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Describing parametric Families of Algebraic Points of Degree at most ℓ on the Hyperelliptic Curve Family $\mathcal{C}_m$ Defined by $y^2 = x^5 + m^2$

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Abstract. We give an explicit description of the set $\mathcal{C}_m^{(\ell)}$ of algebraic points of degree at most ℓ on the family of hyperelliptic curves

$$\mathcal{C}_m : y^2 = x^5 + m^2,$$

where m is a square-free integer. This family was investigated by Jeong *et al.* [21], who studied the arithmetic of its Jacobian. Throughout this paper, we assume that the Jacobian $\mathcal{J}_m(\mathbb{Q})$ has Mordell–Weil rank zero.

The family $\mathcal{C}_m$ is a special case of the more general family of superelliptic curves

$$\mathcal{C}_{q,p,a} : y^q = x^p + a,$$

where p and q are prime numbers and a is an integer, introduced by Jędrzejak [22]. Our objective is to extend the explicit description of rational points (see Proposition 1.1) to algebraic points of bounded degree.

To achieve this, we first determine an explicit $\mathbb{Q}$-basis of the Riemann–Roch spaces $\mathcal{L}(\ell\infty)$ for every $\ell \in \mathbb{N}$. Under the rank assumption above, we combine the structure of the Mordell–Weil group of the Jacobian with a fundamental version of the Abel–Jacobi theorem [15, 17] to characterize principal divisors arising from suitable rational functions. Finally, by introducing a parameter γ, we explicitly construct parametric families of algebraic points whose union coincides with the set $\mathcal{C}_m^{(\ell)}$.

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1 Introduction

1.1 Background and motivation

The study of algebraic points of bounded degree on algebraic curves has been a central theme in arithmetic geometry for several decades. Following the proof of Faltings' theorem [12], which settled Mordell's conjecture, it became clear that although the set of rational points on curves of genus $g \geq 2$ is finite, the question of effectivity and explicit determination of these sets remains highly nontrivial. Vojta later provided alternative approaches through Diophantine approximation techniques [31], while subsequent works by Hindry-Silverman [20], Bombieri–Gubler [5], and Poonen [27] have expanded the theoretical framework surrounding rational and algebraic points on curves. A related strand of research focuses on points of bounded degree, where methods based on the arithmetic of Jacobians [7], Riemann-Roch spaces [1], and Abel–Jacobi maps play a fundamental role [18, 32]. In particular, Frey [14] and Abramovich–Harris [3] studied the structure of low-degree points in connection with isogenies and torsion properties of Jacobians, providing inspiration for further explicit investigations.

1.2 Hyperelliptic curves and explicit parametrization

Within this general context, special families of algebraic curves, and in particular hyperelliptic curves, provide natural test cases in which explicit arithmetic investigations become more tractable. Hyperelliptic curves of the form $y^m = f(x)$ have been extensively investigated due to their rich geometry and their connection with Diophantine equations [2, 4, 26].

In particular, Jeong *et al.* [21] investigated the family of hyperelliptic curves $\mathcal{C}_m$ defined by the affine equation

$$y^2 = x^5 + m^2. \tag{1}$$

They investigated the arithmetic of their Jacobians, including the structure of the torsion subgroup. Moreover, they showed that there exist

infinitely many square-free integers m for which the Mordell–Weil group satisfies

$$\mathcal{J}_m(\mathbb{Q}) \cong \mathbb{Z}/5\mathbb{Z},$$

and, assuming the parity conjecture, infinitely many square-free integers m for which $\mathcal{J}_m(\mathbb{Q}) \cong \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}$. Throughout this paper, we therefore assume that $\operatorname{rank} \mathcal{J}_m(\mathbb{Q}) = 0$, so that $\mathcal{J}_m(\mathbb{Q}) = \mathcal{J}_m(\mathbb{Q})_{\mathrm{tors}} \cong \mathbb{Z}/5\mathbb{Z}$.
For a positive integer ℓ, we denote by

$$\mathcal{C}_m^{(\ell)}(\mathbb{Q}) = \left\{P \in \mathcal{C}_m(\overline{\mathbb{Q}}) : [\mathbb{Q}(P) : \mathbb{Q}] \leq \ell\right\}$$

the set of algebraic points on $\mathcal{C}_m$ of degree at most ℓ over $\mathbb{Q}$.

Proposition 1.1. *Assume that m is a square-free integer such that $\operatorname{rank} \mathcal{J}_m(\mathbb{Q}) = 0$. Then the set of rational points on the curve $\mathcal{C}_m : y^2 = x^5 + m^2$ is given by:*

$$\mathcal{C}_m^{(1)}(\mathbb{Q}) = \{(0, \pm m),\, \infty\}\,.$$

The proof of Proposition 1.1 we will give later. It relies on the explicit description of the rational points together with the assumption that

$$\operatorname{rank} \mathcal{J}_m(\mathbb{Q}) = 0,$$

which implies that the Mordell–Weil group coincides with its torsion subgroup. The family of curves $\mathcal{C}_m$ is a special case of the more general hyperelliptic family $\mathcal{C}_{q,p,a}$ given by

$$y^q = x^p + a, \tag{2}$$

where q and p are prime numbers with $q \geq 2$ and $p \geq 3$, and a is an integer, studied in [22] and [24]. Related investigations on algebraic points of bounded degree on hyperelliptic curves can also be found in the work of Stoll [30]. This broader framework allows one to combine the Riemann–Roch theorem with divisor class group techniques, thereby enabling the explicit determination of algebraic points. Such an approach has already been implemented in the works of Sall [29], Fall [11], and more recently Diallo–Coly [10], where explicit parametrizations have enabled the effective determination of algebraic points. In this context, the main

contribution of the present paper is to provide an explicit parametrization of the set $\mathcal{C}_m^{(\ell)}$ of algebraic points of degree at most ℓ on the family $\mathcal{C}_m$. Assuming that the Jacobian $\mathcal{J}_m(\mathbb{Q})$ has Mordell–Weil rank zero, we combine the explicit structure of the Mordell–Weil group with Riemann–Roch spaces and the Abel–Jacobi theorem to derive explicit families of algebraic points, thereby extending the known description of rational points to algebraic points of bounded degree.

2 Auxiliary Results

Definition 2.1. For a divisor $D \in \mathrm{Div}(\mathcal{C})$, the $\mathbb{Q}$-vector space $\mathcal{L}(D)$ is defined by

$$\mathcal{L}(D) \ := \ \{f \in \mathbb{K}(\mathcal{C})\setminus\{0\} \ \mid div(f) \geq -D\} \cup \{0\}.$$

We begin by recalling a fundamental property of the Jacobian associated with the family of hyperelliptic curves

$$\mathcal{C}_m : y^2 = x^5 + m^2,$$

which will play a crucial role throughout this paper. Under the assumption that the Mordell–Weil rank of $\mathcal{J}_m(\mathbb{Q})$ is zero, the group of rational points on the Jacobian coincides with its torsion subgroup. This structural result will be used repeatedly in the sequel, especially in the characterization of principal divisors and in the proof of the main theorem.

Lemma 2.2. *Assume that m is a square-free integer such that* $\operatorname{rank} \mathcal{J}_m(\mathbb{Q}) = 0$. *Then*

$$\mathcal{J}_m(\mathbb{Q}) = \mathcal{J}_m(\mathbb{Q})_{\mathrm{tors}} \cong \mathbb{Z}/5\mathbb{Z}.$$

Proof. See [23, Theorem 4.1]. □

The above lemma provides the algebraic framework required in the remainder of the paper. In particular, the assumption that the Mordell–Weil group has rank zero ensures that every rational point on the Jacobian is torsion. This fact considerably simplifies the study of divisor classes and allows us to derive explicit descriptions of algebraic points on $\mathcal{C}_m$ by means of the Abel–Jacobi map.

Consider now the affine model of the equation (1) for the curve $\mathcal{C}_m$ given by

$$\mathcal{C}_m : \begin{cases} y^2 = \prod_{\mu=0}^{4} (x + \eta_\mu), \\ x^5 = \prod_{\mu=5}^{6} (y - \kappa_\mu), \end{cases} \tag{3}$$

where $\eta_\mu = \sqrt[5]{m} e^{\frac{(2\mu+1)\pi i}{5}}$ are the complex roots of $x^5 + m^2 = 0$ and $\kappa_\mu = (-1)^\mu m$. Let X, Y, Z denote homogeneous coordinates. The projective model of $\mathcal{C}_m$ is then given by

$$\mathcal{C}_m : \begin{cases} Y^2 Z^3 = \prod_{\mu=0}^{4} (X + \eta_\mu Z), \\ X^5 = Z^3 \prod_{\mu=5}^{6} (Y - \kappa_\mu Z). \end{cases} \tag{4}$$

We define the following projective points associated with $\mathcal{C}_m$:

- $M_\mu = [\eta_\mu : 0 : 1]$ for $\mu \in \{0, \ldots, 4\}$,
- $M_\mu = [0 : \kappa_\mu : 1]$ for $\mu \in \{5, 6\}$,
- the unique point at infinity $\infty = [0 : 1 : 0]$.

Lemma 2.3. *For the curve $\mathcal{C}_m$, the following principal divisors hold:*

i) $\operatorname{div}(x - \eta_\mu) = 2M_\mu - 2\infty$ *for all* $\mu \in \{0, \ldots, 4\}$,

ii) $\operatorname{div}(y) = \sum_{\mu=0}^{4} M_\mu - 5\infty$.

Proof. First, note that, based on the calculations performed in [10], we can deduce the following:

- For $X - \eta_\mu Z = 0$, for $\mu \in \{0, \ldots, 4\}$, from equation (4), we get: $Z^3 Y^2 = 0$, thus giving the points M_μ with multiplicity order equal to 1 for each $\mu \in \{0, \ldots, 4\}$.

- For $Y = 0$, from equation (4), we get: $\prod_{\mu=0}^{4}(X - \eta_\mu Z) = 0$, thus giving the points M_μ with multiplicity order equal to 1 for each $\mu \in \{0, \ldots, 4\}$.
- For $Z = 0$, similarly, from equation (4), we get: $X^5 = 0$, thus giving the point ∞ with multiplicity order equal to 5.

Note that for the divisor calculations, we can observe the following:

$$\begin{aligned}
\operatorname{div}(y^2) = \operatorname{div}\left(\prod_{\mu=0}^{4}(x-\eta_\mu)\right) &\Longleftrightarrow \operatorname{div}(Z^3Y^2) = \operatorname{div}\left(\prod_{\mu=0}^{4}(X-\eta_\mu Z)\right),\\
&\Longleftrightarrow \operatorname{div}(Z^3) + \operatorname{div}(Y^2) = \operatorname{div}\left(\prod_{\nu=0}^{4}(X-\eta_\mu Z)\right),\\
&\Longleftrightarrow 3\operatorname{div}(Z) + 2\operatorname{div}(Y) = \sum_{\mu=0}^{4}\operatorname{div}(X-\eta_\mu Z),\\
&\Longleftrightarrow 3(Z=0)\cdot\mathcal{C}_m + 2(Y=0)\cdot\mathcal{C}_m\\
&\qquad = \sum_{\nu=1}^{2}(2M_\mu + 3\infty)\cdot\mathcal{C}_m,\\
&\Longleftrightarrow 3(5\infty) + 2\left(\sum_{\mu=0}^{4}M_\mu\right) = \sum_{\mu=0}^{4}(2M_\mu + 3\infty).
\end{aligned}$$

Thus, in general, since $\operatorname{div}(w-\gamma) = (W-\gamma Z=0)\cdot\mathcal{C}_m - (Z=0)\cdot\mathcal{C}_m$, where $w \in \{x, y\}$ and $W \in \{X, Y\}$, we deduce that:

$$\begin{aligned}
&i) \quad \operatorname{div}(x-\eta_\mu) = 2M_\mu - 2\infty \qquad \text{where} \quad \mu \in \{0, \ldots, 4\},\\
&ii) \quad \operatorname{div}(y) = \sum_{\mu=0}^{4} M_\mu - 5\infty.
\end{aligned}$$

□

Corollary 2.4. *As a consequence of* ***Lemma 2.3****, we have:*

- $\sum_{\mu=0}^{4} j_m(M_\mu) = 0$,

- $2\,j_m(M_\mu) = 0$ *for all* $\mu \in \{0, \ldots, 4\}$.

Here j_m *denotes the Jacobian embedding, i.e. the map which associates to any point* $P \in \mathcal{C}_m(\mathbb{Q})$ *the divisor class*

$$j_m(P) = \left[\, P - \infty \,\right] \in \mathcal{J}_m(\mathbb{Q}),$$

as described in [19].

Thus, the divisor classes $j_m(M_\mu)$ for $\mu = 0, \ldots, 4$ generate the same subgroup of $\mathcal{J}_m(\mathbb{Q})$.

Remark 2.5. The explicit description of the torsion subgroup of the Jacobian plays a fundamental role in the determination of algebraic points on $\mathcal{C}_m$. Under the assumption that $\operatorname{rank} \mathcal{J}_m(\mathbb{Q}) = 0$, the Mordell–Weil group coincides with its torsion subgroup. Moreover, by [23, Theorem 4.1], a generator of the torsion subgroup is given by

$$\mathcal{J}_m(\mathbb{Q})_{\text{tors}} \cong \left\langle \left[\, M_6 - \infty \,\right] \right\rangle.$$

Lemma 2.6. *From* ***Lemma 2.2*** *and* ***Remark 2.5****, the Mordell–Weil group is explicitly given by*

$$\mathcal{J}_m(\mathbb{Q}) = \left\{\, \gamma j_m(M_6) \mid \gamma \in \{0, \ldots, 4\} \,\right\}.$$

Remark 2.7. From **Lemma 2.6**, we deduce an explicit form: for every $t \in \mathcal{J}_m(\mathbb{Q})$, there exists $\gamma \in \{0, \ldots, 4\}$ such that

$$t = [\, \gamma\infty - \gamma M_6 \,].$$

Lemma 2.8. *A* $\mathbb{Q}$*-basis of* $\mathcal{L}(\ell\infty)$ *is given by*

$$\mathcal{B}_\ell = \left\{ (x - \eta_\mu)^i \;\middle|\; i \leq \mathbb{E}\left[\tfrac{\ell}{2}\right] \right\} \;\cup\; \left\{ y(x - \eta_\mu)^j \;\middle|\; j \leq \mathbb{E}\left[\tfrac{\ell-5}{2}\right] \right\},$$

where $i, j \in \mathbb{N}$ *and* $\mathbb{E}[\cdot]$ *denotes the integer part (floor) function.*

Proof. When $\ell < 2g - 1$ *i.e* $\ell < 3$, the result follows directly from **Lemma 2.3**, together with an application of Clifford's theorem described in [6, 13]. For $m \geq 2g - 1$ *i.e* $m \geq 3$, the statement follows from an application of the Riemann-Roch theorem [16, 25]. A more detailed proof can be found in [8, 9]. □

Proof.[Proof of Proposition 1.1] Let $P \in \mathcal{C}_m(\mathbb{Q})$. By the Abel–Jacobi map, every rational point P determines a divisor class $[P-\infty] \in \mathcal{J}_m(\mathbb{Q})$. Since $\operatorname{rank} \mathcal{J}_m(\mathbb{Q}) = 0$, the Mordell–Weil group is finite and coincides with its torsion subgroup. By Remark 2.5,

$$\mathcal{J}_m(\mathbb{Q}) = \mathcal{J}_m(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/5\mathbb{Z}.$$

The explicit description of the divisor classes obtained in Section 3 shows that the only effective divisor classes of degree one defined over $\mathbb{Q}$ are represented by the points

$$(0, m), \qquad (0, -m),$$

and by the unique point at infinity. Since $0^5 + m^2 = m^2$, the points $(0, \pm m)$ belong to $\mathcal{C}_m(\mathbb{Q})$. Moreover, because the defining polynomial has odd degree, the smooth projective model of $\mathcal{C}_m$ possesses a unique point at infinity, which is rational. Therefore,

$$\mathcal{C}_m^{(1)}(\mathbb{Q}) = \{(0, \pm m), \infty\},$$

as claimed. □

3 Main Result

Theorem 3.1. *Let m be a square-free integer such that $\operatorname{rank} \mathcal{J}_m(\mathbb{Q}) = 0$ and let $\mathcal{C}_m$ be a hyperelliptic curve defined by $y^2 = x^5 + m^2$ over $\mathbb{Q}$. Then, for every integer $\ell \geq 5$, the set of algebraic points of degree at most ℓ is explicitly described as:*

$$\mathcal{C}_m^{(\ell)}(\mathbb{Q}) = \bigcup_{\gamma=0}^{4} \mathcal{F}_\gamma,$$

where each subset $\mathcal{F}_\gamma$ consists of points of the form

$$\mathcal{F}_\gamma = \left\{ \left(x, -\frac{\displaystyle\sum_{i=\gamma+1}^{\frac{\ell+\gamma}{2}} a_i(\omega^i - \xi) - m \sum_{j=\gamma+1}^{\frac{\ell+\gamma-5}{2}} b_j \xi}{\displaystyle\sum_{j=\gamma+1}^{\frac{\ell+\gamma-5}{2}} b_j \omega^j} \right) \;\middle|\; \begin{array}{l} \bullet\ a_i, b_j \in \mathbb{Q}, \quad \omega = x - \eta_\mu, \\ \quad \xi = \displaystyle\sum_{\gamma=0}^{4} (-\eta_\mu)^{i-\gamma} \omega^\gamma, \\ \bullet\ a_{\frac{\ell+\gamma}{2}} \neq 0 \text{ if } \ell \equiv 0 \bmod 2 \\ \bullet\ b_{\frac{\ell+\gamma-5}{2}} \neq 0 \text{ if } \ell \not\equiv 0 \bmod 2 \\ \text{and } x \text{ is a root of } \mathcal{E}_\gamma \end{array} \right\}$$

Here, $\mathcal{E}_\gamma$ denotes the following algebraic equation in ω:

$$\left(\omega^{-\frac{\gamma}{2}}\left(\sum_{i=\gamma+1}^{\frac{\ell+\gamma}{2}} a_i\left(\omega^i - \xi\right) - m\sum_{j=\gamma+1}^{\frac{\ell+\gamma-5}{2}} b_j\xi\right)\right)^2 = \left(\sum_{j=\gamma+1}^{\frac{\ell+\gamma-5}{2}} b_j\omega^{j-\frac{\gamma}{2}}\right)^2 \prod_{\mu=0}^{4}(\omega+2\eta_\mu).$$

Proof. Let $R \in \mathcal{C}_m(\overline{\mathbb{Q}})$ be a point of degree $\ell \geq 5$, such that R is not one of the distinguished points $R \notin \{M_\mu | \mu \in \{0,\ldots,6\}\}$. Let $R_1,\ldots,R_\ell$ denote its Galois conjugates over $\mathbb{Q}$. Define the divisor class

$$t = \left[\sum_{n=1}^{\ell} R_n - \ell\infty\right] \in \mathcal{J}(\mathbb{Q}).$$

According to **Remark 2.7**, the class t admits a representation in the Jacobian as

$$\left[\sum_{n=1}^{\ell} R_n - \ell\infty\right] = \left[\gamma\infty - \gamma M_6\right]. \tag{5}$$

By reducing the classes in the expression (5), we obtain:

$$\left[\sum_{n=1}^{\ell} R_n + \gamma M_6 + (\ell+\gamma)\,\infty\right] = 0. \tag{6}$$

From the vanishing of class (6), and by the Abel-Jacobi theorem [15, 17], there exists a rational function $f_\gamma(x,y)$ defined over $\mathbb{Q}$ such that its divisor is:

$$\operatorname{div}(f_\gamma) = \sum_{n=1}^{\ell} R_n + \gamma M_6 + (\ell+\gamma)\,\infty.$$

Thus, $f_\gamma(x,y)$ belongs to the Riemann–Roch space: $f_\gamma \in \mathcal{L}\left((\ell+\gamma)\,\infty\right)$. By **Lemma 2.8**, $f_\gamma(x,y)$ can be expressed as:

$$f_\gamma(x,y) = \sum_{i=0}^{\frac{\ell+\gamma}{2}} a_i(x-\eta_\mu)^i + \sum_{j=0}^{\frac{\ell+\gamma-5}{2}} b_j(x-\eta_\mu)^j y. \tag{7}$$

Moreover, from the vanishing orders $\operatorname{ord}_{M_6}(f_\gamma) = \gamma$, it follows that:

$$a_\gamma + b_\gamma = -\left(\sum_{i=\gamma+1}^{\frac{\ell+\gamma}{2}} a_i(-\eta_\mu)^{i-\gamma} + \sum_{j=\gamma+1}^{\frac{\ell+\gamma-5}{2}} b_j m(-\eta_\mu)^{j-\gamma}\right), \tag{8}$$

where $\gamma \in \{0, \ldots, 4\}$. By performing the change of variable $\omega = x - \eta_\mu$, we rewrite (7) as:

$$f_\gamma(\omega, y) = \left(\sum_{i=\gamma+1}^{\frac{\ell+\gamma}{2}} a_i \omega^i + \sum_{j=\gamma+1}^{\frac{\ell+\gamma-5}{2}} b_j \omega^j y \right) + \sum_{\gamma=0}^{4} (a_\gamma + b_\gamma)\, \omega^\gamma. \tag{9}$$

Setting $\xi = \sum_{k=0}^{4} (-\eta_\mu)^{i-k} \omega^k$ and substituting the relation (8) into the expression of $f_\gamma(\omega, y)$ given in (9), we obtain:

$$f_\gamma(\omega, y) = \sum_{i=\gamma+1}^{\frac{\ell+\gamma}{2}} a_i \left(\omega^i - \xi\right) + \sum_{j=\gamma+1}^{\frac{\ell+\gamma-5}{2}} b_j \left(y\omega^j - m\xi\right), \tag{10}$$

The coefficients satisfying les condition suivantes:

$$a_{\frac{\ell+\gamma}{2}} \neq 0 \quad \text{if} \quad \ell \text{ is even}, \quad b_{\frac{\ell+\gamma-5}{2}} \neq 0 \quad \text{if} \quad \ell \text{ is odd}.$$

Otherwise, one of the R_n would equal ∞, which is not possible.

Since each R_n is a simple zero of $f_\gamma(\omega, y)$ in (10), we deduce an explicit expression for y in terms of ω, namely:

$$y = -\frac{\displaystyle\sum_{i=\gamma+1}^{\frac{\ell+\gamma}{2}} a_i \left(\omega^i - \xi\right) - m \sum_{j=\gamma+1}^{\frac{\ell+\gamma-5}{2}} b_j \xi}{\displaystyle\sum_{j=\gamma+1}^{\frac{\ell+\gamma-5}{2}} b_j \omega^j}. \tag{11}$$

Substituting (11) into equation (3), we obtain:

$$\left(\sum_{i=\gamma+1}^{\frac{\ell+\gamma}{2}} a_i \left(\omega^i - \xi\right) - m \sum_{j=\gamma+1}^{\frac{\ell+\gamma-5}{2}} b_j \xi \right)^2 = \left(\sum_{j=\gamma+1}^{\frac{\ell+\gamma-5}{2}} b_j \omega^j \right)^2 \prod_{\mu=0}^{4} (\omega + 2\eta_\mu).$$

This leads to the algebraic equation $\mathcal{E}_\gamma$:

$$(\mathcal{E}_\gamma) : \left(\omega^{-\frac{\gamma}{2}} \left(\sum_{i=\gamma+1}^{\frac{\ell+\gamma}{2}} a_i \left(\omega^i - \xi\right) - m \sum_{j=\gamma+1}^{\frac{\ell+\gamma-5}{2}} b_j \xi \right) \right)^2 = \left(\sum_{j=\gamma+1}^{\frac{\ell+\gamma-5}{2}} b_j \omega^{j-\frac{\gamma}{2}} \right)^2 \prod_{\mu=0}^{4} (\omega + 2\eta_\mu).$$

We now check that the defining equation $\mathcal{E}_\gamma$ is indeed of degree ℓ in ω. On the left-hand side we obtain

$$2 \cdot \left(\frac{\ell+\gamma}{2} - \frac{\gamma}{2} \right) = \ell.$$

while on the right-hand side we obtain

$$2 \cdot \left(\frac{\ell+\gamma-5}{2} - \frac{\gamma}{2} \right) + 5 = \ell.$$

hence the total degree is ℓ. Therefore, we define a set $\mathcal{F}_\gamma$ of algebraic points of degree ℓ, with explicitly parametrized coordinates:

$$\mathcal{F}_\gamma = \left\{ \left(x, -\frac{\sum_{i=\gamma+1}^{\frac{\ell+\gamma}{2}} a_i(\omega^i - \xi) - m \sum_{j=\gamma+1}^{\frac{\ell+\gamma-5}{2}} b_j \xi}{\sum_{j=\gamma+1}^{\frac{\ell+\gamma-5}{2}} b_j \omega^j} \right) \middle| \begin{array}{l} \bullet\ a_i, b_j \in \mathbb{Q}, \quad \omega = x - \eta_\mu, \\ \quad \xi = \sum_{\gamma=0}^{4} (-\eta_\mu)^{i-\gamma} \omega^\gamma, \\ \bullet\ a_{\frac{\ell+\gamma}{2}} \neq 0 \text{ if } \ell \equiv 0 \bmod 2 \\ \bullet\ b_{\frac{\ell+\gamma-5}{2}} \neq 0 \text{ if } \ell \not\equiv 0 \bmod 2 \\ \text{and } x \text{ is a root of } \mathcal{E}_\gamma \end{array} \right\}$$

□

Example 3.2. Consider the curve $\mathcal{C}_{101} : y^2 = x^5 + 101^2 = x^5 + 10201$. By [21], we have $\mathcal{J}_{101}(\mathbb{Q}) \cong \mathbb{Z}/5\mathbb{Z}$, and $\mathcal{C}_{101}(\mathbb{Q}) = \{(0, \pm 101), \infty\}$.
To exhibit an algebraic point of degree 5, let ζ_5 be a primitive fifth root of unity and set $\alpha = -\zeta_5 \sqrt[5]{10201}$. Then $\alpha^5 = -\zeta_5^5\, 10201 = -10201$, and hence $\alpha^5 + 10201 = 0$. Therefore, $P = \left(-\zeta_5 \sqrt[5]{10201},\, 0 \right)$ is a point on $\mathcal{C}_{101}$. Moreover, α is a root of the binomial $X^5 + 10201 = X^5 + 101^2$. Since -101^2 is not a fifth power in $\mathbb{Q}$, the binomial $X^5 + 101^2$ is irreducible over $\mathbb{Q}$. Consequently, $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 5$. Since the coordinates of P are α and 0, we obtain $[\mathbb{Q}(P) : \mathbb{Q}] = [\mathbb{Q}(\alpha) : \mathbb{Q}] = 5$. Thus, $P \in \mathcal{C}_{101}^{(5)} \setminus \mathcal{C}_{101}^{(4)}$. This gives an explicit algebraic point of degree exactly 5 on $\mathcal{C}_{101}$.

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