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Original Research Papee

Two Stage and Modified T wo S tage Estimation of the Mean in a Linear Process

S. Sajjadipanah*
University of Applied Science and Technology,

SM. Mirjalili
Velayat University,

Kh. Rezaie
Islamic Azad University,

SM. Mousavialiabadi
Shiraz University.

Abstract. In this paper, we study the two-stage procedure and the class of modified two-stage procedures for estimating the mean of a general linear process, with particular application to the $ARMA(p, q)$ class. We investigate point and confidence interval estimation based on least-squares estimators and establish asymptotic efficiency, asymptotic risk efficiency, and asymptotic consistency of the proposed procedures. We also establish the asymptotic distribution of the resulting estimators. Monte Carlo simulation studies are conducted to assess the finite-sample performance of the proposed procedures and to examine their behavior in relation to the theoretical results. Finally, a real time-series dataset is analyzed to illustrate the practical applicability of the proposed methodology.

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*S. Sajjadipanah

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1 Introduction

Time series analysis plays an important role in a wide range of scientific and practical applications. It encompasses various methods for analyzing temporal data, with the objectives of understanding the underlying data-generating process and obtaining reliable predictions. A common approach in time series analysis is to determine an optimal fixed sample size for parameter estimation and subsequent prediction. However, in many practical settings, the required sample size cannot be specified reliably in advance. Sequential procedures provide an alternative framework in such settings by allowing the sample size to be determined adaptively according to a prespecified stopping rule. Common sequential approaches include purely sequential, two-stage, modified two-stage, and accelerated sequential procedures. The statistical performance of these procedures is commonly assessed through their first- and second-order asymptotic properties, as well as their associated risk and efficiency measures. Numerous studies have investigated sequential procedures for time series models, and the most relevant contributions are reviewed below.

Sequential procedures have been studied extensively in the context of time series models since 1988, including purely sequential, accelerated sequential, two-stage, and modified two-stage procedures; a detailed review of this literature is provided by Sajjadipanah, Mirjalili, and Zanboori (2023) [15]. Among the purely sequential approaches, Karmakar and Mukhopadhyay (2018, 2019) [7, 8] investigated univariate and multivariate random coefficient autoregressive models, while Sriram and Samadi (2019) [20] investigated the second-order behavior of sequential estimation for the autoregressive parameter in a first-order autoregressive models. The two-stage approach has a broader methodological foundation, originating with Stein [21, 22], who introduced a two-stage procedure for interval estimation of an unknown normal mean with a prescribed coverage probability. Compared with purely sequential pro-

cedures, the two-stage approach provides a simpler stopping rule and is therefore more straightforward to implement. Several studies have subsequently extended this framework to dependent observations and time series models. Sajjadipanah, Mahmoudi, and Zamani (2021) [13] studied the two-stage procedure for a first-order autoregressive model, considering both point and confidence interval estimation and comparing its performance with that of a purely sequential procedure. Subsequently, Sajjadipanah, Mahmoudi, and Zamani (2022) [14] introduced a modified two-stage procedure for point and confidence interval estimation of the mean in a first-order autoregressive model. This line of research was further extended by Sajjadipanah, Mirjalili, and Mousavialiabadi (2023) [16], who developed a class of modified two-stage procedures based on the least-squares and Yule-Walker estimators for AR(p) processes. Sajjadipanah, Mirjalili, and Zanboori (2023) [15] subsequently investigated two-stage and modified two-stage procedures for point estimation of threshold autoregressive parameters in a TAR(1) model, using least-squares and Yule-Walker estimators. More recently, Sajjadipanah and Mirjalili (2025) [17] considered two-stage, modified two-stage, and three-stage procedures for random coefficient AR(p) models and evaluated their performance relative to the corresponding optimal fixed-sample-size procedure for both point and confidence-region estimation. Collectively, these studies demonstrate the progressive development of sequential estimation procedures across increasingly diverse autoregressive time series models.

Recent developments in estimation and statistical inference for linear time series models have addressed a range of challenges associated with parameter estimation, sequential inference, and dependence structures. Wheeler and Ionides (2025) [23] revisited likelihood-based inference for ARMA models and showed that conventional maximization procedures may converge to local rather than global optima of the likelihood surface; they proposed a random-initialization algorithm to mitigate this optimization problem. In the context of sequential estimation, Sengupta, Bapat, and Joshi (2024) [12] investigated multiple linear regression models under balanced loss functions and proposed two-stage, three-stage, purely sequential, and batch sequential procedures based on bounded risk criteria. Bhattacharya et al. (2024) [1] developed a sequential pro-

cedure for constructing fixed-accuracy confidence intervals for the variance of a first-order stationary autoregressive process and established its asymptotic efficiency and consistency. Dufour and Wang (2024) [4] introduced a two-stage estimation method for generalized impulse responses in multi-horizon linear projection models, improving efficiency relative to conventional least-squares estimation with HAC inference and accommodating non-stationary data. Zanboori et al. (2025) [24] introduced a flexible Weibull-geometric distribution and investigated several parameter estimation methods, including maximum likelihood, Bayesian, and shrinkage estimation, together with asymptotic and interval estimation results. More recently, Hwang and Vogelsang (2026) [6] proposed a regression-based approach for testing linear hypotheses concerning autocorrelations in stationary time series using HAR variance estimators and fixed-smoothing critical values, thereby avoiding the need for resampling. Despite these recent advances, a systematic two-stage framework for point and fixed-width confidence interval estimation of the mean of a general linear process has not yet been fully developed. In particular, the simultaneous treatment of two-stage and modified two-stage procedures for mean estimation, together with their first- and second-order asymptotic properties, remains largely unexplored.

In sequential analysis, two-stage (or multistage) designs allow an interim assessment to be performed after an initial set of observations, based on which the investigator determines whether additional observations are required. Such sequential strategies can be particularly useful in comparative studies involving multiple independent models, where the amount of information required for reliable statistical decisions may not be known in advance. For example, Mahmoudi, Maleki, and Pak (2017) [10] considered the problem of comparing two independent time series models and established the asymptotic distribution of the differences between their ARMA coefficients, providing a basis for confidence sets and hypothesis testing for model equality. This type of comparative inference highlights the potential value of sequential designs for adaptively determining the amount of data required for statistical decision-making. These considerations motivate the study of two-stage and modified two-stage procedures for estimating the mean of a general linear process. The class of modified two-stage procedures further addresses the prac-

tical issue of specifying the initial sample size by determining the pilot sample size as a function of the sampling cost and a tuning parameter, thereby avoiding reliance on a preassigned initial sample size.

For point estimation, we establish the asymptotic properties of the proposed two-stage and modified two-stage procedures as the cost per observation tends to zero. For fixed-width confidence interval estimation, we consider the corresponding asymptotic regime in which the prescribed interval width tends to zero. Under these asymptotic regimes, we establish asymptotic efficiency, asymptotic risk efficiency, and asymptotic consistency for the proposed procedures. In addition, we derive second-order asymptotic results for the modified two-stage procedures, in connection with the second-order asymptotic framework discussed by Mukhopadhyay and Banerjee (2023) [11]. Theoretical findings are complemented by comprehensive Monte Carlo simulation studies that assess the finite-sample performance of the proposed procedures in terms of stopping sample size, estimation accuracy, risk, and confidence interval coverage. In addition, a real-world time-series dataset is analyzed to illustrate the practical applicability of the proposed procedures.

While previous studies have primarily focused on first-order autoregressive models [13, 14], autoregressive processes of general order [16], threshold autoregressive models [15], and random coefficient autoregressive models [17], the present study extends the two-stage framework to the broader class of general linear processes. This is a significant advancement, as such processes, unlike finite-order autoregressive models, can capture both autoregressive and moving-average dependencies. Consequently, this more general structure necessitates the characterization of the long-run variance through the underlying linear-process representation. Our main contributions are threefold. First, we establish the asymptotic efficiency, asymptotic risk efficiency, and asymptotic consistency of the two-stage and modified two-stage procedures for point estimation of the mean of a general linear process. Second, for fixed-width confidence interval estimation, we derive the asymptotic distribution of the corresponding estimators and establish the asymptotic consistency of the proposed confidence intervals. Third, we establish the second-order asymptotic efficiency of the modified two-stage procedure under a positive lower-bound condition on the long-run standard deviation.

Together, these results extend the existing two-stage framework from specific autoregressive models to the broader class of general linear processes and provide a unified treatment of both point and fixed-width confidence interval estimation.

The remainder of this paper is organized as follows. Section 1 introduces the optimal fixed-sample-size benchmark. Section 2 presents the proposed two-stage and class of modified two-stage procedures for point and fixed-width confidence interval estimation. Section 3 presents the results of the comprehensive simulation studies. Finally, Section 4 considers a real-world time-series dataset to illustrate the applicability of the proposed procedures.

The Problem

We consider a linear process with mean parameter ($\mu \in \mathbf{R}$),

$$X_t = \mu + \sum_{i=0}^{\infty} a_i \varepsilon_{t-i},$$

where $\{\varepsilon_t, t \in \mathbb{Z}\}$ represents a sequence of independent and identically distributed (*i.i.d.*) random variables with an unknown distribution F. Furthermore, we assume that $\mathbb{E}(\varepsilon_t) = 0$, $\mathbb{E}(\varepsilon_t^2) = \sigma^2$, and that the sequence $\{\varepsilon_i, i \geq 1\}$ is independent of the initial variable X_0 with $\mathbb{E}({X_0}^2) < \infty$. Additionally, the sequence of coefficients $\{a_i\}$ is assumed to be geometrically bounded; that is, there exist constants $A > 0$ and $0 < \rho < 1$ such that $|a_i| \leq A\rho^i$ for all $i \geq 0$. Under these assumptions, the process has mean $\mathbb{E}(X_t) = \mu$. This class of processes is a basic object in time series analysis and includes stationary autoregressive moving-average processes of order (p, q), denoted by $ARMA(p, q)$ with mean μ which can be represented as

$$\phi(B)(X_t - \mu) = \theta(B)\varepsilon_t,$$

where $\phi(z) = 1 - \beta_1 z - \cdots - \beta_p z^p$ and $\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$. Here, $lbrace\varepsilon_t\}$ is an i.i.d. sequence with mean zero and variance σ^2. We assume that $\phi(z)$ and $\theta(z)$ have no common zeros and that $\{\phi(z) \neq$

$0, \ |z| \leq 1\}$, so that the ARMA process is stationary and causal. Based on observations $X_1, \ldots, X_n$, we consider the sample mean

$$\hat{\mu} = \bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

as an estimator of the mean parameter μ. It is well known that, under the above conditions, the asymptotic distribution of $\bar{X}_n$ (Brockwell and Davis(1987)[2]) as $n \to \infty$ is given by

$$\sqrt{n} \left(\bar{X}_n - \mu \right) \to N \left(0, \tau^2 \right), \tag{1}$$

where $\tau^2 = (\sum_{i=0}^{\infty} a_i)^2 \sigma^2$. The loss function for estimating μ is

$$L_n(\bar{X}_n, \mu) = a(\bar{X}_n - \mu)^2 + cn, \quad a, c > 0,$$

where $a > 0$ represents the weight assigned to the quadratic estimation loss and $c > 0$ denotes the cost per observation. When τ is known, the corresponding fixed-sample-size risk function can be expressed as

$$R_n = \mathbb{E}\left\{ L_n(\bar{X}_n, \mu) \right\} = a\mathbb{E}\left\{ \bar{X}_n - \mu \right\}^2 + cn = \frac{a\tau^2}{n} + cn + o(n^{-1}).$$

Ignoring the $o(n^{-1})$ term, the optimal fixed-sample size is approximately $n_c \simeq \left(\frac{a}{c} \right)^{1/2} \tau$. Accordingly, the corresponding risk function evaluated at the optimal fixed-sample size is approximately $R_{n_c} \simeq 2cn_c$.

However, in practice, (τ) is generally unknown, the optimal fixed-sample size (n_c) unavailable prior to data collection. This creates a fundamental difficulty in determining the sample size that balances estimation accuracy against sampling cost. The two-stage and class of modified two-stage procedures therefore provide approaches to this problem by using an initial sample to estimate the unknown (τ) and subsequently determining the required sample size based on that estimate.

2 Solution Of The Problem

2.1 Two-stage estimation

To address this issue, we consider the following two-stage sampling scheme. First, an estimator of τ, denoted as $\hat{\tau}_m$, is obtained from a

preliminary sample of size m. Subsequently, an estimator of τ_n^2 denoted by $\hat{\tau}_n^2$ is defined for a given sample size n as follows

$$\hat{\gamma}_n(h) = n^{-1}\sum_{i=1}^{n-h}(X_{i+h} - \bar{X}_n)(X_i - \bar{X}_n), \quad 0 \le h \le h_n; \qquad \hat{\tau}_n^2 = \hat{\gamma}_n(0) + 2\sum_{h=1}^{h_n}\hat{\gamma}_n(h).$$

where $\{h_n\}$ is a sequence of positive integers satisfying

$$\begin{aligned} &(i) \quad \sqrt{n}\rho^{h_n} \longrightarrow 0, \qquad \rho \in (0,1), \\ &(ii) \quad \frac{h_n}{n^{\delta}} \longrightarrow 0 \qquad \delta > 0. \end{aligned}$$

A typical example of such a sequence is $h_n = [(logn)^2]$ where $[u]$ stands throughout for the largest integer less than u. In what follows, the two-stage stopping rule is defined as

$$N_m = \max\{m, \lfloor\left(\frac{a}{c}\right)^{1/2}(\hat{\tau}_m + m^{-\lambda})\rfloor + 1\}, \tag{2}$$

where $\lambda > 0$ is an appropriate constant and $\lfloor x \rfloor$ denotes the largest integer not exceeding x. The term $m^{-\lambda}$ serves as a delay factor. The delay factor $m^{-\lambda}$, incorporated into the estimator of τ, was introduced by Chow and Robbins (1965) [3].

The main asymptotic results of this subsection establish the asymptotic efficiency and asymptotic risk efficiency of the proposed procedure as $c \to 0$. The following lemma provides a technical result needed to establish these properties.

$$n_1 = [n_c(1-\varepsilon)], \qquad n_2 = [n_c(1+\varepsilon)].$$

Lemma 2.1. *Assume that for $p^* > 2$, $\mathbb{E}(|\varepsilon_1|^{2p^*}) < \infty$. Furthermore, let λ as defined in (2), satisfy $\lambda \in \left(0, \frac{p^*-2}{4}\right)$ and suppose that $(a/c)^{\frac{1}{2(1+\lambda)}} \leqslant m = o(c^{-1/2})$. Let $s = p^*/2$. Then for every $\varepsilon > 0$, as $c \to 0$*

$$P(N_m \le n_1) = O\left(c^{s/2(1+\lambda)}\right), \tag{3}$$

$$P(N_m \geq n_2) = O\left(c^{s/2(1+\lambda)}\right). \tag{4}$$

Proof.

$$\begin{aligned}
P(N_m \leq n_1) &= P\left(N_m \leq (1-\varepsilon)(a/c)^{1/2}\tau\right) \\
&\leq P\left(\hat{\tau}_m^2 \leq (1-\varepsilon)^2\tau^2\right) \\
&\leq P\left(\hat{\tau}_m^2 - \tau^2 \leq (\varepsilon^2 - 2\varepsilon)\tau^2\right) \\
&\leq P\left(\left|\hat{\tau}_m^2 - \tau^2\right| \geq \varepsilon(2-\varepsilon)\tau^2\right) \\
&= O\left(m^{-p^*/2h_m^{p^*}}\right) = O\left((c/a)^{s/2(1+\lambda)+\varepsilon}\right) = O\left(c^{s/2(1+\lambda)}\right).
\end{aligned}$$

In view of Lemma 3 of Sriram (1987) [19], the above result follows. To prove equation (4), we have

$$\begin{aligned}
P(N_m \geq n_2) &\leq P\left(\hat{\tau}_m > (1+\varepsilon)\tau - (a/c)^{-1/2}2 - m^{-\lambda}\right) \\
&\leq P\left(\hat{\tau}_m > (1+\varepsilon)\tau - (a/c)^{-1/2}2 - m^{-\lambda} \pm \tau\right) \\
&= P\left(\hat{\tau}_m - \tau > (a/c)^{-1/2}\left(n_2 - (a/c)^{1/2}\tau - m^{-\lambda}\right)\right) \\
&= P\left(\hat{\tau}_m - \tau > (a/c)^{-1/2}(n_2 - n_c) - (a/c)^{-1/2}2 - m^{-\lambda}\right)
\end{aligned}$$

By choosing c sufficiently small, we obtain

$$(a/c)^{-1/2}(n_2 - n_c) - 2(a/c)^{-1/2} - m^{-\lambda} > \frac{\varepsilon\sigma}{2}$$

So,

$$\begin{aligned}
&\leq P\left(\hat{\tau}_m - \tau > \frac{\varepsilon\sigma}{2}\right) \\
&\leq P\left(\left|\hat{\tau}_m^2 - \tau^2\right| > \frac{\varepsilon^2\tau^2}{4}\right) \\
&= O\left(m^{-p^*/2\lambda_m^{p^*}}\right) = O\left((c/a)^{s/2(1+\lambda)+\varepsilon}\right) = O\left(c^{s/2(1+\lambda)+\varepsilon}\right).
\end{aligned}$$

The proof follows directly from Lemma 3 of Sriram (1987) [19]. □

Theorems 2.2 and 2.3 constitute the primary results of this subsection, establishing the asymptotic efficiency and asymptotic risk efficiency of the proposed procedure as $c \to 0$.

Theorem 2.2. *For $p^* > 2$, assume that $\mathbb{E}(|\varepsilon_1|^{2p^*}) < \infty$ and $\lambda \in \left(0, \frac{p^*-2}{4}\right)$. Then, provided that $(a/c)^{1/2(1+\lambda)} \leqslant m = o(c^{-1/2})$, as $c \to 0$*

$$\sqrt{N_m}\left(\bar{X}_{N_m} - \mu\right) \xrightarrow{D} N\left(0, \tau^2\right).$$

Proof. *Letting $\tilde{a}_i = \sum_{j=i+1}^{\infty} a_j$ and $\tilde{\varepsilon}_t = \sum_{i=0}^{\infty} \tilde{a}_i \varepsilon_{t-i}$, which is well defined since $\sum_{i=0}^{\infty} |\tilde{a}_i| < \infty$, we have*

$$X_t - \mu = \sum_{i=0}^{\infty} a_i \varepsilon_t + \tilde{\varepsilon}_{t-1} - \tilde{\varepsilon}_t,$$

In view of above equality and Anscombe's Theorem concerning the behavior of $n^{1/2}\sum_{t=1}^{n} \varepsilon_t$, it suffices to prove that

$$\tilde{\varepsilon}_{N_m}/\sqrt{N_m} \xrightarrow{P} 0,$$

which follows from

$$\tilde{\varepsilon}_n/\sqrt{n} \xrightarrow{a.s} 0, \qquad as\ n \longrightarrow \infty.$$

This completes the proof. □

Theorem 2.3. *For $p^* > 2$, assume that $\mathbb{E}(|\varepsilon_1|^{2p^*}) < \infty$. Moreover, let λ as defined in (2) is such that $\lambda \in \left(0, \frac{p^*-2}{4}\right)$ and $(a/c)^{\frac{1}{2(1+\lambda)}} \leqslant m = o(c^{-1/2})$. Then as $c \to 0$*

$$\frac{N_m}{n_c} \xrightarrow{a.s} 1, \tag{5}$$

$$\mathbb{E}\left(\frac{N_m}{n_c}\right) \to 1, \quad \text{(asymptotic efficiency)}, \tag{6}$$

$$\frac{R_{N_m}}{R_{n_c}} \to 1, \quad \text{(asymptotic risk efficiency)}. \tag{7}$$

Proof. *In view of definition of* (2)*, we have*

$$\left(\frac{a}{c}\right)^{1/2}\hat{\tau}_m + m^{-\lambda} \leq N_m \leq \left(\frac{a}{c}\right)^{1/2}\hat{\tau}_m + m^{-\lambda} + m.$$

By taking the limit and then taking expectations of the above equation as $c \to 0$*, we obtain Equations* (5) *and* (6)*, respectively. To establish Assertion* (7)*, we may assume, without loss of generality (W.L.O.G.), that* $\mu = 0$*. We proceed as follows.*

$$R_{N_m} = \mathbb{E}\left(L_{N_m}(\bar{X}_{N_m}, 0)\right) = a\mathbb{E}(\bar{X}^2_{N_m}) + c\mathbb{E}\left(N_m\right).$$

Then,

$$\frac{R_{N_m}}{R_{n_c}} = \frac{a\mathbb{E}(\bar{X}^2_{N_m}) + c\mathbb{E}(N_m)}{2cn_c}.$$

In view of Equation (6)*, it suffices to show that the first term of the above expression converges to 1/2.*

Furthermore, let $A = [n_1 < N_m < n_2]$*,* $B = [N_m \leq n_1]$ *and* $D = [N_m \geq n_2]$*. For this purpose, we deduce that*

$$\mathbb{E}(n_c\bar{X}^2_{N_m}) \to \tau^2.$$

In view of Theorem 2.2 and Equation (5)*, for any* $\varepsilon \in (0, 1)$

$$\sqrt{n_c}\bar{X}_{N_m} I_A \xrightarrow{D} N(0, \tau^2).$$

Hence, to complete the proof, it suffices to show that, for any $\varepsilon \in (0, 1)$

$$\sqrt{n_c}\bar{X}_{N_m} I_B \to 0, \qquad \sqrt{n_c}\bar{X}_{N_m} I_D \to 0.$$

The above results follow from Lemma 2.1, Lemma 3, and Lemma 4 of [5]. Thus, the proof is complete. □

This subsection is devoted to constructing a fixed-width 2d confidence interval for the population mean μ with a confidence coefficient of at least $1 - \alpha$. It is assumed that $\alpha \in (0, 1)$ and $d > 0$ are two preassigned constants. The proposed fixed-width confidence interval for μ is formally defined as $I = \left(\bar{X}_{k_d} - d, \bar{X}_{k_d} + d\right)$ where $k_d = [d^{-2}\tau^2 z^2_{(1-\alpha)/2}]$.

When τ is unknown, the optimal fixed-sample size procedure cannot be achieved. In this case, as established previously, the two-stage stopping rule is defined

$$N_m^d = \max\{m, \lfloor d^{-2} z^2_{(1-\alpha)/2}(\hat{\tau}_m^2 + m^{-\lambda})\rfloor + 1\}, \tag{8}$$

where $\hat{\tau}_m^2$ represents the least-squares estimator obtained from an initial sample of size m. We denote the confidence interval for μ based on the stopping variable N_m^d by $I_{N_m^d}$.

The second primary result of this subsection is presented in Theorem 2.4. The theorem establishes that the confidence interval $I_{N_m^d}$ has an asymptotic coverage probability of $1-\alpha$ as $d \to 0$. Furthermore, two important consequences of this theorem are the asymptotic consistency and asymptotic efficiency of the proposed procedure.

Theorem 2.4. *Suppose that* $\mathbb{E}(|\varepsilon_1|^{2p^*}) < \infty$, $p^* > 2$. *Moreover, let* $\lambda \in \left(0, \dfrac{p^*-2}{4}\right)$ *where* λ *is defined in* (8). *Then as* $d \to 0$, *we have*

$$\frac{N_m^d}{k_d} \xrightarrow{a.s} 1, \tag{9}$$

$$\mathbb{E}\left(\frac{N_m^d}{k_d}\right) \to 1, \quad \text{(asymptotic efficiency)}, \tag{10}$$

$$P\left(\mu \in I_{N_m^d}\right) \to 1-\alpha, \quad \text{(asmyptotic consistency)}. \tag{11}$$

Proof. From (8), we obtain

$$d^{-2} z^2_{(1-\alpha)/2}\hat{\tau}_m^2 + m^{-\lambda} \le N_m^d \le d^{-2} z^2_{(1-\alpha)/2}\hat{\tau}_m^2 + m^{-\lambda} + m.$$

Taking limits and expectations in the above inequalities as $d \to 0$ yields Equations (9) and (10). Regarding Assertion (11), we establish the following

$$\begin{aligned} P(\bar{X}_{N_m^d} - d < \mu < \bar{X}_{N_m^d} + d) &= P\left(\frac{N_d(\bar{X}_{N_m^d} - \mu)^2}{\hat{\tau}_m^2} < \frac{N_m^d d^2}{\hat{\tau}_m^2}\right) \\ &\ge P\left(\frac{N_d(\bar{X}_{N_m^d} - \mu)^2}{\hat{\tau}_m^2} < \frac{\hat{\tau}_m^2}{\hat{\tau}^2} z^2_{(1-\alpha)/2}\right) \to 1-\alpha. \end{aligned}$$

The proof is complete by applying Anscombe's Theorem, Theorem 2.2 and the Slutsky Theorem. □

2.2 Modified two-stage estimation

This subsection presents a class of modified two-stage procedures based on a two-stage sampling methodology. This methodology specifies two stopping rules and their corresponding conditions. The generalized approach first defines the pilot sample size for the first stage and subsequently imposes conditions on the sampling scheme to formulate the stopping rules. The traditional modified two-stage stopping rule is defined as follows:

$$m = \max\{m_0, \left(\frac{a}{c}\right)^{\frac{1}{2(\gamma+1)}}\},$$

$$N'_m = \max\{m, \lfloor\left(\frac{a}{c}\right)^{1/2}(\hat{\tau}_m + m^{-\lambda})\rfloor + 1\}, \tag{12}$$

where $\gamma \in (1/2, \infty)$ and $m_0 > 2$. The efficiency of point and confidence interval estimation is established through a series of lemmas and theorems. Initially, we present a requisite lemma detailing the tail behavior rate of the stopping rule N'_m. Since its proof follows the same argument as that of Lemma 2.1, the details are omitted.

Lemma 2.5. *Assume that for $p > 2$, $\mathbb{E}(|\varepsilon_1|^{2p}) < \infty$. Moreover, let λ as defined in (12), satisfy $\lambda \in \left(0, \frac{p-2}{4}\right)$ and suppose that $m_0 = o(c^{-1/2})$. Let $s = p/2$. Then, for every $\varepsilon > 0$ as $c \to 0$,*

$$P(N'_m \leq n_1) = O\left(c^{s/2(1+\lambda)}\right),$$

$$P(N'_m \geq n_2) = O\left(c^{s/2(1+\lambda)}\right).$$

Proof. □

Theorem 2.6. *For $p^* > 2$, assume that $\mathbb{E}(|\varepsilon_1|^{2p^*}) < \infty$. Also, λ defined in (12) is such that $\lambda \in \left(0, \frac{p^*-2}{4}\right)$ and $m_0 = o(c^{-1/2})$, then as $c \to 0$*

$$\frac{N'_m}{n_c} \xrightarrow{a.s} 1, \tag{13}$$

$$\mathbb{E}\left(\frac{N'_m}{n_c}\right) \to 1, \quad \text{(asymptotic efficiency)}, \tag{14}$$

$$\frac{R_{N'_m}}{R_{n_c}} \to 1, \quad \text{(asymptotic risk efficiency)}. \tag{15}$$

Proof. From (12), we have

$$\left(\frac{a}{c}\right)^{1/2}\hat{\tau}_m + m^{-\lambda} \leq N'_m \leq \left(\frac{a}{c}\right)^{1/2}\hat{\tau}_m + m^{-\lambda} + mI_{N'_m}.$$

Taking the limit and then the expectation of the above inequality, and using $I_{N'_m} \xrightarrow{a.s} 0$ as $c \to 0$, we obtain Equations (13) and (14), respectively. The proof of Equation (15) follows similarly to that of Theorem 2.3. □

Theorem 2.7. *For $p^* > 2$, assume that $\mathbb{E}(|\varepsilon_1|^{2p^*}) < \infty$ and $\lambda \in \left(0, \frac{p^*-2}{4}\right)$. Then, if $m_0 = o(c^{-1/2})$, as $c \to 0$*

$$\sqrt{N'_m}\left(\bar{X}_{N'_m} - \mu\right) \xrightarrow{D} N\left(0, \tau^2\right).$$

Proof.

$$\sqrt{N'_m}\left(\bar{X}_{N'_m} - \mu\right) = \sqrt{\frac{N'_m}{n_c}}\sqrt{n_c}(\bar{X}_{N'_m} - \bar{X}_{n_c}) + \sqrt{\frac{N'_m}{n_c}}\sqrt{n_c}(\bar{X}_{n_c} - \mu),$$

In view of above inequality, $\sqrt{n_c}|\bar{X}_{N'_m} - \bar{X}_{n_c}| \xrightarrow{p} 0$, Equations (13) (14), and (1) the desired result follows. □

To construct the maximum axis ($2d$), the confidence stopping variable is presented for $\gamma \in (1/2, \infty)$ and $m_0 \geq 2$.

$$m^d = \max\{m_0, [(z^2_{(1-\alpha)/2}/d^2)^{1/(1+\gamma)}] + 1\},$$
$$N'_d = \max\{m^d, \lfloor d^{-2}z^2_{(1-\alpha)/2}(\hat{\tau}^2_{m^d} + (m^d)^{-\lambda})\rfloor + 1\}.$$

A useful theorem is presented, stating that N'_d is as efficient as k_d. Its proof largely mirrors that of Theorem 2.4.

Theorem 2.8. *For $p^* > 2$, assume that $\mathbb{E}(|\varepsilon_1|^{2p^*}) < \infty$. Also, λ is defined such that $\lambda \in \left(0, \frac{p^*-2}{4}\right)$ and $m_0 = o(d^{-2})$. As $d \to 0$,*

$$\frac{N'_d}{k_d} \xrightarrow{\text{a.s}} 1,$$

$$\mathbb{E}\left(\frac{N'_d}{k_d}\right) \to 1, \quad \text{(asymptotic efficiency)},$$

$$P\left(\mu \in I_{N'_d}\right) \to 1-\alpha, \quad \text{(asymptotic consistency)}.$$

Proof.

$$d^{-2}z^2_{(1-\alpha)/2}\hat{\tau}^2_{m^d} + (m^d)^{-\lambda} \leq N'_d \leq d^{-2}z^2_{(1-\alpha)/2}\hat{\tau}^2_{m^d} + (m^d)^{-\lambda} + m^d I_{N'_d}.$$

As in the previous case, taking limits and then expectations in the above inequality and using $I_{N'_d} \xrightarrow{\text{a.s}} 0$ as $d \to 0$ yields the desired results. The final result follows from Anscombe's Theorem, Theorem 2.2, and Slutsky's Theorem. □

We assume that the variance, τ satisfies the lower-bound condition $\tau > \tau_l > 0$ where τ_l is a known positive constant. This assumption naturally raises two questions. First, does the lower-bound condition lead to different stopping rules for point and confidence interval estimation? Second, how does it affect the choice of the pilot sample size? For a given initial sample size $m_0 \geq 2$, the modified pilot sample size is defined as

$$m' = \max\{m_0, [\tau_l(\frac{a}{c})^{1/2}] + 1\},$$

The final sample size is given by

$$N_{m'} = \max\{m', \lfloor\left(\frac{a}{c}\right)^{1/2}(\hat{\tau}_{m'} + m'^{-\lambda})\rfloor + 1\}, \tag{16}$$

We next investigate the asymptotic behavior of the proposed stopping variables. To this end, we first establish Lemma 2.9, followed by Theorem 2.11 and Theorem 2.10.

Lemma 2.9. *Assume that* $p^* > 2$ *and* $\mathbb{E}(|\varepsilon_1|^{2p^*}) < \infty$. *Let,* $\lambda \in \left(0, \frac{p^*-2}{4}\right)$ *and suppose that* $m_0 = o(c^{-1/2})$. *Let* $s = p^*/2$. *Then for every* $\varepsilon > 0$ *as* $c \to 0$,

$$P(N_{m'} \leq n_1) = O\left(c^{s/2(1+\lambda)}\right),$$

$$P(N_{m'} \geq n_2) = O\left(c^{s/2(1+\lambda)}\right).$$

Proof.

The proof follows analogously to that of Lemma 2.1 and is therefore omitted for brevity. □

Theorem 2.10. *For* $p^* > 2$, *assume that* $\mathbb{E}(|\varepsilon_1|^{2p^*}) < \infty$. *Also,* λ *is defined such that* $\lambda \in \left(0, \frac{p^*-2}{4}\right)$ *and* $m_0 = o(c^{-1/2})$, *then as* $c \to 0$

$$\frac{N_{m'}}{n_c} \xrightarrow{a.s} 1, \tag{17}$$

$$\mathbb{E}\left(N_{m'} - n_c\right) \to 0, \quad \text{(secondorder asymptotic efficiency)}, \tag{18}$$

$$\mathbb{E}\left(\frac{N_{m'}}{n_c}\right) \to 1, \quad \text{(asymptotic efficiency)}, \tag{19}$$

$$\frac{R_{N_{m'}}}{R_{n_c}} \to 1, \quad \text{(asymptotic risk efficiency)}. \tag{20}$$

Proof.

In view of definition (16), we have

$$\left(\frac{a}{c}\right)^{1/2}\hat{\tau}_{m'} + m'^{-\lambda} \leq N_{m'} \leq \left(\frac{a}{c}\right)^{1/2}\hat{\tau}_{m'} + m'^{-\lambda} + m' I_{N_{m'}}.$$

In view of $\frac{\hat{\tau}_{m'}}{\tau} \xrightarrow{a.s} 1$, $I_{N_{m'}} \xrightarrow{a.s} 0$ *then dividing the above inequality by* n_c *and passing to the limit yields Equation (17). Furthermore, by taking expectations and applying Equation (17), we obtain Equation (19). To establish Equation (18), we subtract* n_c *from the above inequality and use the rate result provided by Lemma 1 of [5], together with the definition of* m. *Finally, Equation (20) follows by an argument analogous to that used in Theorem 2.3.* □

Theorem 2.11. *For $p^* > 2$, assume that $\mathbb{E}(|\varepsilon_1|^{2p^*}) < \infty$ and $\lambda \in \left(0, \frac{p^*-2}{4}\right)$. Then by supposing $m_0 = o(c^{-1/2})$, as $c \to 0$*

$$\sqrt{N_{m'}}\left(\bar{X}_{N_{m'}} - \mu\right) \xrightarrow{D} N\left(0, \tau^2\right).$$

Proof. *The proof follows analogously to the proofs of Theorem 2.7 and is therefore omitted for brevity* □

We now introduce the confidence interval stopping variable.

$$m'^d = \max\{m_0, [\tau_l z^2_{(1-\alpha)/2}/d^2] + 1\},$$

$$N_{d'} = \max\{m'^d, \lfloor d^{-2} z^2_{(1-\alpha)/2}(\hat{\tau}^2_{m'^d} + (m'^{-d})^{\lambda})\rfloor + 1\}. \tag{21}$$

The theorem indicates that $N_{d'}$ is as efficient as k_d. The proof is similar to that of Theorem 2.4.

Theorem 2.12. *Suppose for $p^* > 2$, $\mathbb{E}(|\varepsilon_1|^{2p^*}) < \infty$ and λ is defined such that $\lambda \in \left(0, \frac{p^*-2}{4}\right)$. As $d \to 0$,*

$$\frac{N_{d'}}{k_d} \xrightarrow{\text{a.s}} 1, \tag{22}$$

$$\mathbb{E}[N_{d'} - k_d] \to 0, \quad \text{(secondorder asymptotic efficiency)}, \tag{23}$$

$$\mathbb{E}\left[\frac{N_{d'}}{k_d}\right] \to 1, \quad \text{(asymptotic efficiency)}, \tag{24}$$

$$P\left(\mu \in I_{N_{d'}}\right) \to 1 - \alpha, \quad \text{(asymptotic consistency)}. \tag{25}$$

Proof.

In view of Definition (21), the following inequality holds:

$$d^{-2} z^2_{(1-\alpha)/2} \hat{\tau}^2_{m'^d} + (m'^d)^{-\lambda} \leq N_{d'} \leq d^{-2} z^2_{(1-\alpha)/2} \hat{\tau}^2_{m'^d} + (m'^d)^{-\lambda} + m'^d I_{N_{d'}}.$$

Since $\frac{\hat{\tau}_{m'^d}}{\tau} \xrightarrow{a.s} 1$, $I_{N_{d'}} \xrightarrow{a.s} 0$, taking limits and expectations in the above inequality, and applying Lemma 1 of [5], yields Equations (22)-(24). Subsequently, Equation (25) follows directly from Theorem 2.4. □

Under the stated lower-bound condition, the modified two-stage procedure also achieves second-order asymptotic efficiency for point and confidence interval estimation, but not for two-stage point estimation or modified two-stage estimation in the absence of this assumption.

3 Simulation Study

In this section, Monte Carlo simulation studies are conducted to evaluate the performance of the point and confidence interval estimation procedures for an $ARMA(1,1)$ model. The $ARMA(1,1)$ model is considered as a representative member of the class of stationary linear time-series models and provides a flexible setting for evaluating the proposed procedures in the presence of both autoregressive and moving-average dependence. Since the primary objective of the simulation study is to assess the performance of the proposed procedures in estimating the mean of the model, the mean parameter is considered at five values, $\mu \in \{-4, -2, 0, 2, 4\}$, thereby covering negative, zero, and positive values, respectively. The remaining model parameters are held fixed at $\beta = 0.3$ and $\theta = 0.5$, while the innovations are generated independently from $N(0, \sigma^2)$, where $\sigma^2 = 0.8$. The values of β, θ and σ^2 are fixed at representative values to isolate the effect of changes in the mean parameter on the performance of the proposed procedures. The performance of the two-stage and the class of modified two-stage procedures is compared with that of the corresponding optimal fixed-sample-size procedure.

To assess the performance of the point-estimation procedures, we consider five performance measures: the ratio of the average stopping sample size of the two-stage procedure to the optimal fixed sample size, the ratio of the average stopping sample size under the class of modified two-stage procedures to the optimal fixed sample size, the root mean squared error (RMSE) of $\bar{X}$, the ratio of the two-stage risk function to the optimal fixed-sample-size risk function, and the ratio of the risk function of the modified two-stage procedure to the corresponding optimal fixed-sample-size risk function. For the class of modified two-stage procedures, $\lambda = 0.7$, $\tau_l = 0.6$ and $a = 1$ are fixed throughout the simulation study. The following pairs of initial sample size and cost per unit observation are considered: $(m, c) = (5, 0.05)$ and $(60, 0.005)$. The two combinations are selected to represent settings with relatively different initial sample sizes and observation costs.

Furthermore, the performance of the two-stage confidence interval procedure and the class of modified two-stage confidence interval procedures is evaluated relative to that of the corresponding optimal fixed-sample-size confidence interval procedure. For this purpose, the initial

sample size is set to $m = 25$, and the values of the tuning parameter are considered as $d \in \{1.2, 0.7\}$. The performance of the confidence interval procedures is assessed using three criteria: the ratio of the average stopping sample size of the two-stage procedure to the optimal fixed sample size, the ratio of the average stopping sample size under the class of modified two-stage procedures to the optimal fixed sample size, and the coverage probability.

Figures 1-8 present the simulation results for the point and confidence interval estimations, respectively. All computations were performed in R based on 10,000 Monte Carlo replications. In a separate simulation experiment aimed at illustrating the asymptotic distributions, histograms are provided for the three point stopping variables under the settings $(m, c) = (50, 0.005)$ and $(100, 0.0005)$, with $\varepsilon_i \sim N(0, 1)$. These histograms provide a graphical assessment of the finite-sample distribution of the stopping variables and its agreement with the corresponding asymptotic distributions.

Figure 1 shows the ratios of the average stopping sample sizes of the two-stage procedure and the class of modified two-stage procedures to the optimal fixed sample size under the two considered (m, c) settings. For $(m, c) = (5, 0.05)$, both ratios are approximately 0.7. Under the second setting, $(m, c) = (60, 0.005)$, both ratios become closer to 1, indicating that the two-stage and class of modified two-stage procedures approach the optimal fixed-sample-size procedure in terms of sample size efficiency. Figure 2 shows that the ratios of the risk functions of the two-stage procedure and the class of modified two-stage procedures to the optimal fixed-sample-size risk function are closer to 1 under the second (m, c) setting. This behavior provides numerical evidence that the proposed procedures approach the risk efficiency of the optimal fixed-sample-size procedure under the considered settings. Figure 3 shows that the RMSEs of the considered point estimators decrease under the second (m, c) setting. Furthermore, Figures 6–8 provide graphical evidence consistent with the asymptotic distributional results established in Theorems 2.2, 2.7, and 2.11.

Overall, the simulation results provide numerical evidence consistent with the asymptotic efficiency, asymptotic risk efficiency, and asymptotic consistency properties established for the two-stage procedure and the

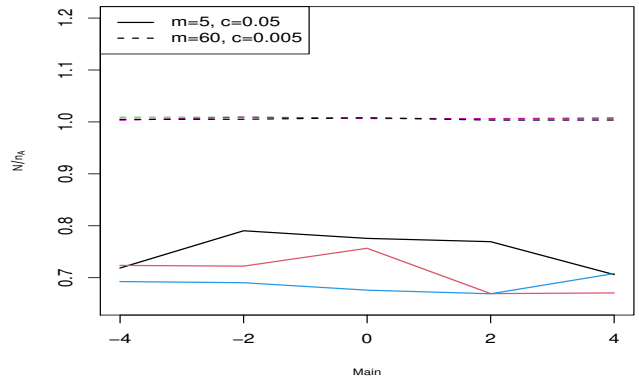


Figure 1: Ratios of Point Stopping Variables of $ARMA(1,1)$, Dash-line and Line Black: Two-stage, Dash-line and Line Red: Modified two-stage, Dash-line and Line Blue: Modified Two-stage with lower bound.

class of modified two-stage procedures. Moreover, the results indicate that the two-stage procedure and the class of modified two-stage procedures exhibit comparable performance in terms of the stopping sample size, the resulting point estimates, the risk function, and the coverage probabilities of the confidence intervals under the distributions considered. The results are broadly stable across the considered values of μ, suggesting that the performance of the proposed procedures is not materially affected by the level or sign of the mean parameter. Thus, the class of modified two-stage procedures appear to retain the statistical performance of the original two-stage procedure while providing greater flexibility in specifying the initial sample size, which may be advantageous in applications involving linear time-series models.

4 Data Analysis

In this section, we investigate the performance of the proposed two-stage procedure and the class of modified two-stage procedures for point and confidence interval estimation using a real dataset. The dataset was previously analyzed by Li et al. (2021) [9], who fitted an $ARMA(0,4)$ model to the data. Following their specification, we adopt the same

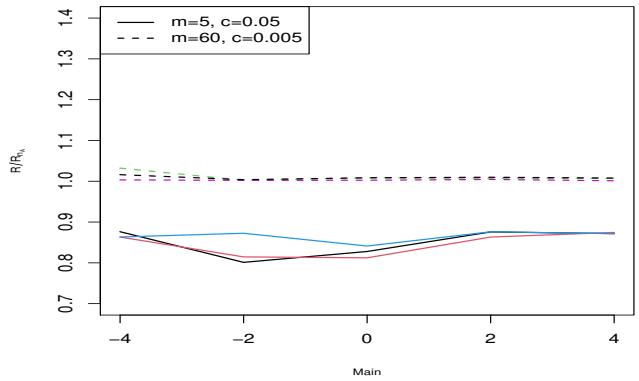


Figure 2: Ratios of Risk Functions of $ARMA(1,1)$, Dash-line and Line Black: Two-stage, Dash-line and Line Red: Modified two-stage, Dash-line and Line Blue: Modified Two-stage with lower bound.

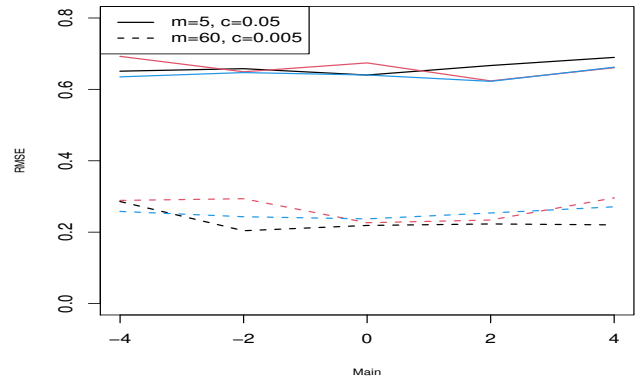


Figure 3: RMSE of Estimators based on Point Stopping Variables of $ARMA(1,1)$, Dash-line and Line Black: Two-stage, Dash-line and Line Red: Modified two-stage, Dash-line and Line Blue: Modified Two-stage with lower bound.

linear time-series structure here to focus on evaluating the proposed sequential estimation procedures rather than on model-order selection. It consists of $n = 84$ monthly observations on highway transportation accidents involving dangerous goods in China, covering the period from 2013 to 2019. We use this dataset to assess the practical performance of the

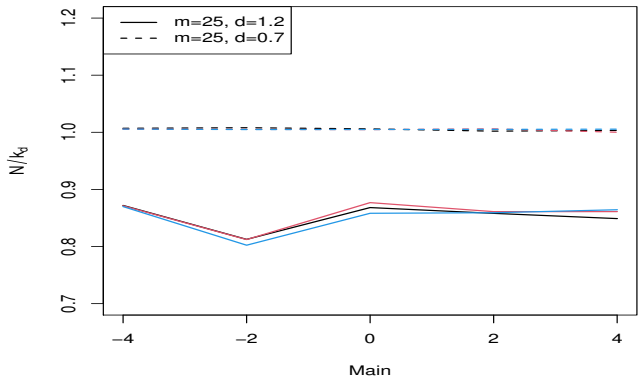


Figure 4: Ratios of Interval Stopping variable of $ARMA(1,1)$, Dash-line and Line Black: Two-stage, Dash-line and Line Red: Modified two-stage, Dash-line and Line Blue: Modified Two-stage with lower bound.

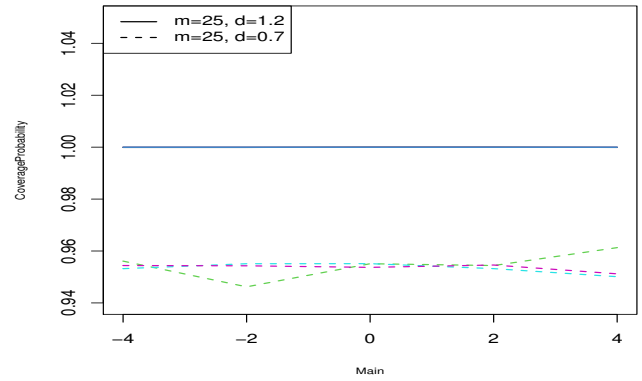


Figure 5: Coverage Probability for $ARMA(1,1)$, Dash-line and Line Black: Two-stage, Dash-line and Line Red: Modified two-stage, Dash-line and Line Blue: Modified Two-stage with lower bound.

proposed procedures by considering the corresponding initial sample-size specifications for the two-stage and class of modified two-stage procedures.

As described previously, the proposed procedures determine the required final sample size based on the information obtained from the ini-

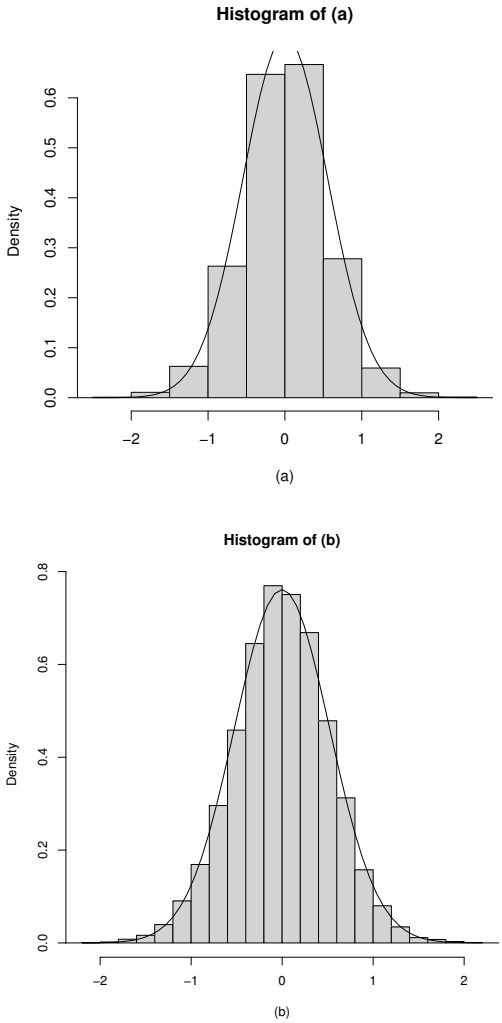


Figure 6: Histogram of Asymptotic Distribution of $\bar{X}_{N_m}$, $(a) : m = 50, c = 0.005$ and $(b) : m = 100, c = 0.0005$.

tial sample. Accordingly, if the stopping rule indicates that additional observations are required, $N_m - m$ additional observations are collected at the second stage for the two-stage procedure, with the corresponding difference defined analogously for the modified procedures

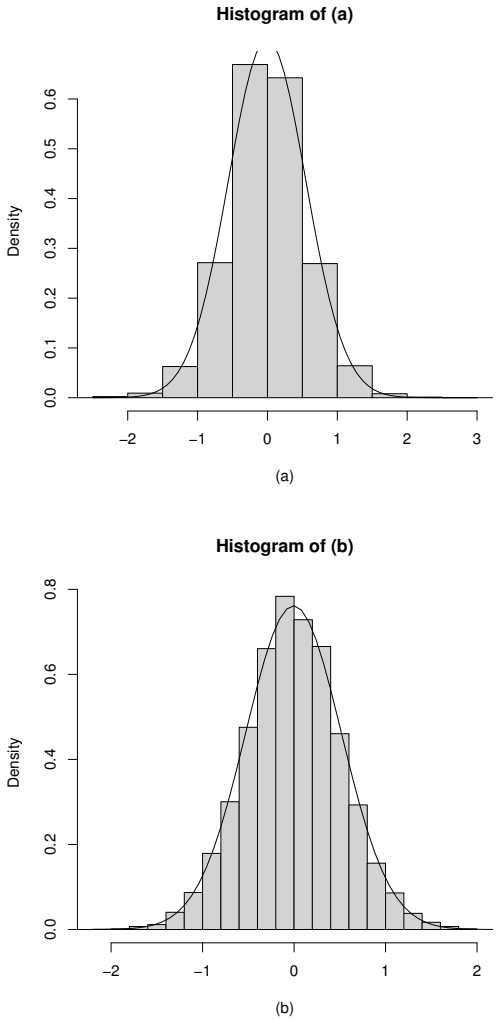


Figure 7: Histogram of Asymptotic Distribution of $\bar{X}_{N'_m}$, $(a) : m = 50, c = 0.005$ and $(b) : m = 100, c = 0.0005$.

Tables 1 and 2 present the point and confidence interval estimation results, respectively. Specifically, these tables report the two-stage stopping variable, the class of modified two-stage stopping variables, and the estimates of the mean and variance of the model, with $h = 1$, $a = 1$,

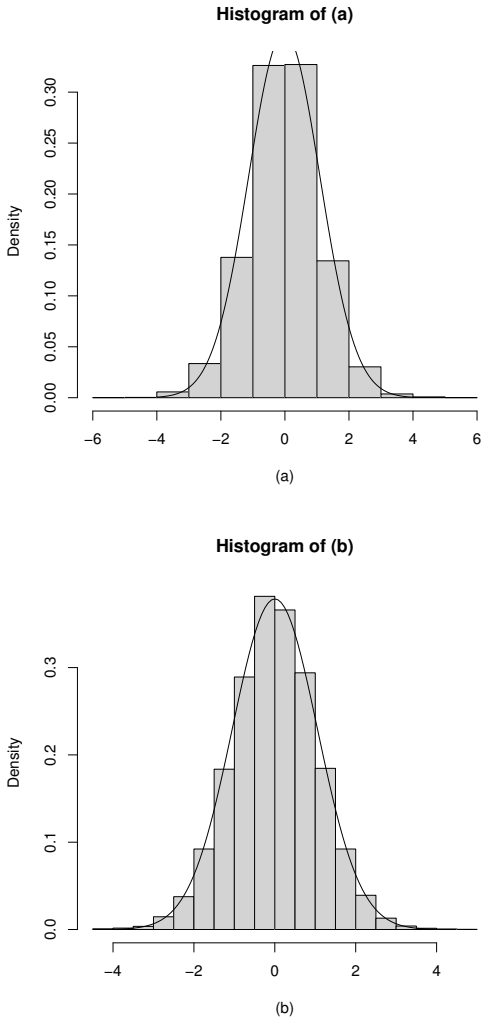


Figure 8: Histogram of Asymptotic Distribution of $\bar{X}_{N_{m'}}$, $(a) : m = 50, c = 0.005$ and $(b) : m = 100, c = 0.0005$.

$\tau_l = 0.6$ and $\lambda = 0.7$. For point estimation, the pairs of the initial sample size and c considered were $(2, 0.5)$, $(6, 0.05)$, and $(8, 0.005)$. Table 2 presents the corresponding interval estimation results for the pairs of the initial sample size and d, namely $(5, 1.2)$, $(5, 0.8)$, and $(5, 0.5)$. The

selected initial sample sizes are used to illustrate the behavior of the proposed procedures under different sampling-cost and precision settings.

Table 1 shows that, across the considered combinations of initial sample size and cost per observation, the stopping variables obtained from the two-stage procedure and the class of modified two-stage procedures become progressively closer and coincide for the considered settings. Since the stopping sample sizes coincide, the resulting of point estimation are identical under the considered settings. These findings indicate that the proposed procedures provide comparable point-estimation performance under the considered settings. Thus, the choice among these procedures may be guided by the practical requirements of the sampling problem without materially affecting the resulting point estimations. Moreover, the class of modified two-stage procedures provides greater flexibility in specifying the initial sample size, which may be advantageous when operational or resource constraints require the pilot sample size to be determined in advance.

Similarly, Table 2 reports the results of the two-stage procedure and the class of modified two-stage procedures for interval estimation. As d decreases, the final sample sizes obtained from the three procedures increase, while they remain identical across the procedures for each considered value of d. This behavior is consistent with the definitions of the corresponding stopping variables, because a smaller value of d requires a higher level of precision and, consequently, a larger final sample size. Furthermore, because the final sample sizes coincide across the procedures, the corresponding point estimators and confidence intervals are identical under the considered settings.

Overall, the real-data results indicate that the class of modified two-stage procedures yields the same point- and interval-estimation results as the two-stage procedure under the considered settings, while providing greater flexibility in specifying the initial sample size. This flexibility may be particularly useful when the initial sample size must be determined in advance because of sampling or resource constraints.

Table 1: Point estimation of two-stage and the class of modified two stage procedures.

(m, c)	N_m	N'_m	$N_{m'}$	$\bar{X}_{N_m}$	$\bar{X}_{N'_m}$	$\bar{X}_{N_{m'}}$	$\widehat{\tau}_{N_m}$	$\widehat{\tau}_{N'_m}$	$\widehat{\tau}_{N_{m'}}$
$(2, 0.5)$	4	4	4	15.5	15.5	15.5	6.8572	6.8572	6.8572
$(6, 0.05)$	30	30	30	23.3	23.3	23.3	88.6449	88.6449	88.6449
$(8, 0.005)$	75	75	75	27.8	27.8	27.8	79.8916	79.8916	79.8916

Table 2: Interval estimation of two-stage and the class of modified two stage procedures.

(m, d)	N_m^d	N'_d	$N_{d'}$	$\bar{X}_{N_m^d}$	$\bar{X}_{N'_d}$	$\bar{X}_{N_{d'}}$	$\widehat{\tau}_{N_m^d}$	$\widehat{\tau}_{N'_d}$	$\widehat{\tau}_{N_{d'}}$
$(5, 1.2)$	20	20	20	24.8	24.8	24.8	123.1663	123.1663	123.1663
$(5, 0.8)$	44	44	44	26.2727	26.2727	26.2727	85.1287	85.1287	85.1287
$(5, 0.5)$	77	77	77	27.9610	27.9610	27.9610	80.2407	80.2407	80.2407

Conclusion

In this study, we developed a two-stage sampling framework and a class of modified two-stage procedures for estimating the mean of a linear time-series model when the final sample size is unknown in advance. Appropriate stopping variables were introduced for both point and confidence interval estimations, providing a systematic means of determining the final sample size in accordance with the desired inferential precision. The theoretical results establish the properties of these procedures under the relevant asymptotic conditions and provide a rigorous foundation for adaptive sample-size determination in dependent-data settings.

An important feature of the proposed framework is the integration of sample size determination with subsequent statistical inference. Rather than treating the required sample size as a fixed determined independently of the estimation objective, the proposed stopping rules allow the sampling process to adapt to a specified precision requirement. Once sampling is terminated, the resulting observations can be used directly for point and confidence interval estimation of the model mean and, when appropriate, for model fitting and analyses. Thus, the proposed approach provides a unified framework linking sampling design, estimation, and subsequent analysis.

The class of modified two-stage procedures further enhances this framework by providing flexibility in specifying the initial sample size. This feature is particularly useful when sampling resources or observational costs must be considered at the design stage. This allows the initial sampling decision to be determined in advance while retaining an adaptive mechanism for determining the final sample size. Such flexibility may be especially valuable when several linear time-series models are to be investigated simultaneously under sampling or cost constraints, where an appropriate estimation precision is required.

Overall, the proposed procedures provide a flexible framework for adaptive sample size determination and inference in linear time-series models. Their practical relevance stems from the ability to incorporate the desired precision, sampling cost, and subsequent inferential requirements into a single sampling scheme. This framework may also provide a useful basis for extending sample size determination to more general dependent data models and settings involving multiple parameters or more complex inferential objectives.

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Soudabe Sajjadipanah
University of Applied Science and Technology
Ph.D graduate of statistics
Tangestan University of Applied Science and Technology
Tangestan, Boushehr, Iran
E-mail: soodabesajadi@yahoo.com

Sayyed Mahmoud Mirjalili
Department of Statistics
Associate Professor of Mathematics
Velayat University
Velayat, Iran
E-mail: mirjalili8@yahoo.com

Khadijeh Rezaie
Department of Art
Master graduate of Art
Islamic Azad University
Yazd, Iran
E-mail: khadijeh.rezaie@gmail.com

Sayyedeh Maryam Mousavialiabadi
Department of Statistics
Master graduate of statistics
Shiraz University
Shiraz, Iran
E-mail: maryam-amar86@yahoo.com