Journal of Mathematical Extension
Vol. 10, No. 1, (2016), 65-86
ISSN: 1735-8299
URL: http://www.ijmex.com

Existence Theorems of Solutions for Mixed Monotone Mappings in Partially Ordered Metric Spaces and Applications

R. Allahyari
Mashhad Branch, Islamic Azad University

Abstract. The purpose of this paper is to present some new couple fixed point theorems for multivalued mappings which satisfy generalized weak contraction in ordered metric spaces. The results of this paper are generalizations of the main results of [3, 5, 9]. As an application, we show existence and uniqueness of solutions of a class of nonlinear integral equations.

AMS Subject Classification: 47H08; 47H10
Keywords and Phrases: Fixed point, partially ordered set, meir-Keeler condition, integral equations, mixed monotone property

1. Introduction

The existence of a fixed point and a couple fixed point for contraction type mappings in partially ordered metric spaces has been considered recently by Agarwal et al.[1], Bhaskar and Lakshmikantham [2], Nieto and Lopez [6, 7], Lakshmikantham and Ćirić [4], Samet [9], Harjani, Lopez and Sadarangani[3], Luong and Thuan[5] and Rus[8], and proved some fixed point and couple fixed point theorems for mappings having monotone property. For various new results see [11, 12, 13, 14, 15, 16, 17]. In this work we prove some new couple fixed point theorems

Received: February 2015; Accepted: July 2015

for multivalued mappings which satisfy generalized weak contraction in ordered metric spaces. Let $(X, \preceq)$ be a partially ordered metric space. For $x, y \in X$, set:

$$[x, y] = \{z \in X | x \preceq z \preceq y\},$$
$$(-\infty, x] = \{z \in X | z \preceq x\},$$
$$[x, \infty) = \{z \in X | z \succeq x\}.$$

Let $(X, \preceq)$ be a partially ordered set and $A : X \times X \longrightarrow X$, we say that A has the mixed monotone property if A is nondecreasing in the first argument and nonincreasing in the second argument, i.e.

$$x_1, x_2, y \in X, \qquad x_1 \preceq x_2 \Longrightarrow A(x_1, y) \preceq A(x_2, y),$$

and

$$y_1, y_2, x \in X, \qquad y_1 \preceq y_2 \Longrightarrow A(x, y_1) \succeq A(x, y_2).$$

A pair $(x, y) \in X \times X$ is called a coupled fixed point of a bivariate mapping A if

$$A(x, y) = x \quad \textit{and} \quad A(y, x) = y.$$

Also a point $x \in X$ is called a fixed point of A if $A(x, x) = x$.

Let (X, d) be a metric space and $\varphi : X \longrightarrow \mathbb{R}$ be a function. Define the relation "$\preceq$" on X by

$$x \preceq y \Longleftrightarrow d(x, y) \leq \varphi(x) - \varphi(y).$$

Then "$\preceq$" is a partial order on X and $(X, \preceq)$ is called an ordered metric space induced by φ (see [10]).

In [10] Zhang proved the following interesting result on the existence coupled fixed point for multivalued mappings.

Theorem 1.1. [10] *Let (X, d) be a complete metric space, $\varphi : X \longrightarrow \mathbb{R}$ be a function bounded below, and $\preceq$ be the order in X induced by φ. Let $F : X \times X \longrightarrow 2^X$ be a multivalued mapping and $M = \{(x, y) | x \preceq y$ and $F(x, y) \cap [x, +\infty) \neq \emptyset$ and $F(y, x) \cap (-\infty, y] \neq \emptyset\}$. Suppose that:*
(i) F is upper semi-continuous, that is, $x_n \in X$, $y_n \in X$ and $z_n \in F(x_n, y_n)$, with $x_n \longrightarrow x_0$, $y_n \longrightarrow y_0$ and $z_n \longrightarrow z_0$ imply $z_0 \in F(x_0, y_0)$;
(ii) For each $(x, y) \in M$, there is $(u, v) \in M$ such that $u \in F(x, y) \cap [x, +\infty)$ and $v \in F(y, x) \cap (-\infty, y]$;

(iii) $M \neq \emptyset$. *Then* F *has a couple fixed point* (x^*, y^*), *i.e.* $x^* \in F(x^*, y^*)$ *and* $y^* \in F(y^*, x^*)$, *and there exist two sequences* $\{x_n\}$ *and* $\{y_n\}$ *with*

$$x_{n-1} \preceq x_n \in F(x_{n-1}, y_{n-1}) \ , \ y_{n-1} \succeq y_n \in F(y_{n-1}, x_{n-1})$$

such that $x_n \longrightarrow x^*$ *and* $y_n \longrightarrow y^*$.

Definition 1.2. *A double sequence of real numbers is a function* $S : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{R}$. *We shall use the notation* $\{S(n,m)\}$ *or* $\{S_{n,m}\}$. *We say that a double sequence* $\{S_{n,m}\}$ *converges to* $a \in \mathbb{R}$ *and we write* $S_{n,m} \longrightarrow a$, *if the following condition is satisfied: For every* $\varepsilon > 0$, *there exists* $N \in \mathbb{N}$ *such that*

$$|S_{n,m} - a| < \varepsilon, \qquad for\ all\ n, m \geq N.$$

Definition 1.3. [3] *An altering distance function is a function* $\theta : [0, \infty) \longrightarrow [0, \infty)$ *satisfy:*
(i) θ *is continuous and nondecreasing,*
(ii) $\theta(t) = 0 \Longleftrightarrow t = 0$.

Definition 1.4. [3] *Let* Ψ *denote all functions* $\psi : [0, \infty) \longrightarrow [0, \infty)$ *satisfy:*
$\lim\limits_{s \longrightarrow p} \psi(s) > 0$ *for all* $p > 0$ *and* $\lim\limits_{s \longrightarrow 0} \psi(s) = 0$.

Notice that if θ is an altering distance function therefore $\theta \in \Psi$. Here we define a new class of control functions.

Definition 1.5. *Let* Φ *denote all functions* $\phi : [0, \infty) \times [0, \infty) \longrightarrow [0, \infty)$ *satisfy:*
(i) ϕ *is nondecreasing for each argument,*
(ii) $\phi(t_1 + t_2, s_1 + s_2) \leq \phi(t_1, s_1) + \phi(t_2, s_2)$,
(iii) $\phi(t_{n,m}, s_{n,m}) \longrightarrow 0 \Longleftrightarrow s_{n,m} \longrightarrow 0 \ , \ t_{n,m} \longrightarrow 0$,
(iv) $\phi(t, s) = 0 \Longleftrightarrow t = s = 0$.

Example 1.6. If $a, b \in \mathbb{R}^+$, $n \in \mathbb{N}$ then $\phi_1(t, s) = at + bs$, $\phi_2 = \max(at, bs)$, $\phi_3 = \ln(at+bs+1)$, $\phi_4 = \sqrt[n]{ax + by}$ and $\phi_5 = \sqrt[n]{\max(ax, by)}$ are in Φ .

Definition 1.7. *Let* $(X, \preceq)$ *be a partially ordered set. We say that* $F : X \times X \longrightarrow 2^X$ *has condition* C *if* $\{x_n\}$ *be an increasing sequence and*

$\{y_n\}$ be a decreasing sequence in X and $z_n \in F(x_n, y_n)$ such that $x_n \longrightarrow x_0$, $y_n \longrightarrow y_0$ and $z_n \longrightarrow z_0$ implies $z_0 \in F(x_0, y_0)$, and $H(x, y) := F(y, x)$ has this property too.

Remark 1.8. Notice that if F is upper semi-continuous then F has condition C but the converse is needs not to be true.

2. Couple Fixed Point Theorem and Meir-Keeler Condition

In this section we introduce the notion of ϕ-Meir-Keeler type functions and prove a couple fixed point theorem. Let

$$M_F = \{(x, y) | F(x, y) \cap [x, +\infty) \neq \emptyset \text{ and } F(y, x) \cap (-\infty, y] \neq \emptyset\}.$$

Definition 2.1. *Let $(X, \preceq)$ be a partially ordered set, d a metric on X and $\phi \in \Phi$. We say that a mapping $F : X \times X \longrightarrow 2^X$ is ϕ-Meir-Keeler function if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $x_1 \preceq x_2$, $y_1 \succeq y_2$, $(x_i, y_i) \in M_F$, $p_i \in F(x_i, y_i) \bigcap [x_i, +\infty)$, $q_i \in F(y_i, x_i) \bigcap (-\infty, y_i]$ for $i = 1, 2$ and*

$$\varepsilon \leq \phi(d(x_1, x_2), d(y_1, y_2)) < \varepsilon + \delta,$$

then we have

$$\phi(d(p_1, p_2), d(q_1, q_2)) < \varepsilon.$$

Theorem 2.2. *Let $(X, \preceq)$ be a partially ordered set and d a complete metric on X. Let $F : X \times X \longrightarrow 2^X$ be a multivalued mapping and $M \neq \emptyset$. Suppose that:*
(i) F has condition C and there exists $\phi \in \Phi$ such that F be a ϕ-Meir-Keeler function;
(ii) For each $(x, y) \in M_F$, there is $(u, v) \in M$ such that $u \in F(x, y) \bigcap [x, +\infty)$ and $v \in F(y, x) \bigcap (-\infty, y]$. Then F has a coupled fixed point (x^*, y^*) and there exist two sequences $\{x_n\}$ and $\{y_n\}$ with

$$x_{n-1} \preceq x_n \in F(x_{n-1}, y_{n-1}) \ , \ y_{n-1} \succeq y_n \in F(y_{n-1}, x_{n-1}),$$

such that $x_n \longrightarrow x^*$ and $y_n \longrightarrow y^*$.

Proof. Take $(x_0, y_0) \in M_F$. From (ii) there exists $(x_1, y_1) \in M_F$ such that $x_1 \in F(x_0, y_0)$, $x_0 \preceq x_1$, $y_1 \in F(y_0, x_0)$ and $y_1 \preceq y_0$. Again from (ii) there exists $(x_2, y_2) \in M_F$ such that $x_2 \in F(x_1, y_1)$, $x_1 \preceq x_2$, $y_2 \in F(y_1, x_1)$ and $y_2 \preceq y_1$. Continuing this procedure, we get two sequences $\{x_n\}$ and $\{y_n\}$ satisfying $(x_n, y_n) \in M_F$,

$$x_{n-1} \preceq x_n \in F(x_{n-1}, y_{n-1}), \qquad n \geq 1,$$

and

$$y_{n-1} \succeq y_n \in F(y_{n-1}, x_{n-1}), \qquad n \geq 1.$$

If $(x_n, y_n) = (x_{n-1}, y_{n-1})$ then $x_n \in F(x_{n-1}, y_{n-1}) = F(x_n, y_n)$ and $y_n \in F(y_{n-1}, x_{n-1}) = F(y_n, x_n)$ which imply (x_n, y_n) is a coupled fixed point. Without restriction of the generality, we can suppose that $(x_n, y_n) \neq (x_{n-1}, y_{n-1})$.

claim 1: $\phi(d(x_n, x_{n+1}), d(y_n, y_{n+1})) \longrightarrow 0$ as $n \longrightarrow \infty$.
Define $\varepsilon_n = \phi(d(x_n, x_{n-1}), d(y_n, y_{n-1}))$ and $\delta_n = \delta(\varepsilon_n)$. By definition of x_n, y_n and $\varepsilon_n < \delta_n + \varepsilon_n$ we have

$$\varepsilon_{n+1} = \phi(d(x_{n+1}, x_n), d(y_{n+1}, y_n)) < \varepsilon_n = \phi(d(x_n, x_{n-1}), d(y_n, y_{n-1})).$$

So ε_n is a positive decreasing sequence of real numbers thus, there is an $r \geq 0$ such that $\varepsilon_n \longrightarrow r$. We shall show that $r = 0$. If $r \neq 0$, there exists N_0 such that $n > N_0$ implies $r \leq \varepsilon_n < r + \delta(r)$, therefore by the definition of ϕ-Meir-Keeler, $\varepsilon_{n+1} < r$ which is a contradiction, so $r = 0$.

claim 2: $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences.
Let $\varepsilon > 0$, it follows from claim1 that there exists $k \in \mathbb{N}$ such that

$$\phi(d(x_n, x_{n+1}), d(y_n, y_{n+1})) < \delta(\varepsilon). \tag{1}$$

Without restriction of the generality, we can suppose that $\delta(\varepsilon) \leq \varepsilon$. Consider the set $\Lambda \subset X \times X$ is defined by:

$$\Lambda := \{(x, y) |\ y \preceq y_k,\ x \succeq x_k, (x, y) \neq (x_k, y_k) \text{ and } \phi(d(x_k, x), d(y_k, y)) < \varepsilon + \delta(\varepsilon)\}.$$

We prove that if $u \in F(x, y) \bigcap [x, +\infty)$, $v \in F(y, x) \bigcap (-\infty, y]$ then

$$(u, v) \in \Lambda \ \text{ whenever } \ (x, y) \in \Lambda. \tag{2}$$

Let $(x,y)\in\Lambda$, by the property of subadditivity of ϕ and triangle inequality, we have:

$$\begin{aligned}\phi(d(x_k,u),d(y_k,v)) \leq &\phi(d(x_k,x_{k+1}),d(y_k,y_{k+1}))+\phi(d(x_{k+1},u),d(y_{k+1},v))\\ <&\delta(\varepsilon)+\phi(d(x_{k+1},u),d(y_{k+1},v)).\end{aligned}$$

Now we distinguish two cases:
I) If $\phi(d(x_k,x),d(y_k,y))\leq\varepsilon$ then by the definition of ϕ-Meir-Keeler we get

$$\phi(d(x_k,u),d(y_k,v))<\delta(\varepsilon)+\varepsilon. \tag{3}$$

II) If $\varepsilon<\phi(d(x,x_k),d(y,y_k))\leq\varepsilon+\delta(\varepsilon)$ then

$$\phi(d(x_{k+1},u),d(y_{k+1},v)<\varepsilon,$$

hence

$$\phi(d(x_k,u),d(y_k,v))\leq\delta(\varepsilon)+\phi(d(x_{k+1}),u)),d(y_{k+1},v))\leq\delta(\varepsilon)+\varepsilon. \tag{4}$$

Therefor by (3), (4) we have

$$\phi(d(x_k,u),d(y_k,v))\leq\delta(\varepsilon)+\varepsilon. \tag{5}$$

Now we show that $u\succeq x$, $v\preceq y$ and $(u,v)\neq(x_k,y_k)$. Since $u\in[x,\infty)$, $v\in(-\infty,y]$ and $(x,y)\neq(x_k,y_k)$ we get

$$x_k\preceq x\preceq u\ ,\ y_k\succeq y\succeq v\ and\ (u,v)\neq(x_k,y_k). \tag{6}$$

By (5) and (6) we deduce that (2) holds. Using (1) and (2) we have $(x_{k+1},y_{k+1})\in\Lambda$. Thus, by induction $(x_n,y_n)\in\Lambda$ for every $n\geq k$. Again using the property of subadditivity of ϕ and triangle inequality we get

$$\begin{aligned}\phi(d(x_n,x_m),d(y_n,y_m)) &\leq \phi(d(x_k,x_m),d(y_k,y_m))+\phi(d(x_k,x_n),d(y_k,y_n))\\ &< 2(\varepsilon+\delta(\varepsilon))\\ &\leq 4\varepsilon,\end{aligned}$$

for all $n,m>k$. Then, $\phi(d(x_n,x_m),d(y_n,y_m))\longrightarrow 0$ and by property (iii) of Definition 1.7 we have $d(x_n,x_m)\longrightarrow 0$ and $d(y_n,y_m)\longrightarrow 0$, hence

$\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Since (X, d) is a complete metric space, there exists (x^*, y^*) such that

$$x_n \longrightarrow x^*, \; y_n \longrightarrow y^* \quad as \quad n \longrightarrow \infty,$$

and with the condition C we have $x^* \in F(x^*, y^*)$ and $y^* \in F(y^*, x^*)$. This shows that F has a coupled fixed point and there exist two sequences $\{x_n\}$ and $\{y_n\}$ with

$$x_{n-1} \preceq x_n \in F(x_{n-1}, y_{n-1}) \quad , \quad y_{n-1} \succeq y_n \in F(y_{n-1}, x_{n-1})$$

such that $x_n \longrightarrow x^*$ and $y_n \longrightarrow y^*$. □

Definition 2.3. *Let $(X, \preceq)$ be a partially ordered, d a metric on X and $\phi \in \Phi$. We say that a mapping $A : X \times X \longrightarrow X$ is ϕ-Meir-Keeler type function if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $x, y, u, v \in X$, $v \preceq y$, $x \preceq u$ and $\varepsilon \leq \phi(d(x, u), d(y, v)) < \varepsilon + \delta$ imply that $\phi(d(A(x, y), A(u, v)), d(A(y, x), A(v, u)) < \varepsilon$.*

Corollary 2.4. *Let $(X, \preceq)$ be a partially ordered set, d a complete metric on X and $\phi \in \Phi$. Let $A : X \times X \longrightarrow X$ be a ϕ-Meir-Keeler type function that has the mixed monotone property. Suppose either:*
(a) *A is continuous, or*
(b) *X has the following properties:*
(i) If (x_n) is a nondecreasing sequence that is convergent to x then $x_n \preceq x$ for all n.
(ii) If (y_n) is a nonincreasing sequence that is convergent to y then $y_n \succeq y$ for all n.
If there exist $x_0, y_0 \in X$ such that $x_0 \preceq A(x_0, y_0)$ and $A(y_0, x_0) \preceq y_0$, then A has couple fixed point (x^*, y^*), and there exist two sequences $\{x_n\}$ and $\{y_n\}$ with $x_{n-1} \preceq x_n$, $y_{n-1} \succeq y_n$ such that $x_n \longrightarrow x^*$ and $y_n \longrightarrow y^*$.

Proof. Define $F : X \times X \longrightarrow 2^X$ such that $F(x, y) = \{A(x, y)\}$.
By the assumption, there exist $x_0, y_0 \in X$ such that $x_0 \preceq A(x_0, y_0)$ and $A(y_0, x_0) \preceq y_0$. Hence, $M \neq \emptyset$. Now for $(x, y) \in M_F$, since A has the mixed monotone property we have

$$A(x, y) \preceq A(A(x, y), A(y, x)) \; and \; A(y, x) \succeq A(A(y, x), A(x, y)),$$

Thus

$$A(A(x,y),A(y,x)) \in F(A(x,y),A(y,x)) \cap [A(x,y),+\infty), \tag{7}$$

and

$$A(A(y,x),A(x,y)) \in F(A(y,x),A(x,y)) \cap (-\infty,A(y,x)]. \tag{8}$$

Now from (7) and (8) we have $(A(x,y),A(y,x)) \in M_F$, $A(x,y) \in F(x,y) \cap [x,+\infty)$ and $A(y,x) \in F(y,x) \cap [-\infty,y)$. It follows that F satisfies condition (ii) of Theorem 2.2. Now we prove that F satisfies condition (i) of Theorem 2.2. For this purpose, let condition (a) holds, so F satisfies condition C. On the other hand, since A is a ϕ-Meir-Keeler type function F satisfies condition (i) of Theorem 2.2.

Now suppose that (b) holds. If $\{x_n\}$ is a nondecreasing sequence, then $\{y_n\}$ is a nonincreasing sequence and $z_n \in F(x_n,y_n)$ such that $x_n \longrightarrow x_0$, $y_n \longrightarrow y_0$ and $z_n \longrightarrow z_0$. We have two cases:

Case 1: There exists a subsequence $\{n(k)\}$ such that $(x_{n(k)},y_{n(k)}) = (x_0,y_0)$. Since $z_{n(k)} \in F(x_{n(k)},y_{n(k)}) = F(x_0,y_0) = \{A(x_0,y_0)\}$ and $z_n \longrightarrow z_0$ therefore $z_0 \in F(x_0,y_0)$.

Case 2: there exists $k > 0$ such that for all $n > k$, $(x_n,y_n) \neq (x_0,y_0)$. Since $z_n = A(x_n,y_n)$, $d(x_n,x_0) \longrightarrow 0$ and $d(y_n,y_0) \longrightarrow 0$ then by property (iii) of Definition 1.7 and the property that A is a ϕ-Meir-Keeler type function, we have

$$\begin{aligned}\phi(d(A(x_0,y_0),A(x_n,y_n)),d(A(y_0,x_0),A(y_n,x_n))) < \phi(d(x_n,x_0),\\ d(y_n,y_0)) \longrightarrow 0,\end{aligned}$$

which implies that

$$d(A(x_0,y_0),A(x_n,y_n)) \longrightarrow 0.$$

So $z_0 = A(x_0,y_0)$ and by a similar reasoning $H(x,y) := F(y,x)$ has this property too. Therefore, A has a couple fixed point (x^*,y^*) and there exist two sequences $\{x_n\}$ and $\{y_n\}$ by $x_{n-1} \preceq x_n$ and $y_{n-1} \geq y_n$ such that $x_n \longrightarrow x^*$ and $y_n \longrightarrow y^*$. $\square$

Let $(x, \preceq)$ be a partially ordered set. We endow the product space $X \times X$ with the following partial order:

$$(x,y),(u,v) \in X \times X, \qquad (u,v) \preceq (x,y) \Longleftrightarrow u \preceq x, v \succeq y.$$

Suppose that the product space $X \times X$ endowed with the above mentioned partial order has the following property:
(H): For $(x,y),(u,v) \in X \times X$ there exists $(z,t) \in X \times X$ that is comparable to (x,y) and (u,v).

Theorem 2.5. *Adding condition (H) to the hypotheses of Corollary 2.4 we obtain uniqueness of the couple fixed point of A.*

Proof. Suppose that (x,y) and (z,t) are couple fixed points of A, that is $x = A(x,y)$, $y = A(y,x)$, $z = A(z,t)$ and $t = A(t,z)$. Let (u,v) be an element of $X \times X$ and comparable to (x,y) and (z,t). Assume that $(x,y) \succeq (u,v)$. We construct sequences (u_n) and (v_n) defined inductively by $u_0 = u$, $v_0 = v$, $u_{n+1} = A(u_n, v_n)$ and $v_{n+1} = A(v_n, u_n)$. Similar to Theorem 4 in [3] we have $(x,y) \succeq (u_n, v_n)$ for all $n \in \mathrm{N}$. Now, since $u_n \preceq x$ and $v_n \succeq y$, using the ϕ-Meir-Keeler condition we have

$$\begin{aligned}\phi(d(x,u_{n+1}),d(y,v_{n+1})) =&\phi(d(A(x,y),A(u_n,v_n)),d(A(y,x),A(v_n,u_n)))\\ <&\phi(d(x,u_n),d(y,v_n)).\end{aligned}$$

Thus $\phi(d(x,u_n),d(y,v_n))$ is decreasing and with similar argument we have $\phi(d(x,u_n),d(y,v_n)) \longrightarrow 0$. Therefore by property (iii) of Definition 1.7 we have $d(x,u_n) \longrightarrow 0$ and $d(y,v_n)) \longrightarrow 0$ this gives us $u_n \longrightarrow x$ and $v_n \longrightarrow y$.
Using a similar argument for (z,t) we can obtain $u_n \longrightarrow z$ and $v_n \longrightarrow t$. Now uniqueness of the limit gives $x = z$ and $y = t$. The proof of the other cases is similar. □

Corollary 2.4 with a condition that x_0, y_0 are comparable gives a fixed point for the mapping A.

Theorem 2.6. *In addition to the hypotheses of Corollary 2.4, suppose that $x_0, y_0 \in X$ are comparable. Then $x^* = y^*$.*

Proof. We assume that $x_0 \preceq y_0$ (a similar argument applies for $x_0 \succeq y_0$). Using the mixed monotone property of A, we have $x_n \preceq y_n$ and by definition of ϕ-Meir-Keeler

$$\phi(d(x_n, y_n), d(x_n, y_n)) < \phi(d(x_{n-1}, y_{n-1}), d(x_{n-1}, y_{n-1})).$$

Therefore $\phi(d(x_n, y_n), d(x_n, y_n))$ is decreasing and converges to some r and similar to Theorem 2.2 we have $r = 0$. So by property (iii) of Definition 1.7 and using the triangular inequality we have

$$d(x^*, y^*) \leq d(x^*, x_n) + d(x_n, y_n) + d(y^*, y_n) \longrightarrow 0,$$

therefore $x^* = y^*$. □

3. Couple Fixed Point Theorem and $T_{(\theta,\phi,\psi)}$ Condition

In this section we give some couple fixed point result for mappings satisfying generalized weak contraction in the setting of complete partially ordered metric spaces.

Definition 3.1. *Let $(X, \preceq)$ be a partially ordered set, d a metric on X, $\phi \in \Phi$, $\psi \in \Psi$ and θ is an altering function. We say that $F : X \times X \longrightarrow 2^X$ has condition $T_{(\theta,\phi,\psi)}$, if $x_1 \preceq x_2$, $y_1 \succeq y_2$, $(x_i, y_i) \in M$, $p_i \in F(x_i, y_i) \bigcap [x_i, +\infty)$ and $q_i \in F(y_i, x_i) \bigcap (-\infty, y_i]$ for $i = 1, 2$ then*

$$\begin{aligned}\theta(\phi(d(p_1, q_1), d(p_2, q_2)) \leq\ & \theta(\phi(d(x_1, x_2), d(y_1, y_2)) \\ & -\psi(\phi(d(x_1, x_2), d(y_1, y_2)).\end{aligned} \tag{9}$$

Theorem 3.2. *Let $(X, \preceq)$ be a partially ordered set and d a complete metric on X. Let $F : X \times X \longrightarrow 2^X$ be a multivalued mapping with $M_F \neq \emptyset$. suppose that:*
(i) F has conditions C and $T_{(\theta,\phi,\psi)}$ for some $\phi \in \Phi$, $\psi \in \Psi$ and θ is an altering function ,
(ii) For each $(x, y) \in M_F$, there is $(u, v) \in M_F$ such that $u \in F(x, y) \bigcap [x, +\infty)$, $v \in F(y, x) \bigcap (-\infty, y]$.

Then F has a coupled fixed point (x^*, y^*) and there exist two sequences $\{x_n\}$ and $\{y_n\}$ with

$$x_{n-1} \preceq x_n \in F(x_{n-1}, y_{n-1}) \text{ and } y_{n-1} \succeq y_n \in F(y_{n-1}, x_{n-1}),$$

such that $x_n \longrightarrow x^*$ and $y_n \longrightarrow y^*$.

Proof. Similar to the proof of Theorem 2.2 we get two sequences $\{x_n\}$ and $\{y_n\}$ satisfying $(x_n, y_n) \in M_F$,

$$x_{n-1} \preceq x_n \in F(x_{n-1}, y_{n-1}), \qquad n \geq 1,$$

and

$$y_{n-1} \succeq y_n \in F(y_{n-1}, x_{n-1}), \qquad n \geq 1,$$

and without restriction of the generality, we can suppose that $(x_n, y_n) \neq (x_{n-1}, y_{n-1})$.

claim 1: $\phi(d(x_n, x_{n+1}), d(y_n, y_{n+1})) \longrightarrow 0$ as $n \longrightarrow \infty$.

Define $\varepsilon_n = \phi(d(x_n, x_{n-1}), d(y_n, y_{n-1}))$. By definition of x_n, y_n and from (i) we have

$$\theta(\varepsilon_n) \leq \theta(\varepsilon_{n-1}) - \psi(\varepsilon_{n-1}) < \theta(\varepsilon_{n-1}).$$

Since θ is nondeceareasing, so ε_n is a positive decreasing sequence of real numbers thus, there is an $r \geq 0$ such that $\varepsilon_n \longrightarrow r$. We shall show that $r = 0$. Suppose, to the contrary, that $r \neq 0$. Taking the limit as $n \longrightarrow \infty$(equivalently, $\varepsilon_n \longrightarrow r$) of both sides of (9) and have in mind that we suppose $\lim_{s \longrightarrow p} \phi(s) > 0$ for all $p > 0$ and θ is continuous, then we have

$$\theta(r) = \lim_{n \longrightarrow \infty} \theta(\varepsilon_n) \leq \lim_{n \longrightarrow \infty} [\theta(\varepsilon_{n-1}) - \psi(\varepsilon_{n-1})] < \theta(r),$$

which is a contradiction, hence $r = 0$.

claim 2: $\phi(d(x_n, x_m), d(y_n, y_m)) \longrightarrow 0$.

Suppose to the contrary, there exists an $\varepsilon > 0$ for which we can find subsequences $\{x_{n(k)}\}$, $\{x_{m(k)}\}$ of $\{x_n\}$ and $\{y_{n(k)}\}$, $\{y_{m(k)}\}$ of $\{y_n\}$ with $n(k) > m(k) \geq k$ such that

$$\phi(d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})) \geq \varepsilon. \tag{10}$$

Further, corresponding to $m(k)$, we can choose $n(k)$, in such a way that it is the smallest integer with $n(k) > m(k) \geq k$ and satisfy (10). Then

$$\phi(d(x_{n(k)-1}, x_{m(k)}), d(y_{n(k)-1}, y_{m(k)})) < \varepsilon. \tag{11}$$

Using (10), (11), and the property of subadditivity of ϕ and triangle inequality, we have:

$$\begin{aligned}
\varepsilon \leq r_k &:= \phi(d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})) \\
&\leq \phi(d(x_{n(k)-1}, x_{m(k)}), d(y_{n(k)-1}, y_{m(k)})) \\
&+ \phi(d(x_{n(k)-1}, x_{n(k)}), d(y_{n(k)-1}, y_{n(k)})) \\
&\leq \varepsilon + \phi(d(x_{n(k)-1}, x_{n(k)}), d(y_{n(k)-1}, y_{n(k)})).
\end{aligned}$$

Letting $k \longrightarrow \infty$ and using claim 1

$$\lim_{k\longrightarrow\infty} r_k = \lim_{k\longrightarrow\infty} \phi(d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})) = \varepsilon. \tag{12}$$

Again, the property of subadditivity of ϕ and triangle inequality

$$\begin{aligned}
&\phi(d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})) \\
\leq\ &\phi(d(x_{n(k)}, x_{n(k)-1}), d(y_{n(k)}, y_{n(k)-1})) \\
+\ &\phi(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})) \\
+\ &\phi(d(x_{m(k)-1}, x_{m(k)}), d(y_{m(k)-1}, y_{m(k)})) \\
\leq\ &\varepsilon_{n(k)} + \varepsilon_{m(k)} \\
+\ &\phi(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1}))
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
&\phi(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})) \\
\leq\ &\phi(d(x_{n(k)}, x_{n(k)-1}), d(y_{n(k)}, y_{n(k)-1})) \\
+\ &\phi(d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})) \\
+\ &\phi(d(x_{m(k)-1}, x_{m(k)}), d(y_{m(k)-1}, y_{m(k)})) \\
\leq\ &\varepsilon_{n(k)} + \varepsilon_{m(k)} \\
+\ &\phi(d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})).
\end{aligned} \tag{14}$$

Thus, by (13), (14), $\phi(d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})) \longrightarrow \varepsilon$ and $\varepsilon_n \longrightarrow 0$ we get

$$\phi(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})) \longrightarrow \varepsilon. \tag{15}$$

Since $n(k) > m(k)$ hence $x_{n(k)} \succeq x_{m(k)}$ and $y_{n(k)} \preceq y_{m(k)}$. Using the property of θ and condition $T_{(\theta,\phi,\psi)}$, we have

$$\begin{aligned}\theta(\phi(d(x_{n(k)},x_{m(k)}),d(y_{n(k)},y_{m(k)}))) &\leq \\ \theta(\phi(d(x_{n(k)-1},x_{m(k)-1}),d(y_{n(k)-1},y_{m(k)-1}))) \\ -\psi(\phi(d(x_{n(k)-1},x_{m(k)-1}),d(y_{n(k)-1},y_{m(k)-1}))).\end{aligned}$$

Letting $k \longrightarrow \infty$, by (12) and (15) we get

$$\theta(\varepsilon) \leq \theta(\varepsilon) - \psi(\varepsilon) < \theta(\varepsilon),$$

which is a contradiction, so $\phi(d(x_n,x_m),d(y_n,y_m)) \longrightarrow 0$.
Claim 2 and property (iii) in Definition 1.7 show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Since (X,d) is a complete metric space, there exists (x^*,y^*) such that

$$x_n \longrightarrow x^*,\ y_n \longrightarrow y^* \quad as \quad n \longrightarrow \infty.$$

By using condition (C) we have $x^* \in F(x^*,y^*)$ and $y^* \in F(y^*,x^*)$. This implies F has coupled fixed point and there exist two sequences $\{x_n\}$ and $\{y_n\}$ with

$$x_{n-1} \preceq x_n \in F(x_{n-1},y_{n-1}),\ y_{n-1} \succeq y_n \in F(y_{n-1},x_{n-1})$$

such that $x_n \longrightarrow x^*$ and $y_n \longrightarrow y^*$. $\square$

Definition 3.3. *Let $(X,\preceq)$ be a partially ordered set, d a metric on X, $\phi \in \Phi$, $\psi \in \Psi$ and θ an altering function. We say that $A : X \times X \longrightarrow X$ has condition $T_{(\theta,\phi,\psi)}$ if $x \preceq u$, $y \succeq v$ then*

$$\begin{aligned}\theta(\phi(d(A(x,y),A(u,v)),d(A(y,x),A(v,u)))) \leq &\theta(\phi(d(x,u),d(y,v))) \\ &- \psi(\phi(d(x,u),d(y,v))).\end{aligned}$$

Corollary 3.4. *Let $(X,\preceq)$ be a partially ordered set, d a complete metric on X, $\phi \in \Phi$, $\psi \in \Psi$ and θ is an altering function. Let $A : X \times X \longrightarrow X$ has condition $T_{(\theta,\phi,\psi)}$ and the mixed monotone property. Suppose either:*
(a) *A is continuous, or*
(b)*$\bar{X}$ has the following properties:*
(i) *if (x_n) is a nondecreasing sequence that is convergent to x then*

$x_n \preceq x$ for all n;
(ii) if (y_n) is a nonincreasing sequence that is convergent to y then $y_n \succeq y$ for all n.
If there exist $x_0, y_0 \in X$ such that $x_0 \preceq A(x_0, y_0)$ and $A(y_0, x_0) \preceq y_0$, then A has couple fixed point and there exist two sequences $\{x_n\}$ and $\{y_n\}$ with $x_{n-1} \preceq x_n$, $y_{n-1} \succeq y_n$ such that $x_n \longrightarrow x^*$ and $y_n \longrightarrow y^*$.

Proof. The proof is similar to the proof of Corollary 2.4. □

As a corollary of Theorem 3.2 we have the main result of [5].

Corollary 3.5. [5] *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \longrightarrow X$ be a mapping having the mixed monotone property on X such that there exist two elements $x_0, y_0 \in X$ with*

$$x_0 \leq A(x_0, y_0) \text{ and } A(y_0, x_0) \leq y_0.$$

Suppose that there exist θ is altering distance function and $\psi \in \Psi$ such that

$$\theta(d(A(x,y), A(u,v))) \leq \frac{1}{2}\theta(d(x,u) + d(y,v)) - \psi(\frac{d(x,u) + d(y,v)}{2})$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$. Suppose either:
(a) *A is continuous, or*
(b)*$\bar{X}$ has the following properties:*
(i) *if (x_n) is a nondecreasing sequence that is convergent to x then $x_n \preceq x$ for all n,*
(ii) *if (y_n) is a nonincreasing sequence that is convergent to y then $y_n \succeq y$ for all n.*
then there exist $x^*, y^* \in X$ such that

$$x^* = A(x^*, y^*) \text{ and } A(y^*, x^*) = y^*.$$

That is, F has a coupled fixed point in X.

As a corollary of Theorem 3.2 we have the main result of [3].

Corollary 3.6. (Theorems 2 and 3 in [3]) *Let* $(X, \preceq)$ *be a partially ordered set and suppose there is a metric* d *on* X *such that* (X, d) *is a complete metric space. Let* $F : X \times X \longrightarrow X$ *be a mapping having the mixed monotone property on* X *such that there exist two elements* $x_0, y_0 \in X$ *with*

$$x_0 \leq A(x_0, y_0) \text{ and } A(y_0, x_0) \leq y_0.$$

Suppose that there exist θ *is altering distance function and* $\psi \in \Psi$ *such that*

$$\theta(d(A(x,y), A(u,v))) \leq \frac{1}{2}\theta(max(d(x,u), d(y,v))) - \psi(max(d(x,u), d(y,v)))$$

for all $x, y, u, v \in X$ *with* $x \succeq u$ *and* $\preceq v$. *Suppose either:*
(a) A *is continuous, or*
(b) X *has the following properties:*

(*i*) *if* (x_n) *is a nondecreasing sequence that is convergent to* x *then* $x_n \preceq x$ *for all* n,
(*ii*) *if* (y_n) *is a nonincreasing sequence that is convergent to* y *then* $y_n \succeq y$ *for all* n.

then there exist $x^*, y^* \in X$ *such that*

$$x^* = A(x^*, y^*) \text{ and } A(y^*, x^*) = y^*.$$

That is, F *has a coupled fixed point in* X.

Theorem 3.7. *Adding condition (H) to the assumptions of Corollary 3.4 we obtain uniqueness of the couple fixed point of* A.

Proof. The proof is similar to the proof of Theorem 2.5. □

Theorem 3.8. *In addition to the assumptions of Corollary 3.4 suppose that* $x_0, y_0 \in X$ *are comparable. Then* $x^* = y^*$.

Proof. The proof is similar to the proof of Theorem 2.6. □

4. Existence of Solution for a Class of Nonlinear Integral Equation

In this section, we study the existence of a unique solution to a class of nonlinear integral equations, as an application of our results.

Definition 4.1. *Let Θ denote all functions $\vartheta : [0, \infty) \longrightarrow [0, \infty)$ satisfy:*
(i) ϑ is nondecreasing;
(ii) There exists $\psi \in \Psi$ such that $\vartheta(x) = x - \psi(x)$.

Consider the following integral equation:

$$\begin{aligned} x(t) = G(t, &\int_0^1 k_1(t,s)\Big(f(s,x(s)) + g(s,x(s))\Big)ds, \\ &\int_0^1 k_2(t,s)\Big(f(s,x(s)) + g(s,x(s))\Big)ds) \end{aligned} \tag{16}$$

for all $t \in [0, 1]$. We will analyze Eq.(16) under the following assumptions:
(i) $G : [0, 1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is nondecreasing in the second argument and nonincreasing in the third argument such that

$$|G(x, y, z) - G(x, y', z')| \leq M_1|y - y'| + M_2|z - z'|, \tag{17}$$

where $0 \leq M_1, M_2 \leq 1$;
(ii) $k_i : [0, 1] \times [0, 1] \longrightarrow \mathbb{R}$ $(i = 1, 2)$ are continuous, $k_1(t, s) \geq 0$ and $k_2(t, s) \leq 0$;
(iii) $f, g : [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions;
(iv) There exist $\lambda, \mu > 0$ and $\vartheta \in \Theta$ such that for all $x, y \in \mathbb{R}$ and $x \geq y$

$$0 \leq f(t, x) - f(t, y) \leq \lambda\, \vartheta(x - y), \tag{18}$$

and

$$-\mu\, \vartheta(x - y) \leq g(t, x) - g(t, y) \leq 0; \tag{19}$$

(v) There exist $\alpha, \beta \in C[0, 1]$ such that $\alpha(t) \leq \beta(t)$,

$$\begin{aligned} \alpha(t) \leq G(t, &\int_0^1 k_1(t,s)\Big(f(s,\alpha(s)) + g(s,\beta(s))\Big)ds, \\ &\int_0^1 k_2(t,s)\Big(f(s,\beta(s)) + g(s,\alpha(s))\Big)ds), \end{aligned}$$

and

$$G(\int_0^1 k_1(t,s)\Big(f(s,\beta(s))+g(s,\alpha(s))\Big)ds,$$
$$\int_0^1 k_2(t,s)\Big(f(s,\alpha(s))+g(s,\beta(s))\Big)ds) \leq \beta(t);$$

(vi) $2\max(\lambda,\mu)\|k_1-k_2\|_\infty \leq 1$ where

$$\|k_1-k_2\|_\infty = \sup\{k_1(t,s)-k_2(t,s) : t,s\in[0,1]\}.$$

Previously, we considered the space $X = C[0,1]$ of continuous function defined on $[0,1]$ with the standard metric given by

$$d(x,y) = \sup_{t\in[0,1]} |x(t)-y(t)|, \qquad for\ x,y\in C[0,1].$$

This space can also be equipped with a partial order given by

$$x,y\in C[0,1], \qquad x\leq y \Longleftrightarrow x(t)\leq y(t), \qquad \text{for any } t\in[0,1].$$

Clearly, in $X\times X$ we can consider the order given by

$$(x,y),(u,v)\in X\times X, \qquad (x,y)\leq(u,v) \Longleftrightarrow x\leq u \text{ and } y\geq v.$$

Now since for any $x,y\in X$ we have that $max(x,y), min(x,y)\in X$, condition (H) is satisfied.
Moreover, in [7] it is proved that $(C[0,1],\preceq)$ satisfies the assumption (16).
Now, we formulate our result.

Theorem 4.2. *Under assumptions* (i) $-$ (vi)*, Eq.(16) has a unique solution in* $C[0,1]$.

Proof. Define $F : X\times X \longrightarrow X$

$$F(x,y)(t) = G(t, \int_0^1 k_1(t,s)\Big(f(s,x(s))+g(s,y(s))\Big)ds,$$
$$\int_0^1 k_2(t,s)\Big(f(s,y(s))+g(s,x(s))\Big)ds)$$

for $t\in[0,1]$. By virtue of our assumptions, F is well defined (this means that for $(x,y)\in X$ then $F(x,y)\in X$). By (18), (19), G is nondecreasing in the second argument and nonincreasing in the third argument. Therefore, F has the mixed monotone property.

Now, for $x \geq u$, $y \leq v$, since F has mixed monotone property, we have

$$\begin{aligned}
d &= d(F(x,y),F(u,v)) \\
&= \sup_{t\in[0,1]} \mid F(x,y)(t) - F(u,v)(t) \mid \\
&= \sup_{t\in[0,1]} \Big| G(t, \int_0^1 k_1(t,s)\Big(f(s,x(s)) + g(s,y(s))\Big)ds, \\
&\quad \int_0^1 k_2(t,s)\Big(f(s,y(s)) + g(s,x(s))\Big)ds) \\
&- G(t, \int_0^1 k_1(t,s)\Big(f(s,u(s)) + g(s,v(s))\Big)ds, \\
&\quad \int_0^1 k_2(t,s)\Big(f(s,v(s)) + g(s,u(s))\Big)ds)\Big|.
\end{aligned}$$

So by (17), (18) and (19) we get

$$\begin{aligned}
d \leq \sup_{t\in[0,1]} \Bigg[M_1 \int_0^1 k_1(t,s) \Bigg[&\Big(f(s,x(s)) - f(s,u(s))\Big) \\
&- \Big(g(s,v(s)) - g(s,y(s))\Big)\Bigg] ds \\
-M_2 \int_0^1 k_2(t,s) \Bigg[&\Big(f(s,v(s)) - f(s,y(s))\Big) \\
&- \Big(g(s,x(s)) - g(s,u(s))\Big)\Bigg] ds \Bigg].
\end{aligned} \tag{20}$$

By our assumptions (notice that $x \geq u$ and $y \leq v$)

$$f(s,x(s)) - f(s,u(s)) \leq \lambda \ \vartheta(x(s) - u(s)),$$

$$g(s,v(s)) - g(s,y(s)) \geq -\mu \ \vartheta(v(s) - y(s)),$$

$$f(s,v(s)) - f(s,y(s)) \leq \lambda \ \vartheta(v(s) - y(s)),$$

$$g(s,x(s)) - g(s,u(s)) \geq -\mu \ \vartheta(x(s) - u(s)).$$

Taking into account these last inequalities, $k_2 \leq 0$ and (20) we get

$$\begin{aligned} d(F(x,y),F(u,v)) \leq \sup_{t\in[0,1]} \Bigg[& M_1 \int_0^1 k_1(t,s)\Big[\lambda\ \vartheta(x(s)-u(s)) \\ + & \ \mu\ \vartheta(v(s)-y(s))\Big] ds \\ + & \ M_2 \int_0^1 -k_2(t,s)\Big[\lambda\ \vartheta(v(s)-y(s)) + \mu\ \vartheta(x(s)-u(s))\Big] ds \Bigg]. \end{aligned}$$

Since $0 \leq M_1, M_2 \leq 1$, so we have

$$\begin{aligned} d(F(x,y),F(u,v)) \leq \max(\lambda,\mu) \sup_{t\in[0,1]} \Bigg[& \int_0^1 (k_1(t,s)-k_2(t,s)) \\ & \vartheta(x(s)-u(s)) + \int_0^1 (k_1(t,s)-k_2(t,s))\vartheta(y(s)-v(s)) \Bigg]. \end{aligned} \tag{21}$$

Define

$$\text{(I)} = \int_0^1 (k_1(t,s)-k_2(t,s))\ \vartheta(x(s)-u(s)),$$

$$\text{(II)} = \int_0^1 (k_1(t,s)-k_2(t,s))\ \vartheta(y(s)-v(s)).$$

Using the Cauchy-Schwartz inequality in (I) we obtain

$$\begin{aligned} (I) &\leq \left(\int_0^1 (k_1(t,s)-k_2(t,s))^2 ds\right)^{\frac{1}{2}} \left(\int_0^1 \vartheta(x(s)-u(s))^2 ds\right)^{\frac{1}{2}} \\ &\leq \ \| k_1 - k_2 \|_\infty .\vartheta(\|u-x\|) \\ &= \ \| k_1 - k_2 \|_\infty .\vartheta(d(x,u)). \end{aligned} \tag{22}$$

Similarly, we can obtain the following estimate for (II):

$$(II) \ = \ \| k_1 - k_2 \|_\infty .\vartheta(d(y,v)). \tag{23}$$

From (21)-(23) we have

$$d(F(x,y),F(u,v)) \leq \max(\lambda,\mu) \parallel k_1 - k_2 \parallel_\infty \Big[\vartheta(d(x,u))$$

$$+ \quad \vartheta(d(y,v))\Big]$$

$$\leq \quad 2\max(\lambda,\mu) \parallel k_1 - k_2 \parallel_\infty \max\Big[\vartheta(d(x,u))), \vartheta(d(y,v))\Big].$$

The last inequality and assumption (vi) give us

$$\begin{aligned} d(F(x,y),F(u,v)) &\leq \max\Big[\vartheta(d(x,u))), \vartheta(d(y,v))\Big] \\ &\leq \vartheta\Big(\max\Big[d(x,u), d(y,v)\Big]\Big). \end{aligned}$$

Hence

$$d(F(x,y),F(u,v)) \quad \leq \quad \vartheta\Big(\max\Big[d(x,u), d(y,v)\Big]\Big).$$

Similarly,

$$d(F(y,x),F(v,u)) \quad \leq \quad \vartheta\Big(\max\Big[d(x,u)), d(y,v))\Big]\Big).$$

Therefore, we have

$$\max\Big[d(F(y,x),F(v,u)), d(F(x,y),F(u,v))\Big] \leq \vartheta\Big(\max\Big[d(x,u)), d(y,v))\Big]\Big).$$

Put $\theta = x$, $\phi(x,y) = \max(x,y)$, obviously, $\phi \in \Phi$, $\psi \in \Psi$ and θ is an altering distance function. This proves that the operator F satisfies the contractive condition appearing in Corollary 3.4. Finally, let α, β be the functions appearing in assumption (v). Then, by (v), we get

$$\alpha \leq F(\alpha,\beta) \text{ and } F(\beta,\alpha) \leq \beta.$$

Theorem 3.7 gives us that F has a unique coupled fixed point $(x,y) \in X \times X$. Since $\alpha \leq \beta$, Theorem 3.8 says us that $x = y$. This implies $x = F(x,x)$ and x is the unique solution of Eq. (16). This completes the proof. $\square$

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Reza Allahyari
Department of Mathematics
Instructor of Mathematics
Mashhad Branch, Islamic Azad University
Mashhad, Iran
E-mail: rezaallahyari@mshdiau.ac.ir