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Original Research Paper

On Topologically Super Weakly Mixing

S. R. Parsa
Yasouj University

M. Asadipour*
Yasouj University

Abstract. This paper introduces and investigates two new classes of operators: *topologically super weakly mixing operators* and *locally topologically super weakly mixing operators*. We explore key topological properties of locally topologically super weakly mixing operators and provide an explicit example to demonstrate that the class of topologically weakly mixing operators is a proper subclass of the class of topologically super weakly mixing operators. Furthermore, we prove a theorem that offers a definitive negative answer to a long-standing open problem in the theory of J-class operators.

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1 Introduction

1.1 Background and preliminaries

It is widely accepted that the field of linear dynamics originated in 1982 with Kitai's seminal Ph.D. thesis at the University of Toronto [20], which

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*Corresponding Author

laid the groundwork for subsequent developments in operator theory. As the name suggests, linear dynamics concerns the iterative behavior of linear transformations. While the theory of linear transformations is well understood in finite-dimensional spaces where the Jordan canonical form offers a complete description many intriguing phenomena arise in infinite-dimensional settings. Notable among these are *dense orbits* and *dense projective orbits* of linear operators, encapsulated by the concepts of *hypercyclicity* and *supercyclicity*, respectively. These concepts have significantly influenced the development of linear dynamics.

To clarify the terminology, let X be a Banach space. An operator T on X is said to be:
- *hypercyclic* if there exists a vector $x \in X$ such that

$$\overline{\mathrm{Orb}(T,x)} := \overline{\{T^j x : j \in \mathbb{N} \cup \{0\}\}} = X,$$

- *supercyclic* if there exists a vector $x \in X$ such that

$$\overline{\mathbb{C} \cdot \mathrm{Orb}(T,x)} = X.$$

Such a vector x is referred to as a *hypercyclic* (or *supercyclic*) vector for T, and the sets of such vectors are denoted by $HC(T)$ and $SC(T)$, respectively.

A foundational result in this context asserts that an operator T on a separable Banach space X is hypercyclic if and only if it is *topologically transitive*; that is, for every pair of nonempty open sets $U, V \subseteq X$, there exists a positive integer n such that $T^n(U) \cap V \neq \emptyset$.

In many classical examples, the set $T^n(U)$ not only intersects V for some n, but continues to do so for all sufficiently large integers n. This persistent behavior motivates the notion of a *mixing* operator. The following chain of implications illustrates the hierarchy:

$$\text{Mixing} \Rightarrow \text{Topological Transitivity} \Leftrightarrow \text{Hypercyclicity} \Rightarrow \text{Supercyclicity}.$$

Between topological transitivity and mixing lies an intermediate property: if the direct sum operator $T \oplus T$ is hypercyclic on $X \oplus X$, then T is called *topologically weakly mixing* (or simply *weakly mixing*). The hierarchy can then be refined as:

$$\text{Mixing} \Rightarrow \text{Topologically Weakly Mixing} \Rightarrow \text{Hypercyclicity}.$$

Additionally, an operator $T \in B(X)$ is weakly mixing if and only if for any pair of nonempty open sets $U, V \subset X$ and any neighborhood W of the origin, the intersection

$$N_T(U, W) \cap N_T(W, V) \neq \emptyset$$

is nonempty, where

$$N_T(U_1, U_2) := \{n \in \mathbb{N} : T^n(U_1) \cap U_2 \neq \emptyset\}.$$

This characterization appears in [15].

A major advancement in the theory of hypercyclicity is the so-called *Hypercyclicity Criterion*, introduced by Kitai [20] and independently by Gethner and Shapiro [15]. Building upon this, Salas formulated the following result, known as the *Supercyclicity Criterion* [7]:

Theorem 1.1 (**Supercyclicity Criterion**). *Let T be a bounded linear operator on a Banach space X. Suppose there exist an increasing sequence of positive integers (n_k), dense subsets $X_0, Y_0 \subset X$, and a sequence of mappings $\widetilde{T}_{n_k} : Y_0 \to X$ such that:*

1. $\|T^{n_k}(x)\| \cdot \|\widetilde{T}_{n_k}(y)\| \to 0$ *as* $k \to \infty$*, for all* $x \in X_0$ *and* $y \in Y_0$*;*
2. $T^{n_k}\widetilde{T}_{n_k}(y) \to y$ *as* $k \to \infty$*, for all* $y \in Y_0$*,*

then $SC(T) \neq \emptyset$*.*

For further developments and in-depth discussions related to the aforementioned phenomena, we refer the reader to [1], [7], [9], [10], and [14], where substantial results and illustrative examples are presented.

In a different direction, as emphasized in [11, 12], the authors explored the dynamics of linear operators via the notion of *extended limit sets*, rather than by directly analyzing operator orbits. Given a bounded linear operator T on a Banach space X, the *extended limit set* of a point $x \in X$ is defined as:

$$J_T(x) = \left\{ z \in X \;:\; \begin{array}{l} \text{there exist sequences } \{x_n\} \subset X \text{ and } \{k_n\} \subset \mathbb{N} \\ \text{with } x_n \to x,\ k_n < k_{n+1}, \text{ such that} \\ T^{k_n} x_n \to z \end{array} \right\}.$$

Subsequently, an operator T is called a *J-class operator* if there exists a nonzero vector $x \in X$ such that $J_T(x) = X$.

It is well known that hypercyclic operators cannot exist on non-separable Banach spaces. Nevertheless, it has been established that topologically transitive operators may exist in such spaces; see, for example, [9]. In contrast, some authors have shown that the non-separable Banach space $\ell^\infty(\mathbb{N})$, consisting of bounded sequences over $\mathbb{C}$, does not admit any topologically transitive operators [8]. This discussion becomes even more compelling when considering the results of [11, 12], which provide examples of non-hypercyclic operators on $\ell^\infty(\mathbb{N})$ that nevertheless belong to the J-class. Although the J-class has been extensively studied by numerous researchers (see [2, 3, 5, 6, 18, 19, 21, 22, 24, 26, 27] for comprehensive studies), we present below the J-class criterion from [2] to familiarize the reader with this important result, as it plays a key role in the current work.

Theorem 1.2 (**J-Class Criterion**)**.** *Let T be a bounded linear operator acting on an infinite-dimensional Banach space X. Suppose there exists a nonzero vector $x \in X$, a strictly increasing sequence of positive integers $\{k_n\}$, and a dense subset $Y \subseteq X$ such that:*

(i) *There exists a sequence $\{x_n\} \subset X$ with $x_n \to x$ and $T^{k_n} x_n \to 0$; that is, $0 \in J_T(x)$.*

(ii) *For every $y \in Y$, there exists a sequence $\{w_n\} \subset X$ such that $w_n \to 0$ and $T^{k_n} w_n \to y$; that is, $y \in J_T(0)$.*

Then x is a J-class vector for T.

Note that the J-class criterion is not only a foundational result in the study of J-class operators, but also serves as the most widely used sufficient condition for verifying that an operator possesses a J-class vector. It is also noteworthy that, based on this criterion, a question was raised in [2], which we will address in the final part of the next section.

1.2 Organization of the paper

The next section, titled *Main Results*, is divided into the following parts:

- The first part introduces a new class of operators on Banach spaces, namely, *topologically super weakly mixing operators*. Then a *locally topologically super weakly mixing operator* is introduced. In the remainder of this part, we examine several of their key topological properties. The part culminates in Theorem 2.10, which offers a localized characterization of topologically super weakly mixing operators.

- The second and final part of Section 2 presents two principal contributions of this paper:

  (i) Example 2.11 constructs an explicit operator showing that the class of topologically weakly mixing operators is a proper subclass of the class of topologically super weakly mixing operators.

  (ii) Theorem 2.14 provides a definitive counterexample to [2, Question 2.21], thereby resolving the open problem in the negative.

2 Main Results

2.1 Topologically super weakly mixing and localization of the concept

Instead of explicitly constructing a supercyclic vector, one can leverage the notion of *topological super transitivity* to establish supercyclicity of an operator. This approach often yields cleaner and more elegant proofs, especially when dealing with intricate operators. We begin with the definition of topologically super transitive operators, followed by a key result (Proposition 2.2) that highlights an important structural property.

Definition 2.1. An operator $T \in B(X)$ is said to be *topologically super transitive* if for every pair of nonempty open sets $U, V \subset X$, there exist a positive integer $n \in \mathbb{N}$ and a complex scalar $\lambda \in \mathbb{C}$ such that

$$\lambda T^n(U) \cap V \neq \emptyset.$$

The following proposition shows that there exists an infinite set of integers n such that the above intersection is nonempty.

Proposition 2.2. *Let $T \in B(X)$. The following statements are equivalent:*

(i) *T is topologically super transitive;*

(ii) *For every pair of nonempty open subsets $U, V \subset X$, the set*

$$SN_T(U,V) := \{n \in \mathbb{N} \,:\, \lambda T^n(U) \cap V \neq \emptyset \text{ for some } \lambda \in \mathbb{C}\}$$

is infinite.

Proof. The implication *(ii)* $\Rightarrow$ *(i)* is immediate. To prove the converse, assume that T is topologically super transitive, and let $U, V \subset X$ be arbitrary nonempty open subsets. Consider the following two cases:

Case 1: $0 \in V$. In this case, taking $\lambda = 0$ immediately yields $\lambda T^n(U) \cap V \neq \emptyset$ for all $n \in \mathbb{N}$, hence $SN_T(U,V) = \mathbb{N}$.

Case 2: $0 \notin V$. By assumption, there exists $n \in \mathbb{N}$ and $\lambda \neq 0$ such that $\lambda T^n(U) \cap V \neq \emptyset$. Let

$$W := \lambda U \cap T^{-n}(V).$$

Then there exists $k \in SN_T(W,W)$, i.e., for some $\mu \in \mathbb{C}$,

$$\mu W \cap T^{-k}(W) \neq \emptyset.$$

Observe that:

$$\mu[\lambda U \cap T^{-n}(V)] \cap T^{-k}(\lambda U \cap T^{-n}(V)) \subseteq \mu\lambda U \cap T^{-k-n}(V),$$

which implies

$$\mu\lambda T^{k+n}(U) \cap V \neq \emptyset.$$

Hence $k + n \in SN_T(U,V)$, and repeating this process yields infinitely many such integers. Therefore, $SN_T(U,V)$ is infinite. $\square$

Next, we introduce the notion of *topologically super weakly mixing* operators. While this can be defined analogously to topological weakly mixing (via $T \oplus T$), we present the definition using open sets, as this form is more suitable for introducing the concept of $\widetilde{SJ}$-class operators.

Definition 2.3. An operator $T \in B(X)$ is said to be *topologically super weakly mixing* if for every pair of nonempty open sets $U, V \subset X$, every neighborhood $W \subset X$ of zero, and every positive integer $N \in \mathbb{N}$, there exist an integer $n > N$ and $\lambda \in \mathbb{C}$ such that

$$\lambda T^n(U) \cap W \neq \emptyset \quad \text{and} \quad \lambda T^n(W) \cap V \neq \emptyset.$$

The following theorem is analogous to [17, Theorem 2.47]. Due to its similarity, we omit the proof.

Theorem 2.4. *An operator $T \in B(X)$ is topologically super weakly mixing if and only if $T \oplus T$ is topologically super transitive.*

We now define the localized version of this concept.

Definition 2.5. For $T \in B(X)$ and $x \in X$, the set $\widetilde{SJ}_T(x)$ is defined as:

$$\widetilde{SJ}_T(x) = \left\{ y \in X \ : \begin{array}{l} \text{for every neighborhood } U_x \text{ of } x, \ U_y \text{ of } y, \text{ and} \\ U_0 \text{ of } 0, \text{ and every } N \in \mathbb{N}, \text{ there exist } n > N \\ \text{and } \lambda_n \in \mathbb{C} \text{ such that } \lambda_n T^n(U_x) \cap U_0 \neq \emptyset \\ \text{and } \ \lambda_n T^n(U_0) \cap U_y \neq \emptyset. \end{array} \right\}. \tag{1}$$

An operator $T \in B(X)$ is called a *$\widetilde{SJ}$-class operator* (or *locally topologically super weakly mixing*) if there exists a nonzero vector $x \in X$ such that $\widetilde{SJ}_T(x) = X$.

The following theorem establishes that any nonzero scalar multiple of an $\widetilde{SJ}$-class operator remains within the $\widetilde{SJ}$-class.

Theorem 2.6. *Let $T \in B(X)$ be a $\widetilde{SJ}$-class operator on a Banach space X, and suppose $\widetilde{SJ}_T(x) = X$ for some nonzero vector $x \in X$. Then, for any nonzero scalar $\alpha \in \mathbb{C}$, it follows that $\widetilde{SJ}_{\alpha T}(x) = X$.*

Proof. For every $y \in X$, integer $N > 0$, and neighborhoods U_x, U_y, U_0 of x, y, and 0 respectively, by (1) there exist $n > N$ and $\lambda_n \in \mathbb{C}$ such that

$$\lambda_n T^n(U_x) \cap U_0 \neq \emptyset \quad \text{and} \quad \lambda_n T^n(U_0) \cap U_y \neq \emptyset.$$

Then:

$$(\lambda_n\alpha^{-n})(\alpha T)^n(U_x)\cap U_0\neq\emptyset,\quad (\lambda_n\alpha^{-n})(\alpha T)^n(U_0)\cap U_y\neq\emptyset.$$

Hence $y\in\widetilde{SJ}_{\alpha T}(x)$, so $\widetilde{SJ}_{\alpha T}(x)=X$. $\square$

The next result not only has intrinsic interest but also reveals an important topological consequence.

Theorem 2.7. *Let $T\in B(X)$, and let sequences $\{x_n\},\{y_n\}\subset X$ converge to x and y, respectively. If $y_n\in\widetilde{SJ}_T(x_n)$ for all $n\in\mathbb{N}$, then $y\in\widetilde{SJ}_T(x)$.*

Proof. Let U_x, V_y and U_0 be neighborhoods of x,y and 0, respectively, and let $N\in\mathbb{N}$ be arbitrary. Since $x_n\to x$ and $y_n\to y$, there exists $n_0>N$ such that $x_{n_0}\in U_x$, $y_{n_0}\in V_y$. Because $y_{n_0}\in\widetilde{SJ}_T(x_{n_0})$, there exist $k>n_0$ and $\lambda_k\in\mathbb{C}$ such that:

$$\lambda_kT^k(U_x)\cap U_0\neq\emptyset\quad\text{and}\quad\lambda_kT^k(U_0)\cap V_y\neq\emptyset.$$

Thus $y\in\widetilde{SJ}_T(x)$. $\square$

Corollary 2.8. *For every $T\in B(X)$ and $x\in X$, both $\widetilde{SJ}_T(x)$ and*

$$\widetilde{SJ}_T:=\{x\in X:\widetilde{SJ}_T(x)=X\}$$

are closed subsets of X.

We now present a topological result concerning the structure of $\widetilde{SJ}_T$.

Theorem 2.9. *For every $T\in B(X)$, the set $\widetilde{SJ}_T$ is connected.*

Proof. We proceed in two steps.

Step 1. We characterize $\widetilde{SJ}_T(x)$ in terms of sequences:

$$\widetilde{SJ}_T(x)=\left\{z\in X:\begin{array}{c}\text{There exist sequences }\{x_n\},\{w_n\}\subset X,\{\lambda_n\}\subset\mathbb{C},\\ \{k_n\}\subset\mathbb{N}\text{ with }x_n\to x,\ w_n\to 0,\ k_n<k_{n+1},\text{ such}\\ \text{that }\lambda_nT^{k_n}x_n\to 0,\ \lambda_nT^{k_n}w_n\to z\end{array}\right\}. \tag{2}$$

To prove this, consider neighborhoods U_x and V_z of x and z, respectively. Let $\epsilon > 0$ be chosen such that

$$B(x, \epsilon) \subset U_x \quad \text{and} \quad B(z, \epsilon) \subset V_z.$$

Assume, for $n = 1, 2, \ldots, j$, there exist complex scalars λ_n, vectors x_n, w_n in $B(x, \frac{\epsilon}{n}), B(0, \frac{\epsilon}{n})$, respectively, and integers k_n satisfying

$$\lambda_n T^{k_n} x_n \in W_n \quad \text{and} \quad \lambda_n T^{k_n} w_n \in B(z, \frac{\epsilon}{n}),$$

where $W_n := B(0, \frac{\epsilon}{n})$, with $1 \leqslant k_{n-1} < k_n$. Next, consider the open balls

$$B(x, \frac{\epsilon}{j+1}), B(z, \frac{\epsilon}{j+1}) \text{ and } W_{j+1} := B(0, \frac{\epsilon}{j+1}).$$

By (1), there exist a complex number λ_{j+1} and an integer $k_{j+1} \geq k_j + 1$ such that the intersections

$$\lambda_{j+1} T^{k_{j+1}}(B(x, \frac{\epsilon}{j+1})) \cap W_{j+1} \quad \text{and} \quad \lambda_{j+1} T^{k_{j+1}}(W_{j+1}) \cap B(z, \frac{\epsilon}{j+1})$$

are nonempty. Consequently, we can select $x_{j+1} \in B(x, \frac{\epsilon}{j+1})$ and $w_{j+1} \in W_{j+1}$ such that

$$\lambda_{j+1} T^{k_{j+1}} x_{j+1} \in W_{j+1} \quad \text{and} \quad \lambda_{j+1} T^{k_{j+1}} w_{j+1} \in B(z, \frac{\epsilon}{j+1}).$$

Hence, by induction, we construct sequences $\{x_n\}, \{w_n\} \subset X$, $\{\lambda_n\} \subset \mathbb{C}$, and $\{k_n\} \subset \mathbb{N}$ with this properties;

$$x_n \to x, \;\; w_n \to 0, \;\; k_n < k_{n+1}, \;\; \lambda_n T^{k_n} x_n \to 0 \;\; \textit{and} \;\; \lambda_n T^{k_n} w_n \to z.$$

This completes the proof of the claim for $\widetilde{SJ}_T(x)$ because the inclusion in the reverse direction is evident.

Step 2. Let $x \in \widetilde{SJ}_T$ and $y \in X$. In light of (2), for any nonzero $\mu \in \mathbb{C}$, there exist sequences $\{x_n\}, \{w_n\} \subset X$, $\{\lambda_n\} \subset \mathbb{C}$, and $\{k_n\} \subset \mathbb{N}$ such that $k_n < k_{n+1}$ and the following conditions are satisfied:

$$x_n \to x, \;\; w_n \to 0, \;\; \lambda_n T^{k_n} x_n \to 0, \;\; \lambda_n T^{k_n} w_n \to \frac{1}{\mu} y$$

and subsequently

$$\mu x_n \to \mu x, \;\; \mu w_n \to 0, \;\; \lambda_n T^{k_n}(\mu x_n) \to 0, \;\; \lambda_n T^{k_n}(\mu w_n) \to y$$

as $n \to \infty$. This yields that $\mu x \in \widetilde{SJ}_T$ or equivalently, $\widetilde{SJ}_T(\mu x) = X$. Additionally, if there exists a sequence $\{\mu_n\}$ of nonzero complex numbers such that $\mu_n \to 0$, then $\widetilde{SJ}_T(0) = X$ by Corollary 2.8. Accordingly, since $\mathbb{C}x$ is connected, we write:

$$\widetilde{SJ}_T = \bigcup_{x \in \widetilde{SJ}_T} \mathbb{C}x,$$

and since

$$0 \in \bigcap_{x \in \widetilde{SJ}_T} \mathbb{C}x,$$

so $\widetilde{SJ}_T$ is a union of connected sets with nonempty intersections, which is desired for us. $\square$

We conclude this subsection with a fundamental characterization theorem.

Theorem 2.10. *Let X be a separable Banach space and $T \in B(X)$. The following statements are equivalent:*

(i) *T is topologically super weakly mixing.*

(ii) *For every $x \in X$, T is locally topologically super weakly mixing at x, i.e., $\widetilde{SJ}_T(x) = X$.*

(iii) *$\widetilde{SJ}_T$ is dense in X.*

Proof. *(i) $\Rightarrow$ (ii):* Let $N \in \mathbb{N}$, $x, y \in X$, and neighborhoods U_x, V_y, W_0 of $x, y, 0$, respectively. By (1), there exists $n > N$ and $\lambda \in \mathbb{C}$ such that:

$$\lambda T^n(U_x) \cap W_0 \neq \emptyset \quad \text{and} \quad \lambda T^n(W_0) \cap V_y \neq \emptyset.$$

Hence $y \in \widetilde{SJ}_T(x)$, so $\widetilde{SJ}_T(x) = X$.

(ii) $\Rightarrow$ (iii): Immediate.

(iii) ⇒ (i): Let $U, V \subset X$ be nonempty open sets, and let W be a neighborhood of 0. By *(iii)*, there exists $x_0 \in U \cap \widetilde{SJ}_T$. Then, for any $N \in \mathbb{N}$, there exist $n > N$ and $\lambda \in \mathbb{C}$ such that:

$$\lambda T^n(U) \cap W \neq \emptyset \quad \text{and} \quad \lambda T^n(W) \cap V \neq \emptyset.$$

Hence T is topologically super weakly mixing. □

2.2 An example and answering an open question

While every topologically weakly mixing operator is trivially topologically super weakly mixing, the converse is nontrivial. This motivates the construction of operators that are topologically super weakly mixing but fail to be weakly mixing. The following example provides such a construction.

Example 2.11. Let B_w denote the weighted backward shift operator acting on $\ell^2(\mathbb{N})$. It is defined by $B_w(e_0) = 0$ and $B_w(e_n) = w_n e_{n-1}$ for $n \geq 1$, where $\{e_n\}_{n\in\mathbb{N}}$ is the canonical basis of $\ell^2(\mathbb{N})$, and $w = \{w_n\}_{n\geq 1}$ is a bounded sequence of positive scalars. We show that $B_w \oplus B_w$ fails to be topologically transitive but remains topologically super transitive under suitable conditions.

Step 1. Assume $w_n \to 0$ as $n \to \infty$. Then we have:

$$\|B_w^n\| = \sup_{i\in\mathbb{N}}(w_{i+1} \cdots w_{i+n}) \to 0 \quad \text{as } n \to \infty.$$

This implies that B_w is not topologically transitive and, therefore, cannot be topologically weakly mixing.

Step 2. Let $X_0 = Y_0 := c_{00}(\mathbb{N}) \oplus c_{00}(\mathbb{N})$, where $c_{00}(\mathbb{N})$ denotes the space of finitely supported sequences. Define the map $S_w \oplus S_w$ on Y_0 by:

$$(S_w \oplus S_w)(e_n \oplus e_m) = (w_{n+1}^{-1} e_{n+1} \oplus w_{m+1}^{-1} e_{m+1}), \quad n, m \in \mathbb{N}.$$

We claim that $B_w \oplus B_w$ satisfies the Supercyclicity Criterion with respect to the sequence $\{n_k\} = \{k\}$ and maps $S_k := S_w^k \oplus S_w^k$. Indeed, for all $x_1 \oplus x_2 \in X_0$, $y_1 \oplus y_2 \in Y_0$, we have:

$$\|(B_w \oplus B_w)^k (x_1 \oplus x_2)\| = \|B_w^k x_1\| + \|B_w^k x_2\| \to 0,$$

and subsequently

$$\|(B_w \oplus B_w)^k (x_1 \oplus x_2)\| \cdot \|S_k(y_1 \oplus y_2)\| \to 0 \quad \text{as } k \to \infty.$$

Moreover, for any $e_n \oplus e_m \in Y_0$, we observe:

$$(B_w \oplus B_w)^k S_k(e_n \oplus e_m) = e_n \oplus e_m.$$

This confirms that $B_w \oplus B_w$ satisfies the Supercyclicity Criterion. Hence, B_w is a topologically super weakly mixing operator that is not topologically weakly mixing.

Remark 2.12. The following observations are essential:

1. Example 2.11 illustrates a portion of the diagram below, highlighting the strict inclusion between the class of topologically weakly mixing operators and topologically super weakly mixing operators.

2. An operator T is hypercyclic if and only if the associated operator $S := T \oplus I$ on $X \oplus \mathbb{C}$ is supercyclic [16]. In particular, the De la Rosa Read hypercyclic operator R (which is not weakly mixing) [13] gives rise to an operator

$$W := R \oplus I,$$

which is supercyclic but not super weakly mixing.

Thus, the following diagram summarizes the relationships among the classes of operators discussed in this paper:

$$\begin{tikzcd}
\textbf{Weakly Mixing} \arrow[rr, Rightarrow, bend left] \arrow[d, Rightarrow, bend right] & & \textbf{Hypercyclicity} \arrow[ll, "/" marking] \arrow[d, Rightarrow, bend left] \\
\textbf{Super Weakly Mixing} \arrow[u, "/" marking] \arrow[rr, Rightarrow, bend right] & & \textbf{Supercyclicity} \arrow[ll, "/" marking] \arrow[u, "/" marking]
\end{tikzcd}$$

In the final theorem of this paper, we provide a negative answer to the following open problem posed in [2]:

Question. If $T \in B(X)$ and $x \in X$ satisfies the J-class criterion, can we conclude that x is a J^{mix}-class vector for T?

To be precise for a given $T \in B(X)$, the subset $J_T^{mix}(x)$ of X, for a vector $x \in X$ is defined as:

$$J_T^{mix}(x) = \left\{ y \in X : \begin{array}{c} \text{for every neighborhoods } U_x, V_y \text{ there exists a} \\ \text{positive integer } N \text{ such that the intersection} \\ T^n(U_x) \cap V_y \text{ is nonempty for any } n > N. \end{array} \right\}. \tag{3}$$

According to the above set, an operator T is called a J^{mix}*-class operator* if $J_T^{mix}(x) = X$ for some nonzero $x \in X$.
Similar results to Theorems 2.10 can be observed regarding the set (3) and the aforementioned class of operators. Thus the reader can reconstruct the arguments for the following theorem.

Theorem 2.13. *Let X be a separable Banach space and $T \in B(X)$. Then the following statements are equivalent.*
(i) T is a topologically mixing operator.
(ii) for every point $x \in X$, the operator T is locally topologically mixing in x, i.e., $J_T^{mix}(x) = X$.
(iii) $J_T^{mix} := \{x \in X : \ J_T^{mix}(x) = X\}$ is dense in X.

Theorem 2.14. *There exist a Banach space X, an operator $T \in B(X)$, and a vector $x_0 \in X$ such that:*

1. *x_0 satisfies the J-class criterion for T;*
2. *x_0 is not a J_T^{mix}-class vector.*

Proof. Let $\lambda \in \mathbb{C}$ with $|\lambda| > 1$, and consider the weighted backward shift operator

$$B_w(x_1, x_2, \dots) = (w_2x_2, w_3x_3, w_4x_4, \dots),$$

acting on the separable Banach space C_0, the space of null sequences in $\ell^\infty(\mathbb{N})$, endowed with the supremum norm, where the weight sequence

$w = (w_n)$ is given by:

$$w = (\lambda, \lambda, \lambda^{-1}, \lambda, \lambda, \lambda^{-1}, \lambda^{-1}, \lambda, \lambda, \lambda, \lambda^{-1}, \lambda^{-1}, \lambda^{-1}, \dots).$$

Let $e_n \in C_0$ be the standard basis vector defined by:

$$e_n = (\delta_{1n}, \delta_{2n}, \delta_{3n}, \dots),$$

where δ_{kn} is the Kronecker delta. One can observe that $B_w \in B(C_0)$, since w is bounded. Define a strictly increasing sequence $\{m_k\} \subset \mathbb{N}$ by:

$$m_k = \begin{cases} 3 & \text{if } k = 1, \\ 7 & \text{if } k = 2, \\ 2m_{k-1} - m_{k-2} + 2 & \text{if } k \geq 3. \end{cases}$$

A computation shows that:

$$B_w^{m_k - 1} x = (x_{m_k}, \dots), \tag{4}$$

because:

$$B_w^n x = (w_2 w_3 \cdots w_{n+1} x_{n+1}, \dots) \ \ \textit{and} \ \ \prod_{t=2}^{m_k} w_t = 1, \quad n, k \in \mathbb{N}.$$

Now consider the open sets:

$$U = \{x \in C_0 : \|x\| < 1\}, \quad V = \{x \in C_0 : |x_1| > 1\}.$$

Then $B_w^{m_k-1}(U) \cap V = \emptyset$ for all $k \in \mathbb{N}$ by (4), showing that B_w is not mixing. By Theorem 2.13, there exists a nonzero vector $x_0 \in C_0$ such that $J_T^{mix}(x_0) \neq C_0$.

We now verify that x_0 satisfies the J-class criterion. Let Y be the dense subset of finitely supported sequences in C_0, and let $\tilde{x} \in Y$. Define:

$$\tilde{z}_k = (0, \tilde{x}_{k,1}/w_2, \tilde{x}_{k,2}/w_3, \dots, \tilde{x}_{k,N_0}/w_{N_0+1}, 0, \dots),$$

so that $x_k := B_w \tilde{z}_k \to \tilde{x}$ and $B_w^{n_k} x_k \to 0$, where $n_k := m_k + k - 1 > N_0$. Thus:

$$0 \in J_{B_w}(\tilde{x}).$$

Now take any sequence $\{\tilde{x}_n\} \subset Y$ with $\tilde{x}_n \to x_0$. By [11, Lemma 2.5], we conclude:

$$0 \in J_{B_w}(x_0). \tag{5}$$

Next, let $y \in Y$, and define $\mu_{(k,y)} := S_w^{n_k} y$, where S_w is the weighted forward shift operator on C_0:

$$S_w(x_1, x_2, \dots) = (0, x_1/w_2, x_2/w_3, \dots).$$

It holds that $B_w S_w = I$, so:

$$B_w^{n_k} \mu_{(k,y)} \to y \quad \text{as } k \to \infty. \tag{6}$$

Also, one verifies $\mu_{(k,y)} \to 0$ as $k \to \infty$, since it can be observed that

$$\prod_{t=2}^{m_k+k} \frac{1}{w_t} = \frac{1}{\lambda^k},$$

implying

$$S_w^{n_k} e_1 = (0, \dots, 0, \prod_{t=2}^{m_k+k} \frac{1}{w_t}, 0, \dots) = (0, \dots, 0, 1/\lambda^k, 0, \dots) \to 0$$

as $k \to \infty$. Thus, for any $j \in \mathbb{N}$,

$$S_w^{n_k} e_j = S_w^{n_k} \left(\prod_{t=2}^{j} w_t \, S_w^{j-1} e_1 \right) = \left(\prod_{t=2}^{j} w_t \right) \, S_w^{j-1}(S_w^{n_k} e_1) \to 0$$

as $k \to \infty$, which establishes

$$\mu_{(k,y)} \to 0 \quad \text{as } k \to \infty. \tag{7}$$

(6) and (7) imply that

$$y \in J_{B_w}(0). \tag{8}$$

In light of (5) and (8), x_0 satisfies the J-class criterion, which is desired for us. $\square$

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References

[1] M. Amouch and O. Benchiheb, "Some versions of supercyclicity for a set of operators," *Filomat*, **35** (2021), no. 5, 1619–1627.

[2] M. Asadipour and B. Yousefi, "Some properties of J-class operators," *Communications of the Korean Mathematical Society*, **34** (2019), no. 1, 145–154.

[3] M. Asadipour, "Local subspace transitivity criterion," *Khayyam Journal of Mathematics*, **8** (2022), no. 1, 33–41.

[4] M. Asadipour, "Some notes on subspace convex-cyclicity," *Journal of Mathematical Extension*, **16** (2022), no. 5, 1–9.

[5] M. R. Azimi, "J-class sequences of linear operators," *Complex Analysis and Operator Theory*, **12** (2018), 293–303.

[6] M. R. Azimi and V. Müller, "A note on J-sets of linear operators," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, **105** (2011), 449–453.

[7] F. Bayart and E. Matheron, *Dynamics of Linear Operators*, Cambridge University Press, Cambridge, 2009.

[8] T. Bermúdez and N. J. Kalton, "The range of operators on von Neumann algebras," *Proceedings of the American Mathematical Society*, **130** (2002), 1447–1455.

[9] J. Bonet, L. Frerick, A. Peris, and J. Wengenroth, "Transitive and hypercyclic operators on locally convex spaces," *Bulletin of the London Mathematical Society*, **37** (2005), 254–264.

[10] S. Charpentier, R. Ernst, and Q. Menet, "Γ-supercyclicity," *Journal of Functional Analysis*, **270** (2016), 4443–4465.

[11] G. Costakis and A. Manoussos, "J-class operators and hypercyclicity," *Journal of Operator Theory*, **67** (2012), 101–119.

[12] G. Costakis and A. Manoussos, “J-class weighted shifts on the space of bounded sequences of complex numbers,” *Integral Equations and Operator Theory*, **62** (2008), 149–158.

[13] M. De la Rosa and C. Read, “A hypercyclic operator whose direct sum $T \oplus T$ is not hypercyclic,” *Journal of Operator Theory*, **61** (2009), 369–380.

[14] M. Faghih-Ahmadi and K. Hedayatian, “A note on supercyclic vectors of Hilbert space operators,” *Journal of Mathematical Analysis and Applications*, **505** (2022), Article 125500.

[15] R. M. Gethner and J. H. Shapiro, “Universal vectors for operators on spaces of holomorphic functions,” *Proceedings of the American Mathematical Society*, **100** (1987), no. 2, 281–288.

[16] M. González, F. León-Saavedra, and A. Montes-Rodríguez, “Semi-Fredholm theory: hypercyclic and supercyclic subspaces,” *Proceedings of the London Mathematical Society*, **81** (2000), 169–189.

[17] K. G. Grosse-Erdmann and A. Peris, *Linear Chaos*, Springer, Berlin, 2011.

[18] S. He and Y. Huang, “JF-class weighted backward shifts,” *International Journal of Bifurcation and Chaos*, **28** (2018), no. 6, Article 1850076.

[19] P. S. Hosseini and B. Yousefi, “On the J-reflexive operators,” *Turkish Journal of Mathematics*, **42** (2018), 1795–1802.

[20] C. Kitai, *Invariant Closed Sets for Linear Operators*, Ph.D. Thesis, University of Toronto, Toronto, 1982.

[21] A. B. Nasseri, “On the existence of J-class operators on Banach spaces,” *Proceedings of the American Mathematical Society*, **140** (2012), no. 10, 3549–3555.

[22] A. B. Nasseri, “Operators on ℓ^∞ with totally disconnected spectrum and applications to J-class operators,” *Journal of Mathematical Analysis and Applications*, **410** (2014), 94–100.

[23] R. Sanders, “Weakly supercyclic operators,” *Journal of Mathematical Analysis and Applications*, **292** (2004), 148–159.

[24] A. Tajmouati and M. E. Berrag, “Disjoint J-class operators,” *Italian Journal of Pure and Applied Mathematics*, **37** (2017), 19–28.

[25] S. Talebi, B. Yousefi and M. Asadipour, “Invertible Subspace-Hypercyclic Operators,” *Journal of Mathematical Extension*, **9** (2015), no. 2, 89–94.

[26] G. Tian and B. Hou, “Limits of J-class operators,” *Proceedings of the American Mathematical Society*, **142** (2014), no. 5, 1663–1667.

[27] A. Zhang, “Locally topological ergodicity and weakly mixing for bounded linear operators,” *Indian Journal of Pure and Applied Mathematics*, **43** (2012), 145–154.

Seyede Razieh Parsa
Ph.D student of mathematics
Department of Mathematics
College of Sciences, Yasouj University
Yasouj, Iran
E-mail: razieh0parsa@gmail.com

Meysam Asadipour
Assistant professor of pure mathematics
Department of Mathematics
College of Sciences, Yasouj University
Yasouj, Iran
E-mail: Asadipour.mey@gmail.com, Asadipour@yu.ac.ir