Journal of Mathematical Extension
Vol. 9, No. 3, (2015), 39-56
ISSN: 1735-8299
URL: http://www.ijmex.com

On Stationary and Non-Stationary M-Band Framelet Packets

F. A. Shah
South Campus, University of Kashmir

Abstract. This paper deals with a construction of both stationary and non-stationary M-band tight framelet packets in $L^2(\mathbb{R})$ using extension principles. The approach here is different from the method described by Shah and Debnath in [Explicit construction of M-band tight framelet packets, *Analysis*, 32 (2012) 281-294] in that we directly decompose the multiresolution space V_J for a fixed level $J > 0$ to the level 0 with any combined wavelet mask $\mathbf{m} = [m_0, m_1, \ldots, m_L]$ satisfying the unitary extension principle condition $\mathcal{M}(\xi)\mathcal{M}^*(\xi) = I_M$, where $\mathcal{M}(\xi) = \left\{m_\ell\left(\xi + \frac{2\pi p}{M}\right)\right\}_{\ell,p=0}^{M-1}$.

AMS Subject Classification: 42C15; 42C40; 65T60
Keywords and Phrases: M-band wavelet, tight wavelet frame, framelet packet, extension principle, fourier transform

1. Introduction

The traditional wavelet frames provide poor frequency localization in many applications as they are not suitable for signals whose domain frequency channels are focused only on the middle frequency region. Therefore, in order to make more kinds of signals suited for analyzing by wavelet frames, it is necessary to extend the concept of wavelet frames to a library of wavelet frames, called *framelet packets* or *wavelet frame packets*. The original idea of framelet packets was introduced by Coifman

Received: February 2015; Accepted: March 2015

et al. [4] to provide more efficient decomposition of signals containing both transient and stationary components. Chui and Li [3] generalized the concept of orthogonal wavelet packets to the case of non-orthogonal wavelet packets so that they can be applied to the spline wavelets and so on. Shen [19] generalized the notion of univariate orthogonal wavelet packets to the case of multivariate orthogonal wavelets such that they may be used in a wider field. Other notable generalizations are the wavelet packets and p-framelet packets on the positive half-line $\mathbb{R}^+$ [15, 16], wavelet packets on locally compact abelian groups [18], the vector-valued wavelet packets [8], the M-band wavelet packets [10] and the tight framelet packets on $\mathbb{R}^d$ [12].

On the other hand, the standard orthogonal wavelets are not also suitable for the analysis of high-frequency signals with relatively narrow bandwidth. To overcome this shortcoming, M-band orthonormal wavelets were created as a direct generalization of the 2-band wavelets [20]. The motivation for a larger $M(M > 2)$ comes from the fact that, unlike the standard wavelet decomposition which results in a logarithmic frequency resolution, the M-band decomposition generates a mixture of logarithmic and linear frequency resolution and hence generates a more flexible tiling of the time-frequency plane than that resulting from 2-band wavelet. The other significant difference between 2-band wavelets and M-band wavelets in construction lies in the aspect that the wavelet vectors are not uniquely determined by the scaling vector and the orthonormal bases do not consist of dilated and shifted functions through a single wavelet, but consist of ones by using M-1 wavelets (see[1, 6, 11]). It is this point that brings more freedoms for optimal wavelet bases.

A tight wavelet frame is a generalization of an orthonormal wavelet basis by introducing redundancy into a wavelet system. Tight wavelet frames have some desirable features such as near translation invariant wavelet frame transforms and it may also be easier to recognize patterns in a redundant transform. A catalyst for this development is the unitary extension principle (UEP) introduced by Ron and Shen [14], which provides a general construction of tight wavelet frames for $L^2(\mathbb{R}^n)$ in the shift-invariant setting, and included the pyramidal decomposition and reconstruction filter bank algorithms. The resulting tight wavelet frames are based on a multiresolution analysis, and the generators are often called *mother framelets*. The theory of tight wavelet frames has been extensively studied and well developed over the recent years. To mention a few references on tight wavelet frames, the reader is referred to [2, 5, 7, 13] and many references therein. In the M-band setting, Han and Cheng [9] have provided the general construction of M-band tight wavelet frames on $\mathbb{R}$ by following the procedure of Daubechies et al. [5] and Petukhov [13] via extension principles.

Recently, Shah and Debnath [17] have introduced a general construction scheme for a class of stationary M-band tight framelet packets in $L^2(\mathbb{R})$ via extension principles. They proved a lemma on the so-called splitting trick and splited the wavelet spaces $W_{j,\ell}, \ell = 0, 1, \ldots, L$ by means of the framelet symbols $m_\ell(\xi), \ell = 0, 1, \ldots, L$ and then by recursive decomposition, constructed various M-band tight framelet packets in $L^2(\mathbb{R})$. In this paper, we construct both stationary and non-stationary M-band tight framelet packets in $L^2(\mathbb{R})$ by decomposing the MRA space V_J directly for a fixed level $J > 0$ to the level 0 with any combined MRA mask $\mathbf{m} = [m_0, m_1, \ldots, m_L]$ satisfying the unitary extension principle condition $\mathcal{M}(\xi)\mathcal{M}^*(\xi) = I_M$, where $\mathcal{M}(\xi) = \left\{ m_\ell \left(\xi + \frac{2\pi p}{M} \right) \right\}_{\ell,p=0}^{M-1}$.

The rest of this paper is organized as follows. In Section 2 we review some basic facts about M-band tight wavelet frames using extension principles. In Section 3 and Section 4, we prove our main results regarding the construction of stationary and non-stationary M-band tight framelet packets.

2. Preliminaries and M-Band Wavelet Frames

We begin this section by reviewing some major concepts concerning M-band wavelet frames. In the rest of this paper, we use $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}$ and $\mathbb{R}$ to denote the sets of all natural numbers, non-negative integers, integers and real numbers, respectively.

The Fourier transform of a function $f \in L^1(\mathbb{R})$ is defined as usual by:

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)\, e^{-i\xi x} dx, \quad \xi \in \mathbb{R},$$

and its inverse is

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi)\, e^{i\xi x} d\xi, \quad x \in \mathbb{R}.$$

For given $\Psi := \{\psi_1, \ldots, \psi_L\} \subset L^2(\mathbb{R})$, define the M-band wavelet system

$$X(\Psi) := \big\{\psi_{\ell,j,k} : j, k \in \mathbb{Z}, 1 \leqslant \ell \leqslant L\big\}, \tag{1}$$

where $\psi_{\ell,j,k} = M^{j/2}\psi_\ell(M^j. - k)$. The wavelet system $X(\Psi)$ is called a *M-band wavelet frame*, or simply a *M-band framelet system*, if there exist positive numbers $0 < A \leqslant B < \infty$ such that for all $f \in L^2(\mathbb{R})$

$$A\big\|f\big\|_2^2 \leqslant \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \big|\langle f, \psi_{\ell,j,k}\rangle\big|^2 \leqslant B\big\|f\big\|_2^2. \tag{2}$$

The largest A and the smallest B for which (2) holds are called wavelet frame bounds. A wavelet frame is a *tight wavelet frame* if A and B are chosen so that $A = B = 1$ and then generators $\psi_1, \psi_2, \ldots, \psi_L$ are often referred as *M-band framelets*. Moreover, if only the upper bound holds in the above inequality, then $X(\Psi)$ is said to be a *Bessel sequence* with Bessel constant B.

The construction of framelet systems often starts with the construction of MRA, which is built on refinable functions. A function $\varphi \in L^2(\mathbb{R})$ is called M-refinable if it satisfies a refinement equation:

$$\varphi(x) = \sum_{k\in\mathbb{Z}} h_0[k]\, \varphi(Mx - k), \tag{3}$$

for some $h_0 \in l^2(\mathbb{Z})$. The Fourier transform of (3) yields

$$\hat{\varphi}\,(\xi) = m_0\left(\frac{\xi}{M}\right)\, \hat{\varphi}\left(\frac{\xi}{M}\right), \tag{4}$$

where

$$m_0(\xi) = \frac{1}{M} \sum_{k \in \mathbb{Z}} h_0[k] e^{ik\xi},$$

is a 2π-periodic measurable function in $L^\infty[-\pi, \pi]$ and is often called the *refinement symbol* of φ. In this paper, we follow [1] for the definition of an M-band MRA. Given a M-refinable function $\varphi \in L^2(\mathbb{R})$ with $\hat{\varphi}(0) \neq 0$, the sequence of subspaces $\{V_j : j \in \mathbb{Z}\}$ defined by

$$V_j = \overline{\operatorname{span}} \left\{ \varphi\big(M^j x - k\big) : k \in \mathbb{Z} \right\}, \quad j \in \mathbb{Z}, \tag{5}$$

will form an MRA for $L^2(\mathbb{R})$. Recall that $\{V_j : j \in \mathbb{Z}\}$ is called an MRA if it satisfies (i) $V_j \subset V_{j+1}$ for every $j \in \mathbb{Z}$; (ii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ and (iii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$. In this paper, we only consider the refinable function $\varphi \in L^2(\mathbb{R})$ satisfying the following properties:

$$\lim_{\xi \to 0} \hat{\varphi}(\xi) = 1, \quad \xi \in \mathbb{R}; \tag{6}$$

and

$$\sum_{k \in \mathbb{Z}} |\hat{\varphi}(\xi + 2k\pi)|^2 \in L^\infty[-\pi, \pi]. \tag{7}$$

Given an MRA generated by the refinable function φ, one can construct (see [5]) a set of MRA-based framelets $\Psi := \{\psi_1, \ldots, \psi_L\} \subset V_1$ which is defined by

$$\hat{\psi}_\ell\,(\xi) = m_\ell \left(\frac{\xi}{M}\right) \hat{\varphi} \left(\frac{\xi}{M}\right), \tag{8}$$

where

$$m_\ell(\xi) = \frac{1}{M} \sum_{k \in \mathbb{Z}} h_\ell[k] e^{ik\xi}, \quad \ell = 1, \ldots, L$$

are the 2π-periodic measurable functions in $L^\infty[-\pi, \pi]$ and are called the *framelet symbols* or *wavelet masks*. The so-called *unitary extension principle* (UEP) provides a sufficient condition on Ψ such that the resulting M-band system $X(\Psi)$ forms a tight frame of $L^2(\mathbb{R})$. In this connection, an explicit construction scheme is provided in [9] for the construction of M-band tight framelets on $\mathbb{R}$.

Theorem 2.1. *Suppose that the refinable function φ and the framelet symbols $m_0, m_1, \ldots, m_L$ satisfy (4)-(7). Define $\psi_1, \ldots, \psi_L$ by (8). Let $\mathcal{M}(\xi) = \{m_\ell\,(\xi + 2\pi p/M)\}_{\ell,p=0}^{M-1}$ such that $\mathcal{M}(\xi)\mathcal{M}^*(\xi) = I_M$, for a.e $\xi \in \sigma(V_0) := \left\{\xi \in [-\pi, \pi] : \sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + 2k\pi)|^2 \neq 0\right\}$, then M-band wavelet system $X(\Psi)$ forms a tight wavelet frame for $L^2(\mathbb{R})$ with frame bound 1.*

In order to prove the main results to be presented in next sections, we need the following lemma (see [17]) which plays a key role in the construction of M-band tight framelet packets.

Lemma 2.2. *Let $g \in L^2(\mathbb{R})$ and $\{g_{j,k} : k \in \mathbb{Z}\}$ be a Bessel sequence in $L^2(\mathbb{R})$ i.e.,*

$$\sum_{k\in\mathbb{Z}} \left|\hat{g}(\xi + 2k\pi)\right|^2 \leqslant B, \qquad \xi \in \mathbb{R}$$

for any fixed $j \in \mathbb{Z}$. Let $m_\ell(\xi), \ell = 0, 1, \ldots, L$ be the framelet symbols associated with the refinable function φ and the tight framelets $\psi_\ell, \ell = 1, \ldots, L$ such that they satisfy the UEP condition $\mathcal{M}(\xi)\mathcal{M}^*(\xi) = I_M$. Suppose

$$g_\ell(x) = M \sum_{n\in\mathbb{Z}} m_\ell(n)\, g(Mx - n),$$

$$S_\ell = \overline{span}\left\{g^\ell_{j-1,k} : k \in \mathbb{Z}\right\},$$

and $S = \overline{span}\,\{g_{j,k} : k \in \mathbb{Z}\}$, for $\ell = 0, 1, \ldots, L$. Then
(i). For $\ell = 0, 1, \ldots, L$, each set $\left\{g^\ell_{j-1,k} : k \in \mathbb{Z}\right\}$ forms a Bessel sequence with $\|g_\ell\|_2^2 \leqslant B$ and $\|g\|_2^2 \leqslant B$.
(ii). For any sequence $d \in l^2(\mathbb{Z})$, there exists $L + 1$ sequences $\{d_\ell\}_{\ell=0}^L$, defined by

$$d_\ell(k) = \sqrt{M} \sum_{n\in\mathbb{Z}} \overline{m}_\ell\big(n - Mk\big)\, d(n), \quad k \in \mathbb{Z} \tag{9}$$

such that

$$\|d\|^2_{l^2(\mathbb{Z})} = \sum_{\ell=0}^{L} \|d_\ell\|^2, \tag{10}$$

and

$$\sum_{k\in\mathbb{Z}} d(k)\, g_{j,k} = \sum_{\ell=0}^{L}\sum_{k\in\mathbb{Z}} d_\ell(k)\;\, g^\ell_{j-1,k}. \tag{11}$$

(iii). In particular for any $f \in L^2(\mathbb{R})$, *let* $d(k) = \langle f, g_{j,k}\rangle,\; k \in \mathbb{Z}$, *then* $d \in l^2(\mathbb{Z})$ *and (9)-(11) gives*

$$d_\ell(k) = \left\langle f, g^\ell_{j-1,k}\right\rangle, \quad k\in\mathbb{Z}, \ell = 0,1,\ldots,L,$$

$$\sum_{k\in\mathbb{Z}} |\langle f, g_{j,k}\rangle|^2 = \sum_{\ell=0}^{L}\sum_{k\in\mathbb{Z}} \left|\left\langle f, g^\ell_{j-1,k}\right\rangle\right|^2,$$

and

$$\sum_{k\in\mathbb{Z}} \langle f, g_{j,k}\rangle g_{j,k} = \sum_{\ell=0}^{L}\sum_{k\in\mathbb{Z}} \left\langle f, g^\ell_{j-1,k}\right\rangle g^\ell_{j-1,k},$$

respectively.
(iv). S has the decomposition

$$S = S_0 + S_1 + \cdots + S_L.$$

By virtue of the Lemma 2.2, Shah and Debnath [17] have constructed various *stationary tight* M*-band framelet packets* on $\mathbb{R}$ by the recursive decomposition of wavelet spaces $W_{j,\ell}, \ell = 0,1,\ldots,L, j \in \mathbb{Z}$. For $n = 0,1,2,\ldots$, the basic M-band framelet packets associated with the refinable function φ are defined as

$$\hat\omega_n(\xi) = \hat\omega_{(L+1)r+\ell}(\xi) = m_\ell\left(\frac{\xi}{M}\right)\hat\omega_r\left(\frac{\xi}{M}\right), \tag{12}$$

where $\ell = 0,1,\ldots,L,\; r = 0,1,2,\ldots$.

3. Stationary M-Band Tight Framelet Packets

Besides the recursive derivation of stationary tight M-band framelet packets introduced in [17], stationary tight M-band framelet packets can also be constructed by decomposing the MRA space V_J directly for a fixed level $J > 0$ to the level 0.

To do so, let $X(\Psi)$ be the M-band tight wavelet frame for $L^2(\mathbb{R})$ constructed via UEP in an MRA $\{V_j : j \in \mathbb{Z}\}$ generated by the M-refinable function φ with combined UEP mask $\mathbf{h} = [h_0, h_1, \ldots, h_L]$. Then, for each $j \in \mathbb{Z}$, we define

$$V_j = \overline{\text{span}}\left\{\varphi_{j,k} : k \in \mathbb{Z}\right\}, \ \ \textit{and} \ \ W_{j,\ell} = \overline{\text{span}}\left\{\psi_{\ell,j,k} : k \in \mathbb{Z}\right\}, \ \ell = 0, 1, \ldots, L.$$

Therefore, in view of tight frame decomposition, we have

$$V_j = V_{j-1} + \sum_{\ell=1}^{L} W_{j-1,\ell}. \tag{13}$$

It is immediate from the above decomposition that these $L+1$ spaces are in general not orthogonal. Therefore, by the repeated applications of (13), we can further split the V_j spaces as:

$$V_j = V_{j-1} + \sum_{\ell=1}^{L} W_{j-1,\ell} = V_{j-2} + \sum_{r=j-2}^{j-1} \sum_{\ell=1}^{L} W_{r,\ell} = \cdots = V_{j_0} + \sum_{r=j_0}^{j-1} \sum_{\ell=1}^{L} W_{r,\ell}$$

$$= \sum_{r=-\infty}^{j-1} \sum_{\ell=1}^{L} W_{r,\ell}. \tag{14}$$

Now, at the first level of decomposition, by Lemma 2.2, V_J is decomposed into the $L+1$ spaces $W_{J-1,\mathbf{r}}, \mathbf{r} \in \Lambda_1$ where

$$\Lambda_1 = \Big\{\mathbf{r} = (r_J, r_{J-1}, \ldots, r_1) : 0 \leqslant r_J \leqslant L,\, r_{J-1} = \cdots = r_1 = 0\Big\}.$$

For this choice of $\mathbf{r} = (r_J, r_{J-1}, \ldots, r_1)$, we define

$$\mathbf{r}(n) = r_n, \quad n = 1, 2, \ldots, J,$$

$$\omega_{\mathbf{r}}(x) = M \sum_{n \in \mathbb{Z}} h_{\mathbf{r}(1)}[n]\, \varphi(Mx - n),$$

and

$$W_{J-1,\mathbf{r}} := \overline{\text{span}}\left\{\omega_{\mathbf{r},J-1,k} : k \in \mathbb{Z}\right\}.$$

Therefore, for any $f \in L^2(\mathbb{R})$, we have

$$\sum_{k\in\mathbb{Z}} |\langle f, \varphi_{J,k}\rangle|^2 = \sum_{\mathbf{r}\in\Lambda_1}\sum_{k\in\mathbb{Z}} |\langle f, \omega_{\mathbf{r},J-1,k}\rangle|^2.$$

At the second level of decomposition, by Lemma 2.2, each space $W_{J-1,\mathbf{r}}$, $\mathbf{r} \in \Lambda_1$ is decomposed with $\mathbf{h}$ into spaces $W_{J-2,\mathbf{r}'}$, $\mathbf{r}' \in \Lambda_2^{\mathbf{r}}$, where $\Lambda_2^{\mathbf{r}}$ is a subset of Λ_2 defined by

$$\Lambda_2^{\mathbf{r}} = \left\{\mathbf{r}' \in \Lambda_2 : \mathbf{r}'(1) = \mathbf{r}(1)\right\},$$

and Λ_2 is a J-tuple index set defined by

$$\Lambda_2 = \left\{\mathbf{r} = (r_J, r_{J-1}, \ldots, r_1) : 0 \leqslant r_{J-1}, r_J \leqslant L,\ r_{J-2} = \cdots = r_1 = 0\right\},$$

$$\omega_{\mathbf{r}'}(x) = M\sum_{n\in\mathbb{Z}} h_{\mathbf{r}'(2)}[n]\,\omega_{\mathbf{r}}(Mx-n),$$

$$W_{J-2,\mathbf{r}'} := \overline{\operatorname{span}}\left\{\omega_{\mathbf{r}',J-2,k} : k \in \mathbb{Z}\right\}.$$

Thus, for any $f \in L^2(\mathbb{R})$, we have

$$\sum_{k\in\mathbb{Z}} |\langle f, \omega_{\mathbf{r},J-1,k}\rangle|^2 = \sum_{\mathbf{r}'\in\Lambda_2^{\mathbf{r}}}\sum_{k\in\mathbb{Z}} |\langle f, \omega_{\mathbf{r}',J-2,k}\rangle|^2.$$

Finally, at the p-th level ($2 \leqslant p \leqslant J$) of decomposition, by Lemma 2.2, each space $W_{J-p+1,\mathbf{r}}$, $\mathbf{r} \in \Lambda_{p-1}$ is decomposed with $\mathbf{h}$ into spaces $W_{J-p,\mathbf{r}'}$, $\mathbf{r}' \in \Lambda_p^{\mathbf{r}}$, where $\Lambda_p^{\mathbf{r}}$ is a subset of Λ_p defined by

$$\Lambda_p^{\mathbf{r}} = \left\{\mathbf{r}' \in \Lambda_p : \mathbf{r}'(n) = \mathbf{r}(n), \quad \text{for} \ \ 1 \leqslant n \leqslant p-1\right\}, \tag{15}$$

and Λ_p is a J-tuple index set defined by

$$\Lambda_p = \left\{\mathbf{r} = (r_J, r_{J-1}, \ldots, r_1) : 0 \leqslant r_{J-p} \leqslant L,\ r_{J-p} = \cdots = r_1 = 0\right\},$$

$$\omega_{\mathbf{r}'}(x) = M\sum_{n\in\mathbb{Z}} h_{\mathbf{r}'(p)}[n]\,\omega_{\mathbf{r}}(Mx-n), \tag{16}$$

$$W_{J-p,\mathbf{r}'} := \overline{\text{span}}\left\{\omega_{\mathbf{r}',J-p,k} : k \in \mathbb{Z}\right\}.$$

Therefore for any $f \in L^2(\mathbb{R})$, we have

$$\sum_{k\in\mathbb{Z}} |\langle f, \omega_{\mathbf{r},J-p+1,k}\rangle|^2 = \sum_{\mathbf{r}'\in\Lambda_p^{\mathbf{r}}} \sum_{k\in\mathbb{Z}} |\langle f, \omega_{\mathbf{r}',J-p,k}\rangle|^2.$$

In particular, at the J-th level of decomposition, by Lemma 2.2, each space $W_{1,\mathbf{r}}$, $\mathbf{r} \in \Lambda_{J-1}$ is decomposed with $\mathbf{h}$ into spaces $W_{0,\mathbf{r}'}$, $\mathbf{r}' \in \Lambda_J^{\mathbf{r}}$, where $\Lambda_J^{\mathbf{r}}$ is a subset of Λ_J defined by

$$\Lambda_J^{\mathbf{r}} = \left\{\mathbf{r}' \in \Lambda_J : \mathbf{r}'(n) = \mathbf{r}(n), \quad \text{for} \quad 1 \leqslant n \leqslant J-1\right\},$$

and Λ_J is a J-tuple index set defined by

$$\Lambda_J = \left\{\mathbf{r} = (r_J, r_{J-1}, \ldots, r_1) : 0 \leqslant r_t \leqslant L,\, 1 \leqslant t \leqslant J\right\}, \tag{17}$$

$$\omega_{\mathbf{r}'}(x) = M \sum_{n\in\mathbb{Z}} h_{\mathbf{r}'(J)}[n]\, \omega_{\mathbf{r}}(Mx - n),$$

$$W_{0,\mathbf{r}'} := \overline{\text{span}}\left\{\omega_{\mathbf{r}',0,k} : k \in \mathbb{Z}\right\}.$$

Thus for any $f \in L^2(\mathbb{R})$, we have

$$\sum_{k\in\mathbb{Z}} |\langle f, \omega_{\mathbf{r},1,k}\rangle|^2 = \sum_{\mathbf{r}'\in\Lambda_J^{\mathbf{r}}} \sum_{k\in\mathbb{Z}} |\langle f, \omega_{\mathbf{r}',0,k}\rangle|^2.$$

Combining all the inner product equations in the above construction, we get

$$\sum_{k\in\mathbb{Z}} |\langle f, \varphi_{J,k}\rangle|^2 = \sum_{\mathbf{r}\in\Lambda_J} \sum_{k\in\mathbb{Z}} |\langle f, \omega_{\mathbf{r},0,k}\rangle|^2, \quad \text{for any} \quad f \in L^2(\mathbb{R}). \tag{18}$$

In other words, we obtain another representation of V_J as

$$V_J := \overline{\text{span}}\left\{\omega_{\mathbf{r},0,k} : \mathbf{r} \in \Lambda_J, k \in \mathbb{Z}\right\}.$$

Theorem 3.1. (See [2]) *Suppose $X(\Psi)$ is a M-band tight wavelet frame constructed via UEP in an MRA and $\mathbf{h} = [h_0, h_1, \ldots, h_L]$ is the combined mask satisfying the UEP condition $\mathcal{M}(\xi)\mathcal{M}^*(\xi) = I_M$. Then for any fixed $J > 0$, the family of functions*

$$\mathcal{F} = \{\omega_{\mathbf{r},0,k} : \mathbf{r} \in \Lambda_J\} \cup \{\psi_{\ell,j,k} : \ell = 1, \ldots, L, j \geqslant J, k \in \mathbb{Z}\},$$

forms a tight frame for $L^2(\mathbb{R})$, where Λ_J is a index set defined in (17).

Proof. Since $X(\Psi)$ is a tight wavelet frame of $L^2(\mathbb{R})$, then by (18), we have

$$\begin{aligned}\|f\|_2^2 &= \sum_{k\in\mathbb{Z}} |\langle f, \varphi_{J,k}\rangle|^2 + \sum_{\ell=1}^{L}\sum_{j\geqslant J}\sum_{k\in\mathbb{Z}} |\langle f, \psi_{\ell,j,k}\rangle|^2 \\ &= \sum_{\mathbf{r}\in\Lambda_J}\sum_{k\in\mathbb{Z}} |\langle f, \omega_{\mathbf{r},0,k}\rangle|^2 + \sum_{\ell=1}^{L}\sum_{j\geqslant J}\sum_{k\in\mathbb{Z}} |\langle f, \psi_{\ell,j,k}\rangle|^2,\end{aligned}$$

for any $f \in L^2(\mathbb{R})$. □

Similar to the recursive construction of stationary tight M-band framelet packets (see [17]), we can obtain a stationary tight M-band framelet packets by performing various disjoint partitions Γ_J of Λ_J with each partition separating Λ_J into disjoint subsets of the form

$$I_{j,\mathbf{r}} = \{(r_J, \ldots, r_{j+1}, r'_j, \ldots, r'_1) \in \Lambda_J : \mathbf{r} = (r_J, \ldots, r_{j+1}, 0, \ldots, 0) \in \Lambda_{J-j}\},$$

$$i.e.,\ \Gamma_J = \Big\{I_{j,\mathbf{r}} : \bigcup I_{j,\mathbf{r}} = \Lambda_J\Big\}. \quad \square \tag{19}$$

Theorem 3.2. (See [2]) *Suppose $X(\Psi)$ is a M-band tight wavelet frame constructed via UEP in an MRA and $\mathbf{h} = [h_0, h_1, \ldots, h_L]$ is the combined mask satisfying the UEP condition $\mathcal{M}(\xi)\mathcal{M}^*(\xi) = I_M$. Let Γ_J be a disjoint partition of Λ_J, where Λ_J and Γ_J are defined in (17) and (19), respectively. Then the collection*

$$\mathcal{F}_{\Gamma_J} = \{\omega_{\mathbf{r},j,k} : I_{j,\mathbf{r}} \in \Gamma_J, k \in \mathbb{Z}\} \cup \{\psi_{\ell,j,k} : \ell = 1, \ldots, L, j \geqslant J, k \in \mathbb{Z}\},$$

generates a tight frame for $L^2(\mathbb{R})$.

Proof. Since Γ_J is a disjoint partition of Λ_J, for any $f \in L^2(\mathbb{R})$, we have

$$\begin{aligned}\sum_{I_{j,\mathbf{r}}\in\Gamma_J}\sum_{k\in\mathbb{Z}}\left|\langle f,\omega_{\mathbf{r},j,k}\rangle\right|^2 &= \sum_{I_{j,\mathbf{r}}\in\Gamma_J}\sum_{\mathbf{r}'\in I_{j,\mathbf{r}}}\sum_{k\in\mathbb{Z}}\left|\langle f,\omega_{\mathbf{r}',0,k}\rangle\right|^2\\ &= \sum_{\mathbf{r}\in\Lambda_J}\sum_{k\in\mathbb{Z}}\left|\langle f,\omega_{\mathbf{r},0,k}\rangle\right|^2.\end{aligned}$$

By applying Theorem 3.1, Theorem 3.2 is proved. □

4. Non-Stationary M-Band Tight Framelet Packets

In this section, we construct the M-band tight framelet packets on $\mathbb{R}$ by recursively decomposing V_J with arbitrarily chosen combined UEP masks to the coarsest scale 0. However, in this case we may change the underlying MRA spaces $\{V_j : j \in \mathbb{Z}\}$ associated with $X(\Psi)$ if one of the low-pass filters in the set of combined UEP masks decomposing V_j does not coincide with the refinement mask of φ which generates MRA and all the tight M-band framelet packets obtained in this way will be called *non-stationary tight M-band framelet packets.*

To do so, let $X(\Psi)$ be the given M-band tight wavelet frame for $L^2(\mathbb{R})$ constructed via UEP in an MRA $\{V_j\}_{j\in\mathbb{R}}$ generated by the M-refinable function φ. Firstly, we decompose $V_J := \overline{\text{span}}\,\{\varphi_{J,k} : k \in \mathbb{Z}\}$ associated with the combined mask $\mathbf{m}_J = [m_{\mathbf{r}} : \mathbf{r} \in \Lambda_1]$ satisfying the UEP condition $\mathcal{M}(\xi)\mathcal{M}^*(\xi) = I_M$, where Λ_1 is a J-tuple index set defined by

$$\Lambda_1 = \big\{(r_J, r_{J-1}, \dots, r_1) : 0 \leqslant r_J \leqslant \mathcal{J},\, r_{J-1} = \cdots = r_1 = 0\big\},$$

in which $\mathcal{J}$ is a positive constant. By invoking Lemma 2.2, we can decompose V_J into spaces $W_{J-1,\mathbf{r}}$, $\mathbf{r} \in \Lambda_1$, where

$$\omega_{\mathbf{r}}(x) = M\sum_{n\in\mathbb{Z}} m_{\mathbf{r}}[n]\,\varphi(Mx-n),$$

$$W_{J-1,\mathbf{r}} := \overline{\text{span}}\big\{\omega_{\mathbf{r},J-1,k} : k \in \mathbb{Z}\big\}.$$

Therefore for any $f \in L^2(\mathbb{R})$, we have

$$\sum_{k\in\mathbb{Z}} \left|\langle f, \varphi_{J,k}\rangle\right|^2 = \sum_{\mathbf{r}\in\Lambda_1}\sum_{k\in\mathbb{Z}} \left|\langle f, \omega_{\mathbf{r},J-1,k}\rangle\right|^2.$$

At the second level of decomposition, by Lemma 2.2, each space $W_{J-1,\mathbf{r}}$, $\mathbf{r} \in \Lambda_1$ is decomposed with a combined UEP mask $\mathbf{m}_{J-1,\mathbf{r}} = [m_{\mathbf{r}'} : \mathbf{r}' \in \Lambda_2^{\mathbf{r}}]$ satisfying the UEP condition $\mathcal{M}(\xi)\mathcal{M}^*(\xi) = I_M$,, where Ł$ambda_2^{\mathbf{r}}$ is a subset of Λ_2 defined by

$$\Lambda_2^{\mathbf{r}} = \left\{\mathbf{r}' \in \Lambda_2 : \mathbf{r}'(1) = \mathbf{r}(1)\right\},$$

and Λ_2 is a J-tuple index set defined by

$$\Lambda_2 = \left\{(r_J, r_{J-1}, \ldots, r_1) : 0 \leqslant r_J \leqslant \mathcal{J},\, 0 \leqslant r_{J-1} \leqslant \mathcal{J}^{(r_J)},\, r_{J-2} = \cdots = r_1 = 0\right\},$$

in which $\mathcal{J}^{(r_J)}$ is a positive constant for each (r_J) into spaces $W_{J-2,\mathbf{r}'}$, $\mathbf{r}' \in \Lambda_2^{\mathbf{r}}$, where

$$\omega_{\mathbf{r}'}(x) = M \sum_{n\in\mathbb{Z}} m_{\mathbf{r}'}[n]\, \omega_{\mathbf{r}}(Mx - n),$$

$$W_{J-2,\mathbf{r}'} := \overline{\mathrm{span}}\left\{\omega_{\mathbf{r}',J-2,k} : k \in \mathbb{Z}\right\}.$$

Thus, for any $f \in L^2(\mathbb{R})$, we have

$$\sum_{k\in\mathbb{Z}} \left|\langle f, \omega_{\mathbf{r},J-1,k}\rangle\right|^2 = \sum_{\mathbf{r}'\in\Lambda_2^{\mathbf{r}}}\sum_{k\in\mathbb{Z}} \left|\langle f, \omega_{\mathbf{r}',J-2,k}\rangle\right|^2.$$

Generally, at the p-th level $(2 \leqslant p \leqslant J)$ of decomposition, by Lemma 2.2, each space $W_{J-p+1,\mathbf{r}}$, $\mathbf{r} \in \Lambda_{p-1}$ is decomposed with a combined UEP mask $\mathbf{m}_{J-p+1,\mathbf{r}} = [m_{\mathbf{r}'} : \mathbf{r}' \in \Lambda_p^{\mathbf{r}}]$ satisfying the UEP condition $\mathcal{M}(\xi)\mathcal{M}^*(\xi) = I_M$,, where $\Lambda_p^{\mathbf{r}}$ is a subset of Λ_p defined by

$$\Lambda_p^{\mathbf{r}} = \left\{\mathbf{r}' \in \Lambda_p : \mathbf{r}'(n) = \mathbf{r}(n), \quad \text{for} \quad 1 \leqslant n \leqslant p-1\right\} \tag{20}$$

and Λ_p is a J-tuple index set defined by

$$\Lambda_p = \Big\{(r_J, r_{J-1}, \ldots, r_1) : 0 \leqslant r_J \leqslant \mathcal{J},\, 0 \leqslant r_{J-t} \leqslant \mathcal{J}^{(r_J, r_{J-1}, \ldots, r_{J-t+1})},$$
$$1 \leqslant t \leqslant p, r_{J-p} = \cdots = r_1 = 0\Big\},$$

in which $\mathcal{J}^{(r_J, r_{J-1}, \dots, r_{J-t+1})}$ is a positive constant for each $(r_J, r_{J-1}, \dots, r_{J-t+1})$ into spaces $W_{J-p,\mathbf{r}'}$, $\mathbf{r}' \in \Lambda_2^{\mathbf{r}}$, where

$$\omega_{\mathbf{r}'}(x) = M \sum_{n\in\mathbb{Z}} m_{\mathbf{r}'}[n]\, \omega_{\mathbf{r}}(Mx-n),$$

$$W_{J-p,\mathbf{r}'} := \overline{\operatorname{span}}\big\{\omega_{\mathbf{r}',J-p,k} : k \in \mathbb{Z}\big\}.$$

Hence, for any $f \in L^2(\mathbb{R})$, we have

$$\sum_{k\in\mathbb{Z}} \big|\langle f, \omega_{\mathbf{r},J-p+1,k}\rangle\big|^2 = \sum_{\mathbf{r}'\in\Lambda_p^{\mathbf{r}}} \sum_{k\in\mathbb{Z}} \big|\langle f, \omega_{\mathbf{r}',J-p,k}\rangle\big|^2.$$

In particular, at the J-th level of decomposition, by Lemma 2.2, each space $W_{1,\mathbf{r}}$, $\mathbf{r} \in \Lambda_{J-1}$ is decomposed with a combined UEP mask $\mathbf{m}_{1,\mathbf{r}} = [m_{\mathbf{r}'} : \mathbf{r}' \in \Lambda_J^{\mathbf{r}}]$ satisfying the UEP condition $\mathcal{M}(\xi)\mathcal{M}^*(\xi) = I_M$,, where $\Lambda_J^{\mathbf{r}}$ is a subset of Λ_J defined by

$$\Lambda_J^{\mathbf{r}} = \big\{\mathbf{r}' \in \Lambda_J : \mathbf{r}'(n) = \mathbf{r}(n), \quad \text{for} \quad 1 \leqslant n \leqslant J-1\big\},$$

and Λ_J is a J-tuple index set defined by

$$\begin{aligned} \Lambda_J = \big\{&(r_J, r_{J-1}, \dots, r_1) : 0 \leqslant r_J \leqslant \mathcal{J}, \\ &0 \leqslant r_{J-t} \leqslant \mathcal{J}^{(r_J, r_{J-1}, \dots, r_{J-t+1})}, 1 \leqslant t \leqslant J\big\}, \end{aligned} \tag{21}$$

in which $\mathcal{J}^{(r_J, r_{J-1}, \dots, r_{J-t+1})}$ is a positive constant for each

$$(r_J, r_{J-1}, \dots, r_{J-t+1}),$$

into spaces $W_{0,\mathbf{r}'}$, $\mathbf{r}' \in \Lambda_J^{\mathbf{r}}$, where

$$\omega_{\mathbf{r}'}(x) = M \sum_{n\in\mathbb{Z}} m_{\mathbf{r}'}[n]\, \omega_{\mathbf{r}}(Mx-n),$$

$$W_{0,\mathbf{r}'} := \overline{\operatorname{span}}\big\{\omega_{\mathbf{r}',0,k} : k \in \mathbb{Z}\big\}.$$

Therefore, for any $f \in L^2(\mathbb{R})$, we have

$$\sum_{k\in\mathbb{Z}} \big|\langle f, \omega_{\mathbf{r},1,k}\rangle\big|^2 = \sum_{\mathbf{r}'\in\Lambda_J^{\mathbf{r}}} \sum_{k\in\mathbb{Z}} \big|\langle f, \omega_{\mathbf{r}',0,k}\rangle\big|^2.$$

Combining all the inner product equations in the above construction, we obtain

$$\sum_{k\in\mathbb{Z}} \left|\langle f, \varphi_{J,k}\rangle\right|^2 = \sum_{\mathbf{r}\in\Lambda_J}\sum_{k\in\mathbb{Z}} \left|\langle f, \omega_{\mathbf{r},0,k}\rangle\right|^2, \quad \text{for any} \quad f \in L^2(\mathbb{R}). \tag{22}$$

In other words, we obtain another representation of V_J as

$$V_J := \overline{\text{span}}\left\{\omega_{\mathbf{r},0,k} : \mathbf{r} \in \Lambda_J, k \in \mathbb{Z}\right\}.$$

Theorem 4.1. (See [2]) *For a given M-band tight wavelet frame $X(\Psi)$, the system*

$$\mathcal{F}_{\mathcal{N}} = \left\{\omega_{\mathbf{r},0,k} : \mathbf{r} \in \Lambda_J\right\} \cup \left\{\psi_{\ell,j,k} : \ell = 1, \ldots, L, j \geqslant J, k \in \mathbb{Z}\right\},$$

is also a tight wavelet frame for $L^2(\mathbb{R})$, where Λ_J is an index set defined in (4.2).

Proof. Using (4.3) and the fact that $X(\Psi)$ is a tight wavelet frame for $L^2(\mathbb{R})$, we have

$$\begin{aligned} \|f\|_2^2 &= \sum_{k\in\mathbb{Z}} \left|\langle f, \varphi_{J,k}\rangle\right|^2 + \sum_{\ell=1}^{L}\sum_{j\geqslant J}\sum_{k\in\mathbb{Z}} \left|\left\langle f, \psi_{\ell,j,k}\right\rangle\right|^2 \\ &= \sum_{\mathbf{r}\in\Lambda_J}\sum_{k\in\mathbb{Z}} \left|\langle f, \omega_{\mathbf{r},0,k}\rangle\right|^2 + \sum_{\ell=1}^{L}\sum_{j\geqslant J}\sum_{k\in\mathbb{Z}} \left|\left\langle f, \psi_{\ell,j,k}\right\rangle\right|^2, \end{aligned}$$

for any $f \in L^2(\mathbb{R})$. This completes the proof of Theorem 4.1.
As in the stationary case constructed above, we can obtain a library of tight M-band framelet packets of $L^2(\mathbb{R})$ by partitioning Λ_J into disjoint subsets of the form

$I_{j,\mathbf{r}} = \left\{(r_J, \ldots, r_{j+1}, r'_j, \ldots, r'_1) \in \Lambda_J : \mathbf{r} = (r_J, \ldots, r_{j+1}, 0, \ldots, 0) \in \Lambda_{J-j}\right\}$, *i.e.*,

$$\Gamma_J = \left\{I_{j,\mathbf{r}} : \bigcup I_{j,\mathbf{r}} = \Lambda_J\right\}. \quad \square \tag{23}$$

Theorem 4.2. (See [2]) *For a given M-band tight wavelet frame $X(\Psi)$, let Γ_J be a disjoint partition Λ_J, where Λ_J and Γ_J are defined in (21) and (23), respectively. Then the system*

$$\mathcal{F}_{\mathcal{N}\Gamma_J} = \big\{\omega_{\mathbf{r},j,k} : I_{j,\mathbf{r}} \in \Gamma_J, k \in \mathbb{Z}\big\} \cup \big\{\psi_{\ell,j,k} : \ell = 1, \ldots, L, j \geqslant J, k \in \mathbb{Z}\big\},$$

also generates a tight frame for $L^2(\mathbb{R})$.

Proof. Since Γ_J is a disjoint partition of Λ_J, for any $f \in L^2(\mathbb{R})$, we have

$$\begin{aligned} \sum_{I_{j,\mathbf{r}} \in \Gamma_J} \sum_{k \in \mathbb{Z}} \big|\langle f, \omega_{\mathbf{r},j,k}\rangle\big|^2 &= \sum_{I_{j,\mathbf{r}} \in \Gamma_J} \sum_{\mathbf{r}' \in I_{j,\mathbf{r}}} \sum_{k \in \mathbb{Z}} \big|\langle f, \omega_{\mathbf{r}',0,k}\rangle\big|^2 \\ &= \sum_{\mathbf{r} \in \Lambda_J} \sum_{k \in \mathbb{Z}} \big|\langle f, \omega_{\mathbf{r},0,k}\rangle\big|^2. \end{aligned}$$

By applying Theorem 4.1, Theorem 4.2 is proved. □

References

[1] N. Bi, X. Dai, and Q. Sun, Construction of compactly supported M-band wavelets, *Appl. Comput. Harmonic Anal.*, 6 (1999), 113-131.

[2] A. Chai and Z. Shen, Deconvolution a wavelet frame approach, *Numer. Math.*, 106 (2007), 529-587.

[3] C. K. Chui and C. Li, Non-orthogonal wavelet packets, *SIAM J. Math. Anal.*, 24 (1993), 712-738.

[4] R. R. Coifman, Y. Meyer, S. Quake, and M. V. Wickerhauser, Signal processing and compression with wavelet packets, *Technical Report, Yale University*, 1990.

[5] I. Daubechies, B. Han, A. Ron, and Z. Shen, Framelets MRA-based constructions of wavelet frames, *Appl. Comput. Harmonic Anal.*, 14 (2003), 1-46.

[6] L. Debnath and F. A. Shah, *Wavelet Transforms and Their Applications*, Birkhäuser, Boston, 2015.

[7] B. Han, Wavelets and framelets within the framework of non-homogeneous wavelet systems, In *Approximation Theory XIII.* Neamtu, M., Schumaker, L., Eds.; Springer, 2012., pp. 121-161.

[8] J. C. Han and Q. J. Chen, Characterizations of a class of orthogonal multiple vector-valued wavelet packets, *Int. J. Wavelets, Multiresolut. Inf. Process.*, 6 (4) (2008), 631-641.

[9] J. C. Han and Z. Cheng, The construction of M-band tight wavelet frames, In *Proceedings of the Third International Conference on Machine Learning and Cybemetics.* Shanghai. 2004, pp. 26-29.

[10] Z. Jiankang, B. Zheng, and J. Licheng, Theory of orthonormal M-band wavelet packets, *J. Elect.*, 15 (3) (1998), 193-198.

[11] T. Lin, S. Xu, Q. Shi, and P. Hao, An algebraic construction of orthonormal M-band wavelets with perfect reconstruction, *Appl. Comput. Harmonic Anal.*, 172 (2006), 717-730.

[12] D. Lu and Q. Fan, A class of tight framelet packets, *Czechoslovak Math. J.,* 61 (2011), 623-639.

[13] A. P. Petukhov, Explicit construction of framelets, *Appl. Comput. Harmonic Anal.,* 11 (2001), 313-327.

[14] A. Ron and Z. Shen, Affine systems in $L^2(\mathbb{R}^d)$: the analysis of the analysis operator, *J. Funct. Anal.*, 148 (1997), 408-447.

[15] F. A. Shah, Construction of wavelet packets on p-adic field, *Int. J. Wavelets, Multiresolut. Inf. Process.*, 7 (5) (2009), 553-565.

[16] F. A. Shah and L. Debnath, p -Wavelet frame packets on a half-line using the Walsh-Fourier transform, *Integral Transf. Spec. Funct.*, 22 (12) (2011), 907-917.

[17] F. A. Shah and L. Debnath, Explicit construction of M-band tight framelet packets, *Analysis*, 32 (2012), 281-294.

[18] F. A. Shah and A. Wahid, Wavelet packets on locally compact Abelian groups, *An. St. Univ. Ovidius Constanta,* 18 (2010), 223-240.

[19] Z. Shen, Non-tensor product wavelet packets in $L^2(\mathbb{R}^s)$, *SIAM J. Math. Anal.*, 26 (4) (1995), 1061-1074.

[20] P. Steffen, P. N. Heller, R. A. Gopinath, and C. S. Burrus, Theory of regular M-band wavelet bases, *IEEE Trans. Sig. Proces.*, 41 (12) (1993), 3497-3510.

Firdous A. Shah
Department of Mathematics
Assistant Professor of Mathematics
University of Kashmir
South Campus, Anantnag-192101
Anantnag, India
E-mail: fashah79@gmail.com