_{<n}, I)$-cofinite A-module. This completes the inductive step.

Now, let i be a non-negative integer. For all $j \leq i$, we observe that $\mathrm{I}_\infty^{i-j,j} = \mathrm{I}_{i+2}^{i-j,j}$, since the A-modules $\mathrm{I}_k^{i-j-k,j-1+k}$ and $\mathrm{I}_k^{i-j+k,j+1-k}$ are

zero for all $k \geq i+2$. Consequently, the modules $\mathrm{I}_{\infty}^{i-j,j}$ are $(\mathrm{FD}_{<n}, I)$-cofinite over A for all $j \leq i$. Furthermore, there exists a finite filtration given by

$$0 = \phi^{i+1}\mathrm{H}^i \subseteq \phi^i \mathrm{H}^i \subseteq \cdots \subseteq \phi^1 \mathrm{H}^i \subseteq \phi^0 \mathrm{H}^i = \mathrm{H}_I^i(F, C),$$

such that for all $j \leq i$, we have $\mathrm{I}_{\infty}^{i-j,j} = \phi^{i-j}\mathrm{H}^i / \phi^{i-j+1}\mathrm{H}^i$. Consequently, the short exact sequences

$$0 \longrightarrow \phi^{i-j+1}\mathrm{H}^i \longrightarrow \phi^{i-j}\mathrm{H}^i \longrightarrow \mathrm{I}_{\infty}^{i-j,j} \longrightarrow 0$$

for all $j \leq i$, imply that $\phi^{i-j}\mathrm{H}^i$ is an $(\mathrm{FD}_{<n}, I)$-cofinite A-module whenever $\phi^{i-j+1}\mathrm{H}^i$ is an $(\mathrm{FD}_{<n}, I)$-cofinite A-module. This inductively establishes that $\mathrm{H}_I^i(F, C)$ is an $(\mathrm{FD}_{<n}, I)$-cofinite A-module, thereby completing the proof. □

Remark 2.3. (a) Unlike [18, Theorem 3.7], which requires the finiteness of the projective dimension of F, our Theorem 2.2 removes this assumption entirely. This represents a considerable advancement, as the absence of finite projective dimension prevents the use of induction based on projective dimension or standard homological techniques. Instead, our proof leverages the convergence of Grothendieck spectral sequences and utilizes the Abelian properties of the category of $(\mathrm{FD}_{<n}, I)$-cofinite modules.

(b) Example 2.1(c) illustrates a module that is not finitely generated but still satisfies the conditions outlined in Theorem 2.2.

(c) Theorem 2.2 assumes the category of $(\mathrm{FD}_{<n}, I)$-cofinite A-modules is Abelian. This assumption may not hold for arbitrary values of n and I; however, it is valid when $\dim(A) \leq n+2$ (see [29, Theorem 4.1]) or when $\dim(A/I) \leq n+1$ (see [27, Corollary 3.3]).

As an application of the previously stated theorem, the following corollary generalizes and enhances [18, Corollary 3.8].

Corollary 2.4. *Given a ring A with Krull dimension at most $n+2$, let F be a finitely generated A-module, and let C be any A-module such*

that $\operatorname{Ext}^i_A(A/I,C)$ *is an* $\mathrm{FD}_{<n}$ *A-module for all* $i \leq n+2$. *Then the generalized local cohomology module* $\mathrm{H}^i_I(F,C)$ *is an* $(\mathrm{FD}_{<n},I)$*-cofinite A-module for all integers* i.

Proof. Since $\dim(A) \leq n+2$ and $\operatorname{Ext}^i_A(A/I,C)$ is an $\mathrm{FD}_{<n}$ A-module for all $i \leq n+2$, it follows that $\mathrm{H}^i_I(C)$ is an $(\mathrm{FD}_{<n},I)$-cofinite A-module for all i, as established in [29, Theorem 3.1]. Additionally, according to [29, Theorem 4.1] and given that $\dim(A) \leq n+2$, the category of $(\mathrm{FD}_{<n},I)$-cofinite A-modules is an Abelian category. Consequently, $\mathrm{H}^i_I(F,C)$ is an $(\mathrm{FD}_{<n},I)$-cofinite A-module for all i, as stated in Theorem 2.2. □

As an application of the aforementioned theorem, the next result generalizes and extends the findings presented in Corollaries 3.10, 3.11, and 3.12 of [18].

Corollary 2.5. *Let A be a ring such that the dimension of A/I is at most $n+1$ and let F be a finitely generated A-module. Let C be any A-module such that* $\operatorname{Ext}^i_A(A/I,C)$ *is an* $\mathrm{FD}_{<n}$ *A-module for all integers* i. *Then the generalized local cohomology module* $\mathrm{H}^i_I(F,C)$ *is an* $(\mathrm{FD}_{<n},I)$*-cofinite A-module for all integers* i.

Proof. Given that $\dim(A/I) \leq n+1$ and that $\operatorname{Ext}^i_A(A/I,C)$ is an $\mathrm{FD}_{<n}$ A-module for all i, it follows from [27, Corollary 4.5] that $\mathrm{H}^i_I(C)$ is an $(\mathrm{FD}_{<n},I)$-cofinite A-module for all i. Furthermore, as indicated in [27, Corollary 3.3], and considering that $\dim(A/I) \leq n+1$, the category of $(\mathrm{FD}_{<n},I)$-cofinite A-modules forms an Abelian category. Therefore, by the results established in Theorem 2.2, $\mathrm{H}^i_I(F,C)$ is an $(\mathrm{FD}_{<n},I)$-cofinite A-module for all i. □

Corollaries 2.4 and 2.5 are complementary. Corollary 2.4 assumes $\dim(A) \leq n+2$ and requires the Ext condition only for $i \leq n+2$, while Corollary 2.5 assumes $\dim(A/I) \leq n+1$ but requires the Ext condition for all i. Neither implies the other; the choice depends on which dimension bound is available.

The following corollary generalizes [18, Theorem 3.7], where F is of finite projective dimension and C is finitely generated.

Corollary 2.6. *Let F be a finitely generated A-module and let C be an arbitrary A-module such that the local cohomology modules $\mathrm{H}^i_I(C)$ (computed over A) are I-cofinite A-modules for all integers i. Additionally, assume that the category of I-cofinite A-modules forms an Abelian category. Then the generalized local cohomology modules $\mathrm{H}^i_I(F, C)$ are I-cofinite A-modules for all integers i.*

Proof. This follows directly from Theorem 2.2 by setting $n = 0$. Indeed, $\mathrm{FD}_{<0}$ means finitely generated, so $(\mathrm{FD}_{<0}, I)$-cofiniteness coincides with classical I-cofiniteness. □

The next corollary improves [18, Corollary 3.8] by eliminating the finite projective dimension condition on F and the finite generation condition on C.

Corollary 2.7. *Given a ring A with Krull dimension at most 2, let F be a finitely generated A-module, and let C be any A-module such that $\mathrm{Ext}^i_A(A/I, C)$ is a finitely generated A-module for all $i \leq 2$. Then the generalized local cohomology module $\mathrm{H}^i_I(F, C)$ is an I-cofinite A-module for all integers i.*

Proof. This results from Corollary 2.4 by setting $n = 0$, since $\dim(A) \leq 2$ becomes $\dim(A) \leq n + 2$, $\mathrm{FD}_{<0}$ indicates finitely generated modules, and $(\mathrm{FD}_{<0}, I)$-cofiniteness aligns with I-cofiniteness. □

The following corollary extends simultaneously [18, Corollaries 3.10, 3.11, and 3.12] by removing the additional assumptions imposed on A, I, F, and C.

Corollary 2.8. *Let A be a ring such that the dimension of A/I is at most 1 and let F be a finitely generated A-module. Let C be any A-module such that $\mathrm{Ext}^i_A(A/I, C)$ is a finitely generated A-module for all integers i. Then the generalized local cohomology module $\mathrm{H}^i_I(F, C)$ is an I-cofinite A-module for all integers i.*

Proof. This directly follows from Corollary 2.5 with $n = 0$, as the condition $\dim(A/I) \leq 1$ coincides with $\dim(A/I) \leq n + 1$, $\mathrm{FD}_{<0}$ indicates finite generation, and $(\mathrm{FD}_{<0}, I)$-cofiniteness is consistent with

I-cofiniteness. □

In the following corollary, we demonstrate that if I satisfies the condition outlined in Question 1.1 (i.e., $\mathrm{H}^i_I(G)$ is I-cofinite for every finitely generated A-module G and all integers i), then it consequently satisfies the condition specified in Question 1.2. The latter condition means that $\mathrm{H}^i_I(F,G)$ is I-cofinite for every finitely generated A-modules F and G, and all integers i. Thus, under this strong hypothesis, the I-cofiniteness property extends from local cohomology modules to generalized local cohomology modules; in particular, the result applies to all finitely generated A-modules F and G. It is important to note that this hypothesis is known to hold in several important cases, including:

- When $\dim(A) \leq 2$ (see [22, Theorem 7.10]);
- When $\dim(A/I) = 1$ (see [23, Corollary 2.12]);
- When $\mathrm{cd}(I) = 1$ (see [22, Corollary 3.14]).

In each of these cases, the hypothesis of the corollary is automatically satisfied, and thus Corollary 2.9 applies. Recall that the cohomological dimension of A at I, denoted $\mathrm{cd}(I)$ (or $\mathrm{cd}(I,A)$), is defined as

$$\mathrm{cd}(I) = \sup\{i \in \mathbb{Z} \mid \mathrm{H}^i_I(A) \neq 0\}.$$

If $\mathrm{cd}(I) = 1$, then the local cohomology modules $\mathrm{H}^i_I(A)$ vanish for all integers $i \geq 2$.

Corollary 2.9. *Suppose that $\mathrm{H}^i_I(G)$ is an I-cofinite A-module for every finitely generated A-module G and all integers i. Then $\mathrm{H}^i_I(F,G)$ is an I-cofinite A-module for all finitely generated A-modules F and G, and all integers i.*

Proof. Since $\mathrm{H}^i_I(G)$ is an I-cofinite A-module for every finitely generated A-module G and for all integers i, it follows from [6, Theorem 5.3] that the category of I-cofinite A-modules is Abelian. Consequently, $\mathrm{H}^i_I(F,G)$ is an I-cofinite A-module for all finitely generated A-modules F and G, and for all integers i, as established in Corollary 2.6. □

The subsequent result generalizes [18, Corollary 3.9], where I is a principle ideal and F is of finite projective dimension. Note that if I is a principle ideal, then $\operatorname{cd}(I) \leq 1$, as established in [9, Theorem 3.3.1].

Corollary 2.10. *Suppose I is an ideal of A with $\operatorname{cd}(I) = 1$. Additionally, assume that F and G are finitely generated A-modules. Then $\mathrm{H}^i_I(F, G)$ is an I-cofinite A-module for all integers i.*

Proof. Since $\operatorname{cd}(I) = 1$, $\mathrm{H}^i_I(N)$ is an I-cofinite A-module for every finitely generated A-module N and all integers i, as established in [22, Corollary 3.14]. Consequently, by applying Corollary 2.9, $\mathrm{H}^i_I(F, G)$ is an I-cofinite A-module for all integers i. □

Concluding remarks

We have extended the classical results on cofiniteness of generalized local cohomology modules to the broader setting of $(\mathrm{FD}_{<n}, I)$-cofiniteness.

Among other things, Theorem 2.2 removes the previous restrictive assumptions, such as finite projective dimension for F and the requirement that C be finitely generated, which were presented in [18, Theorem 3.7]. The key tools are Grothendieck spectral sequences and the Abelianness of the category of $(\mathrm{FD}_{<n}, I)$-cofinite modules.

A natural question for future research is whether the Abelianness assumption on the category of $(\mathrm{FD}_{<n}, I)$-cofinite modules can be derived from weaker conditions, or whether it can be removed entirely.

In Corollary 2.9, we show that if the ideal I meets the criteria outlined in Question 1.1, then it also satisfies the criteria specified in Question 1.2.

Another natural question is: if for any finitely generated A-module G (rather than for all such modules), $\mathrm{H}^i_I(G)$ is I-cofinite for all i, does it follow, perhaps under some weak assumptions, that $\mathrm{H}^i_I(F, G)$ is I-cofinite for every finitely generated A-module F and for all i?

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