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Original Research Paper

Evaluation of Decision-Making Units with Nonhomogeneous Outputs in Different Time: An Application of Network Data Envelopment Analysis

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Abstract. In conventional data envelopment analysis (DEA), it is typically assumed that inputs and outputs are homogeneous and that decision-making units (DMUs) remain present throughout all time periods. However, such assumptions are often violated in real-world applications. This study introduces a new network DEA framework that enables the evaluation of nonhomogeneous DMUs over multiple time periods, in which changes in the type and number of outputs, as well as

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the entry and exit of DMUs across different periods, are explicitly incorporated into the model.In the proposed approach, outputs are first classified into homogeneous subgroups, and the efficiency of each subgroup within the nonhomogeneous network is then computed. This allows the overall efficiency of each DMU to be calculated as a weighted average of the corresponding subnetwork efficiencies. Based on a numerical example and a real case study in the banking industry, the results reveal substantial differences between the efficiency scores obtained under the nonhomogeneous framework and those produced by traditional homogeneous DEA models.Overall, the findings indicate that the proposed model, owing to its high flexibility, provides a more realistic and accurate representation of DMU performance in multi-period environments.

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1 Introduction

Evaluating the performance of organizations operating in dynamic and complex environments has become an increasingly important issue for both researchers and managers. Since its introduction by Charnes, Cooper, and Rhodes [6], Data Envelopment Analysis (DEA) has been widely recognized as a powerful and flexible tool for measuring relative efficiency. Despite its extensive development, traditional DEA models are built upon the assumption that all decision-making units (DMUs) are homogeneous and operate under a common production technology. These assumptions mainly include the homogeneity of inputs and outputs across DMUs and the continuous presence of all DMUs throughout the entire time horizon. Such assumptions impose significant limitations on the applicability of conventional DEA models in real-world settings. In practice, particularly in sectors such as banking, service industries, and production networks, DMUs often operate under nonhomogeneous conditions. The composition, type, and even the number of outputs may vary across units or over time. These differences are typically driven by technological changes, regulatory reforms, macroeconomic policies, or strategic organizational decisions. Moreover, the entry of new units into the system and the exit of existing ones in subsequent periods (temporal heterogeneity) is a common phenomenon in such environments. Recent studies indicate that ignoring output heterogeneity and temporal heterogeneity can lead to biased efficiency estimates and, consequently, misleading managerial implications. As a result, addressing heterogeneity has become one of the main research directions in DEA. Early studies in this area identified the limitations imposed by the homogeneity assumption and proposed preliminary conceptual frameworks to mitigate its effects. Subsequently, researchers developed various approaches for evaluating nonhomogeneous units, including output grouping, handling missing data, and related techniques. Although these efforts yielded notable advances, most of them remained confined to single-period analyses and were unable to capture the temporal dynamics inherent in real-world systems. In parallel, the development of Network DEA (NDEA) as an extension of classical DEA enabled the analysis of complex systems with multi-stage, parallel, or hybrid structures. This approach has proven particularly effective in modeling real production processes that involve internal interactions among different stages. However, most existing NDEA models still assume homogeneous network structures and a stable set of DMUs, which limits their applicability in environments characterized by nonhomogeneous or evolving internal structures (Kao, [21]; Yan,[31]). In recent years, researchers have also turned their attention to heterogeneity in network structures. These studies acknowledge that DMUs may differ not only in terms of outputs, but also with respect to the number of stages, the types of internal components, and the nature of interconnections among them. While such approaches represent an important step toward enhancing model realism, they have largely remained restricted to single-period settings. Conversely, multi-period DEA models have been developed to analyze efficiency changes over time, enabling intertemporal comparisons and long-term performance evaluation. Nevertheless, these models typically maintain the assumptions of output homogeneity and a fixed set of DMUs, and thus do not account for changes in output composition or the entry and exit of units (An & Min, [1]). Accordingly, although nonhomogeneous DEA, network DEA, and multi-period DEA have each been developed independently, an integrated framework capable of simultaneously modeling output heterogeneity, network structures, and multi-period dynamics in the presence of DMU entry and exit remains largely unexplored in the literature. This gap is particularly critical in industries such as banking and services, where product portfolios and institutional participation change continuously over time. To address this gap, the present study proposes a novel multi-period network DEA framework for evaluating the efficiency of nonhomogeneous DMUs over time. In this framework, both output heterogeneity and temporal heterogeneity (defined as changes in the presence or absence of DMUs across periods) are explicitly considered. The proposed model allows for variations in the type and number of outputs across different periods, as well as the entry and exit of DMUs. To this end, outputs are first classified into homogeneous subsets, and the efficiency of each subset is evaluated within its corresponding sub-network. Subsequently, the overall efficiency of each DMU is computed through a weighted aggregation of sub-network efficiencies. The proposed framework has several key advantages over existing approaches. First, it provides an integrated model that simultaneously addresses output heterogeneity, network structures, multi-period dynamic analysis, and the entry and exit of DMUs. Second, by preserving the network structure of the units, it offers a more realistic representation of complex production processes. Third, the model enables decision-makers to identify the strengths and weaknesses of the units in relation to specific output categories across all time periods. The efficiency of the proposed framework has been evaluated through a numerical example and an empirical study in the banking industry. The results indicate significant differences between the efficiency scores obtained from the proposed model and traditional homogeneous DEA models, clearly highlighting the importance of explicitly considering heterogeneity in efficiency analysis. The structure of the rest of the article is organized as follows: First, the next section reviews the related literature. The third section is dedicated to presenting and explaining the main research problem. In the fourth section, the proposed framework and models are introduced. The fifth section provides a real-world example to demonstrate the application of the presented models. Finally, the article concludes with a summary of the findings and suggestions for future research.

2 Literature Review

Data Envelopment Analysis (DEA) has been widely recognized as a powerful tool for evaluating the performance and efficiency of decision-making units (DMUs). This method was first introduced by Charnes, Cooper, and Rhodes [6] for a set of homogeneous DMUs operating under a common technology, and since then, numerous studies have been conducted in this field (Panwar et al. [29], Kyrgiakos et al. [22], and Hosseinzadeh Lotfi et al. [18]).

A fundamental assumption in classical DEA models is the homogeneity of DMUs in terms of input–output structure and production technology; that is, all DMUs are assumed to use the same set of inputs to produce similar outputs. However, in real-world applications, this assumption often does not hold. Temporal, spatial, technical, and environmental differences among DMUs give rise to heterogeneity (Haas & Murphy[16]). Consequently, the investigation and management of heterogeneity in DEA have become major research themes in recent years. With the growing application of DEA to real-world problems, researchers gradually became aware of the limitations imposed by the homogeneity assumption and identified heterogeneity as a key challenge in efficiency evaluation. Dyson et al. [12] were among the first to systematically examine the consequences of violating the homogeneity assumption in DEA, demonstrating that ignoring heterogeneity can lead to misleading efficiency scores and rankings. Subsequently, Castelli et al. [5] showed that in many organizations, DMUs consist of sub-units with different structures and performance characteristics, rendering homogeneous DEA models inappropriate. Cook et al. [8, 9], focusing on output heterogeneity, proposed a framework in which DMUs are classified into homogeneous subgroups based on their output production patterns, and efficiency is evaluated separately within each subgroup. This approach has formed the basis for many subsequent studies on nonhomogeneous DEA. Continuing along this research stream, Li et al. [24] and Lin et al. [25] extended these models under the assumption of variable returns to scale (VRS) and employed directional distance functions to introduce greater flexibility in evaluating nonhomogeneous DMUs. At the same time, Chen et al. [7] addressed the issue of input instability and proposed methods for handling DMUs with variable inputs. Nevertheless, many of these models still face limitations in situations where inputs are non-separable or when the analysis is confined to a single time period. In addition to output heterogeneity, part of the literature has focused on heterogeneity on the input side. Li et al. [23] examined settings in which DMUs utilize non-homogeneous inputs, while Zhu et al. [34] extended the concept of cross-efficiency evaluation to nonhomogeneous units. Chen et al. [7] incorporated heterogeneity arising from fluctuations in input values into the analysis and proposed a DEA-based framework to address this type of uncertainty. These studies indicate that heterogeneity may stem from multiple sources and is not limited to outputs alone.

On the other hand, heterogeneity may arise not only at the DMU level but also within their internal structures. This observation has led to the development of Network Data Envelopment Analysis (Network DEA). As an extension of traditional DEA, Network DEA enables the evaluation of systems with multi-stage, series, parallel, and hybrid structures [20, 33]. In this context, Huang et al. [19] proposed nonhomogeneous two-stage models for evaluating hotel efficiency, and Du et al. [11] incorporated heterogeneity into parallel network structures. Subsequent studies such as Singh and Ranjan [30], Barat et al. [3, 4], and Yan et al. [31] further developed NDEA models for DMUs with nonhomogeneous, dynamic internal structures and undesirable outputs. Despite the substantial expansion of DEA and NDEA models to address heterogeneity, most of these studies are static in nature and rely on data from a single time period. In many practical applications, however, performance evaluation requires multi-period analysis. To meet this need, multi-period DEA models have been developed (Ahn and Min, [1]). Kao and Liu [21] measured DMU efficiency in each period as well as over the entire time horizon, while Hajiagha et al. [17] employed Chebyshev inequality bounds for multi-period efficiency evaluation. More recent studies, including Esmailzadeh and Matin [13], Günaydın et al. [15], Moghaddas et al. [26], Asadi et al. [2], and Gazori-Nishabouri et al. [14], have investigated network, dynamic, and feedback structures within multi-period DEA frameworks.

Nevertheless, a major limitation of most existing multi-period DEA models is their neglect of DMU heterogeneity over time. In practice, the composition of DMUs, the set of outputs, and even production conditions may change across periods. To address this gap, very recent studies have explicitly focused on temporal heterogeneity. Dastani et al. [10] developed a multi-period DEA approach that incorporates output heterogeneity over time into DMU efficiency analysis, demonstrating that ignoring this type of heterogeneity can lead to biased performance estimates. Zotti [35] employed a Conditional DEA framework to incorporate regional heterogeneity into the efficiency analysis of schools, highlighting the importance of environmental and structural factors. Moreover, Mohsenirad and Triantis [27] proposed a comprehensive methodological framework for testing and identifying heterogeneity in DEA, which can serve as a foundation for the development of more advanced models in this area.

Despite these advances, a comprehensive model that simultaneously accounts for output heterogeneity and temporal heterogeneity within a multi-period framework has not yet been fully developed. This gap is particularly critical in settings where DMUs may exit or enter the system in certain periods, or where the set of outputs changes over time. Accordingly, the present study aims to fill this research gap by developing a model for evaluating the multi-period efficiency of nonhomogeneous DMUs. The proposed model incorporates heterogeneity in both outputs and time periods, enabling a more realistic assessment of DMU performance in dynamic and complex environments. This approach represents a meaningful step toward enhancing the analytical capabilities of DEA and expanding its applicability to real-world problems.

3 Description of the research problem

To illustrate the problem under investigation, we first present a hypothetical example. Suppose that three automobile manufacturers (DMU_1, DMU_2, DMU_3) are evaluated over four time periods. The inputs remain homogeneous throughout all periods, whereas the type and composition of outputs may vary across different periods. Moreover, over time and due to technological advancements, some production lines may be eliminated or reconstructed,or new production lines may be introduced into the manufacturing process.

Numerical example: Suppose Iran Khodro has three production units in different cities, comprising five nonhomogeneous outputs and two homogeneous inputs.
x_1: steel consumption (tons)
x_2: labor hours (thousand hours)
y_1: Peugeot 207 production
y_2: Samand production
y_3: Dena production
y_4: Rana production
y_5: Tara production
These five outputs are all passenger cars belonging to the same industry, but they represent product variety. In Table 1, this scenario is illustrated for three companies ($DMUs$) with different production lines, homogeneous inputs, and nonhomogeneous outputs over four time periods.
Nonhomogeneous in output: The companies do not necessarily produce all models in every period, or they may introduce new products. For example, DMU_1 does not produce the Dena in periods t_1 and t_2 but starts its production in t_3, while the Samand remains unproduced. In t_4, the production of the 207 ceases, whereas other models continue to be manufactured. Changes in unit presence over time: DMU_3 becomes active and starts production from period t_2. DMU_2 has no output in t_4 due to production line renovation.

Table 1: Homogeneous inputs and nonhomogeneous outputs over several time periods

Time	Homogeneous	Outputs of DMU_1	Outputs of DMU_2	Outputs of DMU_3
t_1	x_1, x_2	y_1, y_2, y_4, y_5	y_1, y_3, y_4, y_5	-
t_2	x_1, x_2	y_1, y_2, y_4, y_5	y_1, y_3, y_4, y_5	y_3, y_4, y_5
t_3	x_1, x_2	y_1, y_3, y_4, y_5	y_3, y_4, y_5	y_2, y_3, y_4, y_5
t_4	x_1, x_2	y_2, y_3, y_4, y_5	-	y_3, y_4, y_5

In this scenario, DMU_1, DMU_2, and DMU_3 are the three companies under investigation in the automotive manufacturing plant. In addition, x_1 and x_2 are the homogeneous inputs of the plant, whereas y_1, y_2, y_3, y_4, and y_5 represent the nonhomogeneous outputs of the companies in each period.

In Table 1, we can observe that in the first year of operation, DMU_1 and DMU_2 are active. In the second and third years, all DMUs are active. In the fourth year, DMU_1 and DMU_3 are active (in production). In addition, DMUs may not produce the same outputs during their periods of activity. For example, DMU_1 produces outputs y_1, y_2, y_4, and y_5 in the first and second periods, but in the next period, output y_2 is not produced; instead, output y_3 is introduced into the production cycle. In period t_4, output y_1 is not produced, but output y_2 is reintroduced into the production cycle. Similarly, DMU_2, which produces products y_1, y_3, y_4, and y_5 in the first and second periods, produces products y_3, y_4, and y_5 in the third period. Even the production of a production line may not change throughout the evaluation periods. In the fourth column of Table 1, we observe the nonhomogeneous outputs of DMU_3 in periods t_2, t_3, and t_4.

So as to calculate the efficiency of each DMU, (each company in its production line) over all periods of operation, we first categorize the DMU_{j^t} s (the j^{th} Decision Making unit at time t) on the basis of their outputs independent of their periods of operation, according to Cook et. al [8]. This means that DMU_{j^t} s with the same outputs are grouped into p DMU groups N_p, $p = A, B, C, \ldots,$. Thus, DMU_{j^t} s are categorized into P distinct groups. In the provided example, the number of DMU_{j^t} s is 10, which are categorized (based on their outputs) into four distinct subgroups named N_A, N_B, N_C and N_D. For instance, N_A includes DMU_{1^1} and DMU_{1^2} because they produce products y_1, y_2, y_4, and y_5. Meanwhile, DMU_{2^2}, DMU_{2^1}, and DMU_{1^3} in other N_B produce products y_1, y_3, y_4, and y_5. Similarly, instances follow this pattern. Therefore, the number of categories in this example is four, and each subgroup uses similar inputs.

Definition 3.1. (Temporalv Nonhomogeneity). Temporal nonhomogeneity refers to the variation in the composition of DMUs and their operational status across different time periods. Specifically, the set of DMUs and their levels of activity are not constant over time. Some DMUs may enter or exit the system in certain periods, leading to changes in the number of operating units. For example, a new DMU may be introduced in a later period (e.g., DMU_3 is absent in period t_1 but becomes operational from period t_2 onward). Similarly, a DMU may be inactive in a particular period (e.g., DMU_2 produces no output in period t_4 due to a temporary shutdown or production line reconstruction), effectively rendering it non-operational during that interval. Consequently, the set of active DMUs may differ from one period to another.

Definition 3.2. (Definition of $\boldsymbol{N_p}$). N_p denotes the set of all DMU_{j^t} that produce identical outputs and therefore belong to category p, where $p = A, B, C, \ldots$. In fact, the DMUs are partitioned according to their output types, and any DMU_{j^t} with the same output is placed in the same group.

Table 2: Production Outputs in Each Group

Groups/ Outputs	$\mathbf{y_1}$	$\mathbf{y_2}$	$\mathbf{y_3}$	$\mathbf{y_4}$	$\mathbf{y_5}$
$\mathbf{N_A}$	✓	✓	-	✓	✓
$\mathbf{N_B}$	✓	-	✓	✓	✓
$\mathbf{N_C}$	-	✓	✓	✓	✓
$\mathbf{N_D}$	-	-	✓	✓	✓

Now, we analyze the overall scenario and survey the situation shown in Table 2. In this scenario, a group of production lines from automotive industry companies produce a specific set of products at different times, but not all products are produced at all times and in all the production lines. In this example, the automotive industry companies are divided into 4 exclusive and mutually exclusive groups ($P = 4$) based on output (according to production lines). The groups are named N_A, N_B, N_C and N_D respectively. The group N_A includes DMU_{1^1} and DMU_{1^2}. The group N_B includes DMU_{2^2}, DMU_{2^1} and DMU_{1^3}. The group N_C includes DMU_{3^3} and DMU_{1^4} and group N_D includes DMU_{3^4}, DMU_{2^3} and DMU_{3^2}. (In some cases, the symbol $j^t \in N_p$ is used to represent $DMU_{j^t} \in N_p$).

In Data Envelopment Analysis (DEA), it is sometimes necessary to divide outputs into distinct and exclusive categories. This classification helps to identify the strengths and weaknesses of DMUs in the production of each type of output; with this categorization, the performance of DMUs in each category can be analyzed separately.

Definition 3.3. (Definition of R_k). The set R_k denotes a subset of outputs whose members are generated by exactly the same set of DMU_{j^t} units, that is, they share an identical DMU_{j^t} production profile. Accordingly, R_k represents a group of homogeneous outputs produced by a common subset of decision-making units DMU_{j^t}.
According to Definition 3.3, if two outputs r_1 and r_2 belong to R_k, then their corresponding production profiles coincide. Hence, if output r_1 is produced by the DMU_{j^t} units in groups N_A and N_D, then output r_2 is also produced by precisely the same groups of DMU_{j^t}, namely N_A and N_D.
Each set R_k is defined to be maximal with respect to this property; that is, there exists no output r in R_k whose DMU_{j^t} production profile is identical to that of the members of R_k.
It follows that, for a given collection of DMU_{j^t} production profiles, there exist K distinct output subsets. Let y_k denote the outputs contained in R_k. Based on the categorization framework proposed by Cook et al. (2013), the outputs in the illustrative example are partitioned as follows:

$$R_k, \quad k = 1, 2, \ldots, K$$
$$R_1 = \{1\}, \quad R_2 = \{2\}, \quad R_3 = \{3\}, \quad R_4 = \{4, 5\}$$

Subsequently, the DMU_{j^t} units are classified according to the output subgroups they generate. For example, those units producing output subgroups R_1, R_2, and R_4 are assigned to group N_A. In this illustrative case, R_1 contains output y_1, R_2 contains output y_2, and R_4 contains outputs y_4 and y_5. The inputs of each DMU_{j^t} are therefore allocated across the production activities associated with R_1, R_2, and R_4.
A separate DEA model is subsequently formulated and solved for each output subgroup using the corresponding inputs and outputs. This procedure enables an accurate assessment of the performance of the unit under evaluation within each output subgroup.

Definition 3.4 (Definition of L_{N_p}). Let $\boldsymbol{L_{N_p}}$ denote the collection of output subgroups R_k associated with category N_p, such that these

subgroups comprise all outputs generated by the decision-making units DMU_{j^t} belonging to N_p.
For instance, according to the above table, we have:

$$L_{N_A} = \{R_1, R_2, R_4\}, \quad L_{N_B} = \{R_1, R_3, R_4\},$$

$$L_{N_C} = \{R_2, R_3, R_4\}, \quad L_{N_D} = \{R_3, R_4\}$$

Definition 3.5 (Definition L_j)**.** For each decision-making unit DMU_{j^t}, let L_j denote the set of all output subsets produced by DMU_{j^t} throughout all evaluation periods in which it is active.
For example, over their respective active periods,

$$L_2 = \{R_1, R_3, R_4\}, \quad L_3 = \{R_2, R_3, R_4\}.$$

Definition 3.6 (Definition M_{R_K})**.** Let $\boldsymbol{M_{R_K}}$ denote the set of all categories N_p that produce output subgroup R_K. Equivalently, it represents the collection of all categories that include R_K as a member of their associated output set.

$$M_{R_K} = \left\{ N_P \middle| R_K \in L_{N_p} \right\}$$

For example, the hypothetical sets corresponding to $\mathrm{M}_{\mathrm{R}_1}$ and $\mathrm{M}_{\mathrm{R}_2}$ are defined as follows:

$$M_{R_1} = \{N_A, N_B\}, \quad M_{R_2} = \{N_A, N_C\}$$

Definition 3.7 (Definition of $M^o_{R_K}$)**.** Let $M^o_{R_K}$ represent the set of all categories N_{P^o} in which the decision-making unit DMU_o produces output R_K over multiple evaluation periods.

$$M^o_{R_K} = \left\{ N_{P^o} \middle| o^t \in N_{P^o} \text{ for } N_{P^o} \in M_{R_K} \right\}$$

For example, $M^2_{R_3} = \{N_B, N_D\}$ entails the categories that include DMU_2 at different time periods and produce the output R_3. N_B and N_D, thus $M^2_{R_3} = \{N_B, N_D\}$.

In this study, based on Figure 1, each decision-making unit (DMU_o) is regarded as a target network throughout its operating periods. The components inside the networks are referred to as “sub-networks” (sub).

In the o^{th} network (DMU_o), the subs consist of all N_{P^o} that produce output R_k. In other words, the subs include those clusters (N_{P^o}) that contain output R_k in the o^{th} network.Accordingly, in the o^{th} network, the clusters N_{P^o} that contain output R_1 are defined as the first sub-network. Similarly, the clusters that generate outputs $R_2, (R_3, and R_4)$ are defined as the second, third, and fourth sub-networks, respectively. Each N_{P^o} may belong to several subs. Therefore, based on the above classification, we have:

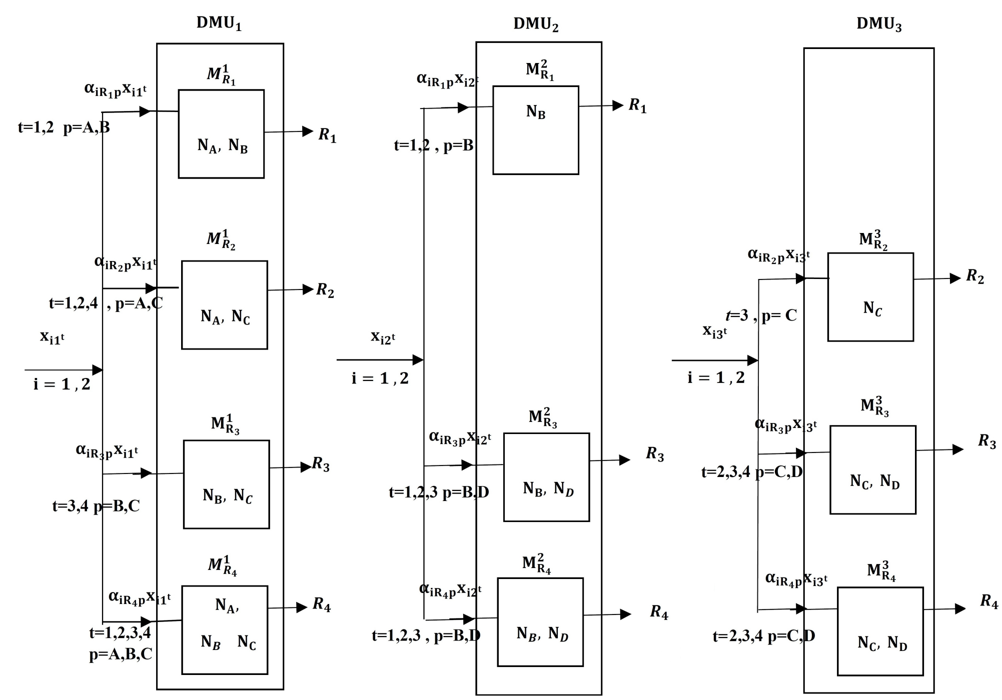


Figure 1: Structure of DMUs

As shown in Figure 1, we are dealing with parallel and independent networks that include various types of subs, and these subs are not necessarily homogeneous. In addition, the outputs of these networks are also nonhomogeneous. This means that the networks are pairwise nonhomogeneous, parallel and independent.

Moreover, the internal structure of the networks is also nonhomogeneous, meaning that the number of subs and the number of (N_p) s in each sub are not the same. In other words, the networks are not only nonhomogeneous when compared with each other, but also have internal nonhomogeneity. Therefore, we generalize the concept of parallel networks in the presence of nonhomogeneity in a more comprehensive and complex manner, such that, the networks have an internally nonhomogeneous structure with homogeneous inputs and nonhomogeneous outputs. In considering the existing non-homogeneity in the networks, how can we calculate the efficiency of DMUs during their activity periods? In the next section, this issue will be examined and answered. In this paper, by generalizing the idea of Cook et al. [8], a method is presented to evaluate the performance changes of DMUs with homogeneous inputs and nonhomogeneous outputs over multiple time periods. Given that, in practice, units may change over time and produce nonhomogeneous outputs, using a method that can evaluate units under these conditions is of great importance.

In this study, the internal structure of the network is used as an important element to get a better understanding of nonhomogeneity and to provide an appropriate method for evaluating DMUs with homogeneous inputs and nonhomogeneous outputs over different time periods. The internal structure of the network also investigates nonhomogeneity from other perspectives. In this study, nonhomogeneity is examined in two areas:

1) nonhomogeneity between different time periods

2) nonhomogeneity in output.

Both factors lead to nonhomogeneity in the network of DMUs. This paper presents a method for evaluating the efficiency of DMUs in nonhomogeneous networks, which can be used and generalized for analyzing and optimizing the performance of units under various conditions.

4 Evaluation of nonhomogeneous units in output and over multiple time periods

We examine the problem in general terms. To evaluate units with nonhomogeneous outputs over multiple time periods, we consider a set of n (DMUs) across T different time periods, where each unit has m homogeneous inputs in all periods and different outputs. Our goal is to calculate the efficiency of these DMUs. For this purpose, we use the method presented in the works of Cook et al. [8] and Brat et.al [**?**], and calculate the efficiency of the DMUs in three stages. In the first stage, we determine the contribution of inputs to the production of output subgroups in all periods in such a way that the total efficiency of the unit under evaluation is maximized. In the second stage, we calculate the efficiency of the output subgroups for the network related to the evaluated DMU_o. In the third stage, we calculate the overall efficiency of the DMU_o throughout its entire operational period using the weighted average of the efficiency of the output subgroups (subs) of the network obtained in the second stage.

In summary, the proposed method for calculating the efficiency of DMUs over multiple time periods consists of three steps:

Stage One: In this stage, Computing the share of inputs in the production of output subgroups for the evaluated DMU_o across all periods and relevant categories.

Stage Two: The efficiency of each output subgroup (network sub-efficiency) of the evaluated DMU_o (for the output subgroups belonging to L_O) is calculated.

Stage Three: In this stage, for the decision-making unit DMU_o, the overall efficiency score is calculated using the weighted average of the efficiencies of the output subgroups (subs), i.e., $M^o_{R_K}$, obtained in Stage 2. In other words, the final efficiency is constructed as a convex combination of the efficiencies of the output subgroups, where each subgroup contributes to the calculation with a specific weight. These weights can be assigned according to the importance or role of the corresponding variables. The flowchart of the proposed approach is as follows:

Flowchart Stage 1: Classification of DMU_{j^t} and calculation of input shares

This flowchart illustrates how DMUs are classified across all periods based on their output patterns, and how the share of each input in producing each output category is determined.

Start:

Step 1: First, all $DMUs$ in all periods are read. Inputs are homoge-

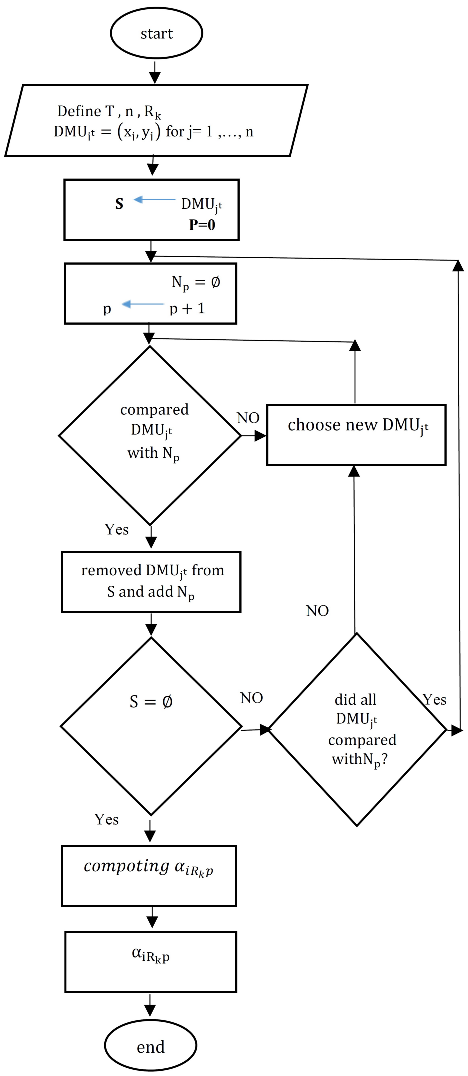
neous, while outputs are nonhomogeneous.
Step 2: Determination of output subgroups R_k In this step, the set of outputs is classified according to the "production profile": Outputs that are consistently produced by the same set of $DMUs$ are grouped into the same subgroup R_k.
For example, y_4 and y_5 are always produced together by the same $DMUs$ Therefore, $R_4 = y_4, y_5$. This classification is necessary to avoid comparing outputs that are not produced in certain periods.
Step 3: Formation of the set S (the list of all $DMUs$ across all periods): This set includes all DMU_{j^t}; each DMU in each period is considered a separate member. **Step 4:**Formation of classification sets $N_p,\ p = A, B, \ldots$
Objective of this step: To classify DMU_{j^t} solely based on the types of outputs they produce.
Procedure:
1. A DMU_{j^t} is selected from S.
2. Its outputs are extracted.
3. A defined output pattern is identified, and a set N_p is created for this pattern.
4. Any other DMU_{j^t} with the same output pattern is added to N_p.
5. DMU_{j^t} with identical patterns are removed from S.
6. This process continues until S is empty.

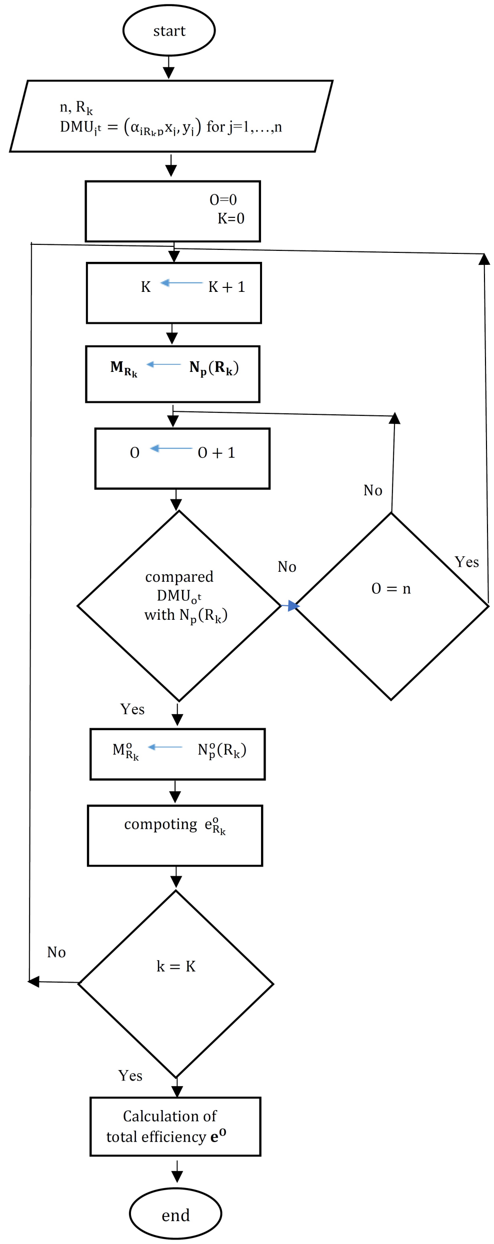
Step 5: Solving the DEA model to calculate $\alpha^*_{iR_kp}$ At this stage, Model (4) is executed. The output of this model is $\alpha^*_{iR_kp}$ (the share of input i in producing output R_k within the category N_p). This step allocates inputs among different outputs in a way that maximizes the efficiency of the evaluated DMU. End of Flowchart 1. A complete table of α values for all $DMUs$ is obtained (Table 7-4).

Flowchart 2: Calculation of Output Subgroup Efficiencies and Overall Efficiency (This flowchart covers the second and third stages of the method.)

Start
Step 1: Identify the effective inputs for each output subgroup using the values obtained from the previous stage.
Step 2: Construct the sets M_{R_k} and $M^o_{R_k}$.

M_{R_k} : Includes all categories (N_p) that generate output R_k.
$M^o_{R_k}$: Includes only those categories in which the DMU under evaluation is present and that generate output $R_k(N^o_p)$.
Step 3: Calculate the efficiency of DMU_O based on each output subgroup. For every output subgroup $R_k an$ independent CCR model is solved, and the value $e^o_{R_k}$ is obtained. This value indicates how efficiently DMU_O has performed within output subgroup R_k across all its active periods.
Step 4: Determine the weight of each output subgroup $W^o_{R_K}$
These weights represent the relative importance of the corresponding output subgroups.
Step 5: Compute the overall efficiency of DMU_O.
The final efficiency e^O is obtained as a weighted average of the efficiencies of the output subgroups.
End.





flowchart (1) and (2)

Let's examine the problem in general terms. In general, each DMU can be considered a network, over its entire period of activity, using homogeneous inputs but producing nonhomogeneous outputs. The DMUs are divided into p exclusive groups across all time periods (based on the explanations in the previous section), denoted by N_p. Each input is distributed among the produced output subgroups. The input share for an output that is not produced is considered as zero, meaning that, the inputs are not used for producing output subgroups that are not generated. In addition, in this study, it is assumed that all inputs are separable across all periods.

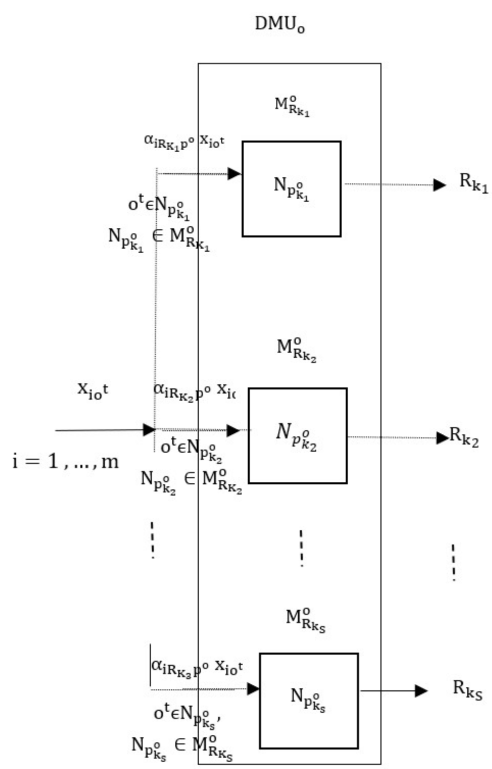


Figure 2: Structure of DMU_o with Nonhomogeneous Outputs across Different Time Periods

The situation depicted in Figure 2 shows that each network has as many subgroups as there are output subgroups, throughout its entire period of activity (in Figure 2, it is assumed that DMU_o has S output subgroups, and therefore, it has S subs). The members of the subs are the categories ($N_{p^o_{k_s}}$) that include DMU_{o^t} with the output subgroup R_{kS}. Figure 3 shows the structure DMU$_2$. It can be stated that in this study, we propose an approach in which the network performance is obtained from the weighted average of its sub-networks. In the following, the details of the proposed approach are presented, and each of the three stages is explained.

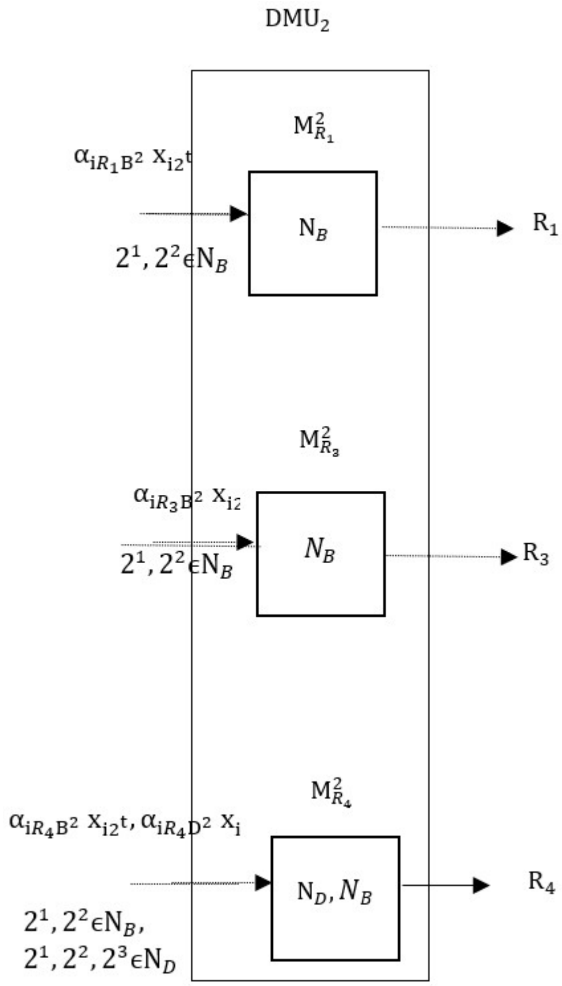


Figure 3: Structure of DMU_2 with Nonhomogeneous Outputs across Different Time Periods

4.1 Proposed Approach

As stated, the goal is to determine the overall efficiency of DMUs. On the basis of this, DMUs are categorized independently for the period on the basis of their outputs across all periods. Then, the input shares in the production of the outputs are determined; and a separate DEA model is executed for each output subgroup to obtain the efficiency of the output subgroups. The overall efficiency of each DMU is obtained using a convex combination of the efficiencies of the output subgroups, across all periods of activity. To achieve this goal, the details of the proposed approach are as follows:

Step 1: Calculating the Share of Inputs in Output Production

In this step, for each network (DMU_o under evaluation) or to be assessed, we determine the part of its inputs for each time period, that should be allocated to each of the output subgroups in the relevant and specific categories like N_{p^o} (categories including DMU_{o^t}). This share is denoted by $\alpha_{iR_kp^o}$. The goal of this resource allocation for output production is to select the best value of α that maximizes the network's efficiency. These shares are calculated by considering the cumulative efficiency of DMU_o under evaluation as a convex combination of the cumulative efficiencies of the output subgroups R_k related to the subs ($M^o_{R_K}$).

To calculate the efficiency rate of each network, we use a convex combination of the efficiencies of the output subgroups R_k across all periods of the DMU activity, under evaluation. In this method, we first calculate the input shares in output production. For this purpose, we use an input-oriented radial prediction model that assumes constant returns to scale (CRS), which can be generalized to variable returns to scale (VRS). The following model, based on Data Envelopment Analysis

(DEA), expresses this idea.

$$
\begin{aligned}
&\max \sum_{R_k \in L_o} W^o_{R_K} \frac{\sum_{N_{p^o} \in M^o_{R_K}} \sum_{o^t \in N_{Po}} \sum_{r \in R_k} u_r y_{ro^t}}{\sum_{N_{p^o} \in M^o_{R_K}} \sum_{o^t \in N_{Po}} \sum_i v_i \alpha_{iR_k p^o} x_{io^t}} \\
&\text{s.t} \\
&\sum_{R_k \in L_j} W^j_{R_K} \frac{\sum_{N_p \in M^j_{R_K}} \sum_{j^t \in N_p} \sum_{r \in R_k} u_r y_{rj^t}}{\sum_{N_p \in M^j_{R_K}} \sum_{j^t \in N_p} \sum_i v_i \alpha_{iR_k p} x_{ij^t}} \le 1, \quad \forall j \\
&\sum_{N_p \in M^j_{R_K}} \sum_{j^t \in N_p} \sum_{r \in R_k} u_r y_{rj^t} - \sum_{N_p \in M^j_{R_K}} \sum_{j^t \in N_p} \sum_i v_i \alpha_{iR_k p} x_{ij^t} \le 0, \quad \forall j, R_k \in L_j \\
&\sum_{j^t \in N_p} \sum_{r \in R_k} u_r y_{rj^t} - \sum_{j^t \in N_p} \sum_i v_i \alpha_{iR_k p} x_{ij^t} \le 0, \quad \forall j, R_k \in L_j,\ N_p \in M^j_{R_K} \\
&\sum_{r \in R_k} u_r y_{rj^t} - \sum_i v_i \alpha_{iR_k p} x_{ij^t} \le 0, \qquad \forall j^t \in N_p, R_k \in L_j, p \\
&\sum_{R_k \in L_{N_p}} \alpha_{iR_k p} = 1, \quad \forall i, p = A, B, \ldots, P \\
&a_{iR_k p} \le \alpha_{iR_k p} \le b_{iR_k p} \quad \forall i, R_k \in L_{N_P},\ p = A, B, ..., P \\
&u_r, v_i, \alpha_{iR_k p} \ \ge 0 \quad \forall i, R_k, p, r
\end{aligned}
\tag{1}
$$

Model (1) is feasible. To prove this, the coefficients u_r and v_i can be chosen as $u_r = 0, \forall r$ and $v_i = 1, \forall i$, respectively. Additionally, the coefficients $\alpha_{iR_k p}$ are selected such that both $\sum_{R_{k N_p}} \alpha_{iR_k p} = 1$ and $a_{iR_k p} \le \alpha_{iR_k p} \le b_{iR_k p}$ are satisfied. These two conditions create a bounded and convex feasible region, and as long as the upper and lower bounds do not conflict with the normalization constraint, at least one feasible value for $\alpha_{iR_k p}$ will exist. Therefore, the feasible region of $\alpha_{iR_k p}$ is non-empty, and the existence of a feasible solution and the well-definedness of the model are guaranteed. In particular, a simple choice to prove the feasibility of the model can be given as follows:

$$
\alpha_{iR_k p} = \frac{1}{|L_{N_p}|} \qquad \forall i, R_k \in L_{N_P}, p = A, B, \ldots, P
$$

In this case, we have:

$$a_{iR_kp} \leq \frac{1}{|L_{N_p}|} \leq b_{iR_kp}$$

Where $b_{iR_kp} \quad = 1$, $a_{iR_kp} = 0$.

In Model (1), N_p includes all DMUs that generate the same output, and $M^o_{R_K}$ denotes the subset of N_p (including the unit under evaluation) consisting of categories that produce output R_K.
Furthermore, v_i represents the weight of input variable i, and $\alpha_{iR_kp^o}$ denotes the share of input i allocated to each output subgroup R_K with in category p for the DMU under evaluation o at time t. The total allocated shares must sum to one, which is enforced through the corresponding constraint in Model (1). The term $\alpha_{iR_kp^o}x_{\mathrm{io}^t}$ represents the amount of input i consumed by output subgroup R_K in period t for the DMU under evaluation. Finally, u_r denotes the output weights, which are variable in the above fractional model. The outputs R_k produced over all operating periods of DMU_j belong to the *set* L_j. The parameters $a_{iR_kp} and b_{iR_kp} \quad$ are the lower and upper bounds for α_{iR_kp} , respectively.
In this section, weights denoted as $W^o_{R_K}$ are introduced, which are assigned to the output subgroup R_k in the unit under evaluation (in the network in question). Different weights can be defined to represent the importance of the output subgroups. To define various weights for illustrating the importance of output subgroups, we can use different approaches. One approach can be fundamentally based on the significance and share of the inputs consumed by each output subgroup. For example, an output subgroup that consumes the largest share of inputs can receive a higher weight. This approach can be logically sound from an accounting perspective, as the ratio of inputs allocated to each output subgroup can indicate the importance of that subgroup within the entire unit or system.

Another approach to defining the importance of output subgroups can be based on the proportion of total output produced by each subgroup. In other words, a subgroup that generates the highest value or output for the unit under evaluation can receive a higher weight. This approach can be logically sound from an economic or valuation perspective, as a subgroup that creates the most value for the unit indicates its importance within the entire unit. The weights ($w^o_{R_K}$) are actually used to calculate the overall efficiency rate. The method of assigning weights in this study is as follows: The ratio of inputs consumed by the output subgroup R_k for the unit under evaluation (network o), across all periods that have output R_k to the total inputs consumed for all output subgroups of the unit under evaluation (network o), across all periods is calculated.

$$W^o_{R_K} = \frac{\sum_{N_{p^o} \in M^o_{R_K}} \sum_{o^t \in N_{Po}} \sum_i v_i \alpha_{iR_k p^o} x_{io^t}}{\sum_{R_k \in L_o} \sum_{N_{p^o} \in M^o_{R_K}} \sum_{o^t \in N_{Po}} \sum_i v_i \alpha_{iR_k p^o} x_{io^t}} \tag{2}$$

In model (1):

1. First Constraint: The coefficients should be set such that, the output-to-input ratio does not exceed one.

2. Second constraint: The output-to-input ratio for each output subgroup, across all time periods and the corresponding categories, must not exceed one.

3. Third constraint: The output-to-input ratio for each output subgroup, within each category and across all time periods, must be less than or equal to one.

4. Fourth Constraint: The output-to-input ratio for each DMU_{o^t} should not exceed one.

5. Fifth Constraint: The total allocated share of inputs, in each category, must equal one.

6. Final Constraint: Sets an upper and lower bound for the variable α.

Model (1) is a nonlinear model. However, with the following transformations, model (1) can be linearized.

$$
\begin{aligned}
&\sum_{R_k \in L_{N_p}} \alpha_{iR_kp} = 1 \quad \Rightarrow v_i \sum_{R_k \in L_{N_p}} \alpha_{iR_kp} = v_i \\
&v_i \ \alpha_{iR_kp} = z_{iR_kp}, \sum_{R_k \in L_{N_p}} z_{iR_kp} = v_i, \\
&f = \frac{1}{\sum_i v_i x_{ij^t}} \Rightarrow \quad \mu_r = u_r f, \quad \gamma_{iR_kp} = f \ z_{iR_kp}, \quad \vartheta_i = f v_i
\end{aligned} \tag{3}
$$

To calculate the input shares of subs in output production, we use the following linear model by eliminating redundant constraints:

$$
\begin{aligned}
&\max \sum_{R_k \in L_o} \sum_{N_{p^o} \in M^o_{R_K}} \sum_{o^t \in N_{P^o}} \sum_{r \in R_k} \mu_r y_{ro^t} \\
&\text{s.t} \\
&\sum_{R_k \in L_o} \sum_{N_{p^o} \in M^o_{R_K}} \sum_{o^t \in N_{P^o}} \sum_i \gamma_{iR_kp^o} x_{io^t} = 1 \\
&\sum_{r \in R_k} \mu_r y_{rj^t} - \sum_i \gamma_{iR_kp} x_{ij^t} \leq 0, \quad \forall j^t, R_k \in L_j, p \\
&\sum_{R_k \in L_{N_p}} \gamma_{iR_kp} = \vartheta_i \quad \forall i, p = A, B, \ldots, P \\
&\vartheta_i \ a_{iR_kp} \leq \gamma_{iR_kp} \leq \vartheta_i \ b_{iR_kp} \quad \forall i, R_k \in L_{N_P}, p \\
&\mu_r, \vartheta_i, \gamma_{iR_kp} \ \geq \varepsilon \quad \forall i, p, r, R_k
\end{aligned} \tag{4}
$$

From the linear model (4), the corresponding values of α for each DMU, in all time periods and categories can be obtained. Inputs are not used for output subgroups that are not produced, meaning that, if the output subgroups are not produced, the corresponding value of α is defined as zero ($\alpha = 0$).

Step 2: Calculating the Efficiency Scores of Output Subgroups

After determining the α values for each DMU, across all time periods, the optimal shares of inputs corresponding to each output subgroup of the unit under evaluation are calculated as follows:

$$\tilde{x}^{k}_{io^t} = \alpha^{*}_{iR_kp^o} x_{io^t} \tag{5}$$

Where: $\alpha^{*}_{iR_kp^o} = \frac{\gamma^{*}_{iR_kp^o}}{\vartheta^{*}_{i}}$

Here, the CCR model is used for each output subgroup in all periods for the subs ($M^{o}_{R_K}$) of the network under evaluation; and the network efficiencies are calculated on the basis of the weighted average of the output R_k subgroups of the subs. In fact, the following model is used to calculate the efficiency of each output subgroup, of the subs, of the network under evaluation.

$$\begin{aligned}
&e^{o}_{R_{k^o}} = \max \sum_{N_{P^o} \in M^{o}_{R_{k0}}} w_{p^o} \frac{\sum_{o^t \in N_{P^o}} \sum_{r \in R_{k0}} \mu_r y_{ro^t}}{\sum_{o^t \in N_{P^o}} \sum_i \vartheta_i \tilde{x}^{k0}_{io^t}} \\
&\text{s.t} \\
&\sum_{N_p \in M^{j}_{R_{k0}}} w_{p^j} \frac{\sum_{j^t \in N_p} \sum_{r \in R_{k0}} \mu_r y_{rj^t}}{\sum_{j^t \in N_p} \sum_i \vartheta_i \tilde{x}^{k0}_{ij^t}}, \qquad \forall\ j\ for\ M^{j}_{R_{k^o}} \neq \emptyset \\
&\frac{\sum_{j^t \in N_p} \sum_{r \in R_{k0}} \mu_r y_{rj^t}}{\sum_{j^t \in N_p} \sum_i \vartheta_i \tilde{x}^{k0}_{ij^t}} \leq 1, \qquad \forall j\ for\ M^{j}_{R_{k^o}} \neq \emptyset,\ N_p \in M^{j}_{R_{k^o}} \\
&\frac{\sum_{r \in R_{k0}} \mu_r, y_{rj^t}}{\sum_i \vartheta_i \tilde{x}^{k0}_{ij^t}} \leq 1, \qquad \forall j^t \in N_P \text{ for } N_P \in M^{j}_{R_{k^o}}, p = 1, ..., P \\
&\mu_r, \vartheta_i \geq \varepsilon \qquad \forall i, r
\end{aligned} \tag{6}$$

In considering the importance of the subgroups based on output R_{k^o} across all periods, we can regard these values as weights w_{p^o}. From an accounting perspective, an appropriate method for selecting weights, is to consider the ratio of inputs consumed by DMU_{o^t} units within category N_{P^o} for producing the output subgroup R_{k^o} relative to all categories that, produce output R_{k^o} in the network under evaluation.

Therefore, the greater the share of resources allocated to the output subgroups within the categories, the higher the weight that the output subgroup receives. Given that, our model is input-oriented, for simplicity, we apply the first perspective, which is based on the ratio of cumulative inputs consumed by the DMU_{o^t} units within category N_{P^o} the defined weight is as follows, mathematically:

$$w_{p^o} = \frac{\sum_{o^t \in N_{P^o}} \sum_i \vartheta_i \tilde{x}_{io^t}^{k^0}}{\sum_{N_{P^o} \in M_{R_{k^o}}^o} \sum_{o^t \in N_{P^o}} \sum_i \vartheta_i \tilde{x}_{io^t}^{k^0}} \tag{7}$$

In this definition, $\sum_{o^t \in N_{P^o}} \sum_i \vartheta_i \tilde{x}_{io^t}^{k^0}$ represents the inputs consumed by DMU_o under evaluation (network) to produce the subgroup of outputs R_{k^0} over all periods in category N_{P^o}.
Moreover, $\sum_{N_{P^o} \in M_{R_{k^o}}^o} \sum_{o^t \in N_{P^o}} \sum_i \vartheta_i \tilde{x}_{io^t}^{k^0}$ denotes the total inputs consumed across all periods to produce the output subgroup R_{k^0} by DMU_o in all categories in which the output R_{k^0} is present.
Given the defined weights, the model can be rewritten as follows, which allows computing the efficiency of each output subgroup across all periods for the unit under evaluation. Model (8) calculates the efficiency of the output subgroup R_{k^o} for DMU_o.

$$\begin{aligned}
&\max \sum_{N_{P^o} \in M_{R_{k^o}}^o} \frac{\sum_{o^t \in N_{P^o}} \sum_{r \in R_{k^0}} \mu_r y_{ro^t}}{\sum_{N_{p^o} \in M_{R_{k^o}}^o} \sum_{o^t \in N_{P^o}} \sum_i \vartheta_i \tilde{x}_{io^t}^{k^0}} \\
&\sum_{N_P \in M_{R_{k^o}}^j} \frac{\sum_{j^t \in N_P} \sum_{r \in R_{k^0}} \mu_r y_{rj^t}}{\sum_{N_P \in M_{R_{k^o}}^j} \sum_{j^t \in N_P} \sum_i \vartheta_i \tilde{x}_{ij^t}^{k^0}} \qquad \forall j \ for \ M_{R_{k^0}}^j \neq \emptyset, \\
&\frac{\sum_{j^t \in N_p} \sum_{r \in R_{k^0}} \mu_r y_{rj^t}}{\sum_{j^t \in N_p} \sum_i \vartheta_i \tilde{x}_{ij^t}^{k^0}} \leq 1 \qquad \forall j, \ for \ M_{R_{k^0}}^j \neq \emptyset, \ N_P \in M_{R_{k^o}}^j \\
&\frac{\sum_{r \in R_{k^0}} \mu_r y_{rj^t}}{\sum_i \vartheta_i \tilde{x}_{ij^t}^{k^0}} \leq 1 \qquad \forall j^t \in N_P \ for \ N_P \in M_{R_{kO}} \\
&\mu_r, \vartheta_i \geq \varepsilon \qquad \forall i, r
\end{aligned} \tag{8}$$

The objective of Model (8) is to maximize the ratio of the output subgroup R_{kO} to the allocated inputs across all periods. At this stage, the inputs have already been distributed among the subgroups (in the preceding models), and now only the output-to-input ratio for each subgroup over all periods is evaluated. This model is solved only for those DMUs that produce the output R_{kO}.

The first constraint states that, for each DMU producing R_{kO}, the weighted amount of this output must not exceed the sum of its weighted inputs.

The second constraint specifies that the weighted ratio of R_{kO} to the total weighted inputs must not exceed one not only at the level of all units, but also within each group in which R_{kO} appears.

The final constraint indicates that, for all DMU_{o^t} that produces this output, the ratio of the output subgroup R_{kO} to the total inputs must be at most equal to one.

By applying an appropriate variable transformation, the nonlinear Model (8) is converted into the linear Model (9), which computes the efficiency of the output subgroup R_{kO} over all periods (i.e., the efficiency of the output subgroups R_{kO} for the unit o throughout its entire activity periods):

$$\max \sum_{N_{p^o} \in M^o_{R_{ko}}} \sum_{o^t \in N_{P^o}} \sum_{r \in R_{k0}} \mu_r y_{ro^t}$$

s.t

$$\sum_{N_{p^o} \in M^o_{R_{ko}}} \sum_{o^t \in N_{P^o}} \sum_{i} \vartheta_i \tilde{x}^{k0}_{io^t} = 1$$

$$\sum_{N_p \in M^j_{R_{ko}}} \sum_{j^t \in N_p} \sum_{r \in R_{k0}} \mu_r y_{rj^t} - \sum_{N_p \in M^j_{R_{ko}}} \sum_{j^t \in N_p} \sum_{i} \vartheta_i \tilde{x}^{k0}_{ij^t} \leq 0 \qquad \forall j \ for \ M^j_{R_{k0}} \neq \emptyset \tag{9}$$

$$\sum_{j^t \in N_p} \sum_{r \in R_{k0}} \mu_r y_{rj^t} - \sum_{j^t \in N_p} \sum_{i} \vartheta_i \tilde{x}^{k0}_{ij^t} \leq 0 \quad \forall j \ for \ M^j_{R_{k0}} \neq \emptyset, \ N_P \in M^j_{R_{ko}}$$

$$\sum_{r \in R_{k0}} \mu_r, y_{rj^t} \ - \sum_{i} \vartheta_i \tilde{x}^{k0}_{ij^t} \leq 0 \qquad \forall j^t \in N_P \ for \ N_P \in M_{R_{kO}}$$

$$\mu_r, \vartheta_i \geq \varepsilon \qquad \forall i, r$$

The model aims to maximize the total outputs belonging to the output subgroup R_{kO} for the DMU under evaluation.

The first constraint is a normalization condition that ensures the total weighted inputs of the DMU being assessed equal 1.

The second constraint compares DMUs that have the output R_{kO}; the weighted output of each DMU must not exceed its weighted input.

This requirement represents the input–output efficiency condition for the output subgroup R_{kO}.

In the third constraint, for each set N_P, the sum of weighted outputs for each subgroup R_{kO} must be less than or equal to the sum of the weighted inputs corresponding to that subgroup, which includes the DMU_{j^t}s.

The fourth constraint states that for each DMU_{j^t}, the total weighted outputs (i.e., outputs multiplied by their weights) for each subgroup R_{kO} in each set N_P must not exceed the total weighted inputs corresponding to that subgroup. In this constraint, μ_r represents the output weights and V_i represents the input weights. Finally, by removing redundant constraints, the simplified form of the model is obtained.

$$
\begin{aligned}
&\max \sum_{N_{p^o} \in M^o_{R_{k^o}}} \sum_{o^t \in N_{P^o}} \sum_{r \in R_{k0}} \mu_r y_{ro^t} \\
&\text{s.t} \\
&\sum_{N_{p^o} \in M^o_{R_{k^o}}} \sum_{o^t \in N_{P^o}} \sum_{i} \vartheta_i \tilde{x}^{k0}_{io^t} = 1 \qquad (10) \\
&\sum_{r \in R_{k0}} \mu_{r,} y_{rj^t} - \sum_{i} \vartheta_i \tilde{x}^{k0}_{ij^t} \leq 0 \qquad \forall j^t \in N_P \ for\ N_P \in M_{R_{kO}} \\
&\mu_{r,} \vartheta_i \geq \varepsilon \qquad \forall i, r
\end{aligned}
$$

Model (10) obtains the efficiency of the output subgroup R_{kO} for DMU_o at all times when it produces the output R_{kO}.

Step 3: Calculation of Overall Efficiency $\mathbf{e}^O$

In this step, we calculate the overall efficiency of the network, using the weighted average of the efficiency rates of the output subgroups, obtained from step two, according to Cook et.al [8].

$$
e^O = \sum_{R_k} e^o_{R_K} w^o_{R_K}
$$

Where $e^o_{R_K}$ is the efficiency of the output subgroup R_k for DMU_o, and $w^o_{R_K}$ is the logical weight (importance) of that subgroup. The overall efficiency is constructed from the combination of the efficiencies of the output subgroups; therefore, each output subgroup contributes to the final performance

Theorem 4.1. *A necessary and sufficient condition for a DMU to be efficient is that all of its output subgroups under evaluation are efficient.*

Proof.
Necessity Condition: If a Decision Making Unit (DMU) is efficient, it means that none of its output subgroups has an efficiency level below the optimal value. This is because the objective function value in the model is the weighted sum of the outputs of all subgroups:

$$Z = \max \sum_{N_{P^o} \in M^o_{R_K}} \sum_{o^t \in N_{P^o}} \sum_{r \in R_k} \mu_r y_{ro^t}$$

Now, suppose there exists an output subgroup whose efficiency is less than the optimal value (i.e., less than one). In that case, we can write:

$$\sum_{r \in R_{k0}} \mu_r y_{\mathrm{ro}^t} - \sum_i v_i \widetilde{x}^{k^0}_{\mathrm{io}^t} < 0$$

Under such circumstances, it becomes possible to increase the output of that subgroup or improve its output-to-input ratio while keeping the inputs fixed. This implies that there exists unused potential to enhance the overall efficiency of the DMU, which contradicts the initial assumption that the DMU is fully efficient. Therefore, the efficiency of a DMU requires that all of its output subgroups be efficient as well.
Sufficient Condition: Now suppose that all output subgroups are efficient, meaning that no subgroup of their outputs has an efficiency score less than one. If every output subgroup is efficient, then the value of the objective function reaches 1.

$$\sum_{r \in R_k} \mu_r y_{\mathrm{ro}^t} = \sum_i v_i \widetilde{x}^{k}_{\mathrm{io}^t}$$

This means that there is no possibility of improvement or optimization in any of the output subgroups. In other words, the weighted output of each subgroup is exactly equal to the weighted input of that subgroup.

Thus, it is neither less nor improvable; that is, the subgroup is efficient. Now, we sum these equalities over all subgroups:

$$\sum_{N_{P^o}\in M^o_{R_K}} \sum_{o^t\in N_{P^o}} \sum_{r\in R_k} \mu_r y_{ro^t} = \sum_{N_{P^o}\in M^o_{R_K}} \sum_{o^t\in N_{P^o}} \sum_{i} v_i \widetilde{x}^k_{\mathrm{io}^t}$$

That is, the total weighted outputs of all subgroups equal the total weighted inputs of all subgroups. Now we use the normalization constraint:

$$\sum_{N_{P^o}\in M^o_{R_K}} \sum_{o^t\in N_{P^o}} \sum_{i} v_i \widetilde{x}^k_{\mathrm{io}^t} = 1$$

Hence, the DMU is efficient. □

One of the most important advantages of the method proposed in this study is its ability to evaluate nonhomogeneous decision-making units within a multi-period setting. This feature enables a more precise analysis of performance in line with periodic fluctuations and organizational behavioral changes, thereby providing a more comprehensive picture of unit efficiency.The use of nonhomogeneous outputs across different time periods is beneficial in a variety of applications. For example, in production planning, analyzing outputs over multiple time intervals helps better understand changes in demand and market needs, leading to more optimal production decisions. In general, incorporating nonhomogeneous outputs across different time periods in fields such as banking, industry, and other organizations can contribute effectively to risk management, performance evaluation, and the improvement of operational processes. To validate the proposed method, a real-world case from the banking industry is examined in the following section, shedding light on the practical and applicable aspects of this approach.

5 Practical Example

In this section, the efficiency of 20 branches of a commercial bank in Iran is evaluated over four consecutive years, from 2019 to 2022, using the approach introduced in the previous section. The inputs and outputs considered in the analysis are as follows:

x_1: Administrative expenses

x_2 Personnel expenses
y_1: Qarz-al-Hasan loans
y_2: Partnership loans
y_3: Number of bank cheque payments
y_4: Demand deposit accounts
y_5: Short-term deposit accounts
This bank has two inputs, x_1 and x_2, which are homogeneous across all periods. However, the outputs y_1, y_2, y_3, y_4 and y_5 are nonhomogeneous. The heterogeneity in outputs arises from the fact that certain branches do not offer some services during particular time periods. For example, some branches may not issue loans in a given period, while others may introduce new loan products. Moreover, heterogeneity also exists across time periods—meaning that one or more branches may not be active during certain years. This may occur when new branches are established or when some branches are merged. The data for these 20 branches are presented in Table 3.

Table 3: Input and Output Information for 20 Bank Branches Under Study Across Four Periods

DMUs	**Periods**	**Input**		**Outputs**				
DMU	**Year**	x_1	x_2	y_1	y_2	y_3	y_4	y_5
n=1	2019	0.886	0.633	-	0.312	0.478	0.144	0.152
	2020	0.770	0.743	-	-	0.469	0.028	0.092
	2021	0.162	0.303	-	-	0.37	0.103	0.375
	2022	0.672	0.997	-	-	0.457	0.031	0.175
n=2	2019	0.469	0.400	-	-	0.472	0.128	0.362
	2020	0.253	0.437	-	-	0.154	0.031	0.034
	2021	0.359	0.383	-	-	0.88	0.274	0.106
	2022	0.320	0.387	-	-	0.035	0.028	0.156
n=3	2019	0.469	0.400	-	-	0.02	0.029	0.267
	2020	-	-	-	-	-	-	-
	2021	0.514	0.310	0.001	-	0.162	0.029	0.171
	2022	0.378	0.317	0.068	-	1	0.961	0.078
n=4	2019	0.305	0.217	0.023	-	0.053	0.03	0.201
	2020	0.401	0.515	0.004	0.132	-	0.002	0.017
	2021	0.471	0.505	-	-	0.076	0.029	0.288
	2022	0.251	0.164	0.002	0.202	-	0.005	0.056
n=5	2019	0.442	0.309	-	0.609	0.450	0.151	0.187
	2020	-	-	-	-	-	-	-
	2021	0.898	0.647	-	0.483	0.534	0.03	0.231
	2022	0.543	0.55	-	0.127	0.313	0.005	0.044
n=6	2019	0.222	0.245	-	-	0.142	0.03	0.043
	2020	0.09	0.318	-	0.313	0.617	0.121	0.242
	2021	0.459	0.363	0.002	-	0.05	0.017	0.062
	2022	-	-	-	-	-	-	-
n=7	2019	0.773	0.728	-	0.11	0.327	0.004	0.063
	2020	0.680	0.877	-	0.145	0.013	0.017	0.14
	2021	0.552	0.629	-	0.363	0.342	0.029	0.184
n=8	2019	0.331	0.304	0.001	0.193	-	0.004	0.05
	2020	-	-	-	-	-	-	-
	2021	0.072	0.301	0.001	-	0.026	0.003	0.024
	2022	0.222	0.234	-	-	0.103	0.016	0.196
n=9	2019	0.199	0.555	-	-	0.167	0.031	0.032
	2020	0.997	0.484	0.023	0.447	-	0.002	0.024
	2021	0.633	0.553	0.002	-	0.03	0.028	0.076
	2022	-	-	-	-	-	-	-
n=10	2019	0.651	0.570	0.008	-	0.044	0.004	0.072
	2020	0.327	0.400	-	0.357	0.049	0.028	0.264
	2021	0.241	0.440	0.008	1	-	0.016	0.122
	2022	-	-	-	-	-	-	-

As shown in Table 3, the outputs—as well as the bank branches ($DMUs$)—exhibit significant heterogeneity. Therefore, applying traditional models for efficiency evaluation does not yield reliable results. Under such conditions, it is necessary to employ the approach proposed in this study. Accordingly, the three steps described in the previous section are implemented sequentially. In the first step, consistent with the initial phase of the proposed method, the contribution of each input to the production of the outputs is determined. Based on the grouping structure presented in Table 2, and by solving model (4), the optimal values $\alpha^*_{iR_k\ p}$ are computed for all output subgroups of each DMU_{jt} within the categories N_p, where $p = A, B, C, D$. The linear programming models are solved using GAMS software, and the results obtained from running model (4) are reported by subgroup in Tables 4–7.

Table 4: $\alpha^*_{iR_kp}$ Values resulting from model (4)— N_A

$\mathbf{DMU_{j^t}}$	X1 K=1	X1 K=2	X1 K=3	X1 K=4	X2 K=1	X2 K=2	X2 K=3	X2 K=4
4^2	0.1	0.677	-	0.223	0.1	0.8	-	0.1
4^4	0.1	0.677	-	0.223	0.1	0.8	-	0.1
8^1	0.1	0.677	-	0.223	0.1	0.8	-	0.1
9^2	0.378	0.1	-	0.522	0.1	0.8	-	0.1
10^3	0.1	0.335	-	0.565	0.1	0.8	-	0.1
11^2	0.1	0.1	-	0.8	0.1	0.8	-	0.1
11^4	0.1	0.1	-	0.8	0.1	0.8	-	0.1
14^1	0.1	0.568	-	0.322	0.1	0.8	-	0.1
19^1	0.165	0.1	-	0.735	0.1	0.8	-	0.1
19^2	0.165	0.1	-	0.735	0.1	0.8	-	0.1
19^3	0.165	0.1	-	0.735	0.1	0.8	-	0.1
19^4	0.165	0.1	-	0.735	0.1	0.8	-	0.1
20^4	0.333	0.333	-	0.333	0.208	0.692	-	0.1

Table 5: $\alpha^{*}_{iR_kp}$ Values resulting from model (4)— N_B

$\mathbf{DMU_{jt}}$	X1 K=1	X1 K=2	X1 K=3	X1 K=4	X2 K=1	X2 K=2	X2 K=3	X2 K=4
1^2	-	0.8	0.1	0.1	-	0.214	0.686	0.1
7^2	-	0.8	0.1	0.1	-	0.412	0.285	0.303
7^3	-	0.8	0.1	0.1	-	0.412	0.285	0.303
7^4	-	0.8	0.1	0.1	-	0.412	0.285	0.303
5^1	-	0.8	0.1	0.1	-	0.444	0.456	0.1
5^3	-	0.8	0.1	0.1	-	0.444	0.456	0.1
5^4	-	0.8	0.1	0.1	-	0.444	0.456	0.1
6^2	-	0.8	0.1	0.1	-	0.213	0.515	0.272
10^2	-	0.8	0.1	0.1	-	0.657	0.1	0.243
11^3	-	0.8	0.1	0.1	-	0.596	0.105	0.229
12^1	-	0.8	0.1	0.1	-	0.632	0.1	0.268
12^4	-	0.8	0.1	0.1	-	0.632	0.1	0.268
13^3	-	0.8	0.1	0.1	-	0.294	0.128	0.578
14^3	-	0.543	0.1	0.357	-	0.33	0.1	0.57
14^4	-	0.543	0.1	0.357	-	0.33	0.1	0.57
15^2	-	0.8	0.1	0.1	-	0.321	0.305	0.374
16^2	-	0.671	0.229	0.1	-	0.317	0.1	0.583
16^4	-	0.671	0.229	0.1	-	0.317	0.1	0.583
17^2	-	0.8	0.1	0.1	-	0.270	0.1	0.63
17^4	-	0.8	0.1	0.1	-	0.270	0.1	0.63
18^3	-	0.1	0.1	0.8	-	0.1	0.1	0.8
20^1	-	0.333	0.333	0.333	-	0.8	0.1	0.1

Table 6: $\alpha^{*}_{iR_{k}p}$ Values resulting from model (4)— N_C

$\mathbf{DMU_{j^t}}$	X1 K=1	X1 K=2	X1 K=3	X1 K=4	X2 K=1	X2 K=2	X2 K=3	X2 K=4
1^2	-	-	0.9	0.1	-	-	0.1	0.9
1^3	-	-	0.9	0.1	-	-	0.1	0.9
1^4	-	-	0.9	0.1	-	-	0.1	0.9
2^1	-	-	0.1	0.9	-	-	0.709	0.291
2^2	-	-	0.1	0.9	-	-	0.709	0.291
2^3	-	-	0.1	0.9	-	-	0.709	0.291
2^4	-	-	0.1	0.9	-	-	0.709	0.291
3^1	-	-	0.1	0.9	-	-	0.1	0.9
4^3	-	-	0.1	0.9	-	-	0.1	0.9
6^1	-	-	0.1	0.9	-	-	0.58	0.42
8^4	-	-	0.1	0.9	-	-	0.1	0.9
9^1	-	-	0.1	0.9	-	-	0.115	0.885
11^1	-	-	0.1	0.9	-	-	0.118	0.882
13^4	-	-	0.1	0.9	-	-	0.1	0.9
16^3	-	-	0.1	0.9	-	-	0.1	0.9

Table 7: $\alpha^*_{iR_kp}$ Values resulting from model (4)— N_D

$\mathbf{DMU_{jt}}$	X1 K=1	X1 K=2	X1 K=3	X1 K=4	X2 K=1	X2 K=2	X2 K=3	X2 K=4
3^3	0.1	-	0.1	0.8	0.1	-	0.1	0.8
3^4	0.1	-	0.1	0.8	0.1	-	0.1	0.8
4^1	0.1	-	0.1	0.8	0.697	-	0.1	0.203
6^3	0.1	-	0.8	0.1	0.1	-	0.488	0.412
8^3	0.1	-	0.1	0.8	0.697	-	0.1	0.203
9^3	0.1	-	0.8	0.1	0.469	-	0.134	0.397
10^1	0.1	-	0.8	0.1	0.474	-	0.108	0.417
12^3	0.1	-	0.8	0.1	0.461	-	0.1	0.439
13^2	0.1	-	0.147	0.753	0.571	-	0.128	0.329
15^1	0.1	-	0.611	0.289	0.396	-	0.1	0.504
15^3	0.1	-	0.611	0.289	0.396	-	0.1	0.504
17^3	0.1	-	0.1	0.8	0.697	-	0.1	0.203
18^2	0.189	-	0.1	0.711	0.8	-	0.1	0.1
18^4	0.189	-	0.1	0.711	0.8	-	0.1	0.1
20^3	0.333	-	0.333	0.333	0.8	-	0.1	0.1

Tables 4 through 7 present the optimal values of $\alpha^*_{iR_k\ p}$, which are the results obtained from solving model (4) for the four categories $N_A N_B N_C$ and N_D . As previously explained, these coefficients represent the relative contribution of each input to the production of the outputs associated with subgroup R_K within each set N_P. Therefore, numerical variations in $\alpha^*_{iR_k\ p}$ directly reflect the structural differences among output groups as well as the behavioral heterogeneity of the decision-making units over the study period.

In each of these tables, the columns correspond to different input–output combinations (for the two inputs X_1 and X_2 and the four output subgroups $k = 1, \ldots, 4$), while each row represents a specific decision-making unit in a specific period. For example, the notation 20^3 refers to Branch 20 of the bank in the third year (i.e., 2021). The symbol "–" indicates that the corresponding input does not contribute to the production of that output subgroup. For instance, in Table 3, DMU_8 in the third year (2021) has no output in subgroup R_2; therefore, no input is considered for its production. Moreover, a larger value of $\alpha_{iR_k\ p}$ indicates

a higher contribution of the input to the production of the corresponding output subgroup, whereas a smaller value of $\alpha_{iR_k\ p}$ signifies lower dependence of that subgroup on the given input.The present approach is fundamentally different from the method proposed by Cook et al. [8]. First, it examines output heterogeneity across multiple time periods; second, it accounts for the behavioral heterogeneity of the decision-making units throughout the study horizon.
Next, the second stage—namely, the computation of the corresponding weights and the efficiency scores of the output subgroups for the bank branches—is carried out. For this purpose, Equation (2) is applied. The results obtained from implementing Equation (2) are presented in Table 8 below.

Table 8: $W^0_{R_{kj}}$ Values resulting from model (2)

DMU	**K=1**	**K=2**	**K=3**	**K=4**
1	-	0.113	0.559	0.32
2	-	-	0.449	0.551
3	0.073	-	0.1	0.827
4	0.108	0.345	0.053	0.494
5	-	0.45	0.45	0.1
6	0.044	0.078	0.52	0.358
7	-	0.508	0.24	0.253
8	0.184	0.322	0.056	0.439
9	0.195	0.237	0.094	0.475
10	0.218	0.431	0.084	0.266
11	0.051	0.59	0.052	0.307
12	0.163	0.413	0.108	0.316
13	0.188	0.07	0.112	0.631
14	0.042	0.505	0.058	0.395
15	0.257	0.105	0.225	0.414
16	-	0.295	0.132	0.573
17	0.178	0.233	0.1	0.49
18	0.228	0.031	0.1	0.641
19	0.101	0.784	-	0.114
20	0.34	0.474	0.085	0.101

In this table, the weights of the output subgroups for the bank branches are calculated across the entire period of their operation. For some output subgroups, no value of $W^o_{R_K}$ is reported. The reason is that the corresponding branch does not produce that specific output subgroup. In the third step of the proposed approach, the overall efficiency of the banking network is computed by taking the weighted average of the efficiency scores of the output subgroups over the four-year time horizon. The results of calculating the efficiency scores for each subgroup, as well as the final overall efficiency, are presented in Table 9.

Table 9: Subunit scores ($\mathbf{e^O}$) and overall efficiency scores

DMU	K=1	K=2	K=3	K=4	Overall Score
1	-	0.47	0.043	0.192	0.14
2	-	-	0.249	0.196	0.22
3	0.263	-	0.393	0.193	0.218
4	0.047	0.15	0.034	0.163	0.14
5	-	0.378	0.207	0.547	0.318
6	0.014	1	0.074	0.233	0.201
7	-	0.141	0.106	0.174	0.141
8	0.003	0.184	0.152	0.171	0.143
9	0.019	0.398	0.045	0.035	0.119
10	0.019	0.623	0.029	0.298	0.355
11	0.37	0.506	0.391	0.303	0.431
12	0.014	0.128	0.049	0.13	0.101
13	0.0002	1	0.047	0.063	0.115
14	1	0.156	0.056	0.171	0.191
15	0.003	0.47	0.07	0.132	0.12
16	-	0.415	0.157	0.299	0.314
17	0.019	0.232	0.029	0.086	0.102
18	0.15	0.661	0.085	0.255	0.226
19	0.058	0.472	-	0.066	0.384
20	0.776	0.111	0.008	0.155	0.333

In Table 9, for example, the overall efficiency of DMU_1 is obtained using the formula:

$$e^O = \sum_{R_k} e^o_{R_K} w^o_{R_K} \tag{11}$$

For this purpose, the weights obtained in Table 8 were multiplied by the efficiency of the corresponding output subgroups obtained in Table 9 and finally aggregated.

$$\begin{aligned} e^1 &= 0.47 \times 0.113 + 0.043 \times 0.559 + 0.192 \times 0.32 \\ &= 0.14 \end{aligned} \tag{12}$$

Table 9 clearly demonstrates that the proposed method is capable not only of estimating the efficiency score of each output subgroup independently, but also of providing an aggregated index of overall performance. According to the findings, Branch 14 achieves the highest efficiency in the first output subgroup; Branches 6 and 13 perform best in the second subgroup; and Branches 3 and 5 attain the highest efficiency levels in the third and fourth subgroups, respectively. The overall efficiency scores are reported in the sixth column of Table 9. As shown, Branch 11 exhibits the best four-year performance, whereas Branch 12 demonstrates the weakest performance. Furthermore, based on the result of Theorem (4-1), none of the branches can be considered fully efficient, since no branch is simultaneously efficient across all output subgroups.The main advantage of this study lies in its ability to incorporate two types of heterogeneity simultaneously: first, heterogeneity in outputs, and second, heterogeneity across time periods. In the existing data envelopment analysis (DEA) literature, prior research has generally followed two separate directions: some studies have focused on output heterogeneity without accounting for temporal dynamics and changes in unit participation, while others have extended multi-period approaches but assumed homogeneous decision-making units, overlooking variations in outputs as well as the entry and exit of units.
In contrast, the present study:

- enables the entry and exit of decision-making units across different periods—an aspect largely overlooked in previous research;

- provides a more accurate understanding of the production mechanism by employing a nonhomogeneous network structure and parallel sub-networks;

- allows not only for the computation of overall efficiency but also for the independent evaluation of the efficiency of output sub-networks, thereby deepening managerial analysis;
- from an applied perspective, is suitable for performance assessment in real-world systems such as banking networks, manufacturing industries, and dynamic operational environments.

Overall, this study bridges the gap between research on heterogeneity and multi-period analyses and offers a comprehensive and practical framework—representing a notable advancement in the data envelopment analysis literature.

6 Conclusion and Suggestions

This study proposes a novel network-based data envelopment analysis (DEA) framework that enables the efficiency evaluation of decision-making units (DMUs) operating in environments characterized by output heterogeneity and temporal discontinuity. While most existing models have addressed these challenges separately, the proposed approach integrates both dimensions simultaneously, thereby providing a more comprehensive and realistic representation of unit performance.One of the main strengths of the model lies in its network structure and the decomposition of outputs into independent subnetworks, which allows efficiency analysis not only at the overall system level but also at the level of its constituent components. This feature facilitates a deeper understanding of the internal operational mechanisms and offers a valuable analytical tool for managers and decision-makers. The empirical application of the model to real banking sector data indicates that both the efficiency scores and the ranking patterns of the units differ substantially from those obtained using conventional methods, highlighting the model's ability to capture the inherent complexities of real-world operational processes.Overall, the findings demonstrate that the proposed framework effectively fills an existing gap in the efficiency analysis literature, particularly in dynamic and multi-period environments, and can serve as a flexible and reliable analytical tool. Owing to its general formulation, the model is not confined to a specific industry and can be applied across various sectors facing heterogeneity and temporal variation.

Despite these contributions, several avenues for future research remain open. These include extending the model to settings in which inputs are also nonhomogeneous or time-dependent, as well as developing the framework to accommodate undesirable outputs alongside desirable ones. Moreover, incorporating non-radial approaches or directional distance functions may lead to a more precise identification of inefficiency sources. Investigating more complex network structures, such as series–parallel configurations, and conducting comparative empirical studies between the proposed model and machine learning– or artificial intelligence–based methods could also provide promising directions for future research.

References

[1] Ahn, Y. H., & Min, H. (2014). Evaluating the multi-period operating efficiency of international airports using data envelopment analysis and the Malmquist productivity index. *Journal of Air Transport Management*, 39, 12-22.

[2] Asadi, F., Kordrostami, S., Amirteimoori, A., & Jahani Sayyad Noveiri, M. (2023). An inverse multi-period FDH model with undesirable outputs. *Croatian Operational Research Review* , 14(2), 111-124.

[3] Barat M, Tohidi G, Sanei, M. (2018). DEA for nonhomogeneous mixed networks. *Asia Pac Manag Rev* .

[4] Barat, M., Tohidi, G., Sanei, M., & Razavyan, S. (2019). Data envelopment analysis for decision making unit with nonhomogeneous internal structures: An application to the banking industry. *Journal of the Operational Research Society*, 70(5), 760-769.

[5] Castelli, L., Pesenti, R., and Ukovich, W. (2004). DEA-like models for the efficiency evaluation of hierarchically structured units. *European journal of operational research*, 154(2), 465-476.

[6] Charnes, A., Cooper, W.W., Rhodes, E. (1978). Measuring the efficiencies of DMUs, Eur. *J. Oper. Res.* 2 (6) 429–444.

[7] Chen, M., Ang, S., Yang, F., & Jiang, L. (2021). Efficiency evaluation of non-homogeneous DMUs with inconsistent input quality. Computers &*Industrial Engineering* , 158, 107418.

[8] Cook WD, Harrison J, Imanirad R, Rouse P, Zhu J (2013) Data envelopment analysis with nonhomogeneous DMUs.*Oper Res* , 61(3):666–676.

[9] Cook WD, Harrison J, Rouse P, Zhu J (2012) Relative efficiency measurement: the problem of a missing output in a subset of decision making units.*Euro J Oper Res.* 220(1):79–84

[10] Dastani, F., Tohidi, G., Sanei, M., Lotfi, F. H., & Razavyan, S. (2025). Multi-period efficiency evaluation of nonhomogeneous decision-making units: Assessing efficiency before, during, and after COVID-19. *Decision Science Letters*, 14(4), 861–876. https://doi.org/10.5267/j.dsl.2025.8.003

[11] Du, J., Chen, Y., & Huo, J. (2015). DEA for non-homogenous parallel networks. Omega, 56, 122-132.

[12] Dyson, R. G., Allen, R., Camanho, A. S., Podinovski, V. V., Sarrico, C. S., & Shale, E. A. (2001). Pitfalls and protocols in DEA.*European Journal of operational research* , 132(2), 245-259.

[13] Esmaeilzadeh, A., & Matin, R. K. (2019). Multi-period efficiency measurement of network production systems. *Measurement*, 134, 835-844.

[14] Gazori-Nishabori, A., Khalili-Damghani, K., & Hafezalkotob, A. (2023). A Nash bargaining game data envelopment analysis model for measuring efficiency of dynamic multi-period network structures.*Journal of Modelling in Management* , 18(2), 241-284.

[15] Günaydın, Y., Correia, A., & Kozak, M. (2022). Comparing efficiency in all-inclusive and bed and breakfast hotel businesses: a

multi-period data envelopment analysis in Turkey. *European Journal of Management and Business Economics* , 31(4), 439-452.

[16] Haas, D. A., & Murphy, F. H. (2003). Compensating for nonhomogeneity in decision-making units in data envelopment analysis. *European Journal of Operational Research* , 144(3), 530-544.

[17] Hajiagha, S. H. R., Hashemi, S. S., Mahdiraji, H. A., & Azaddel, J. (2015). Multi-period data envelopment analysis based on Chebyshev inequality bounds. *Expert systems with applications* , 42(21), 7759-7767.

[18] Hosseinzadeh Lotfi, F., Allahviranloo, T., Shafiee, M., & Saleh, H. (2023). Data Envelopment Analysis. In Supply Chain Performance Evaluation: Application of Data Envelopment Analysis (pp. 179-241). *Cham: Springer International Publishing.*

[19] Huang, C. W., Chiu, Y. H., Tu, C. H., Luo, Z. Y., & Wang, Z. (2016). Using the nonhomogeneous frontier two-stage DEA model to assess the efficiencies of expense utilization and operation of the Taiwanese hotel industry. *International Transactions in Operational Research*, 23(6), 1067-1087.

[20] Kao, C. (2016). Network data envelopment analysis: Foundations and extensions (Vol. 240). *springer.*

[21] Kao, C., & Liu, S. T. (2014). Multi-period efficiency measurement in data envelopment analysis: The case of Taiwanese commercial banks. *Omega* , 47, 90-98.

[22] Kyrgiakos, L. S., Kleftodimos, G., Vlontzos, G., & Pardalos, P. M. (2023). A systematic literature review of data envelopment analysis implementation in agriculture under the prism of sustainability. *Operational Research* , 23(1), 7.

[23] Li, W., Liang, L., Cook, W. D., & Zhu, J. (2016). DEA models for non-homogeneous DMUs with different input configurations. *European Journal of Operational Research*, 254(3), 946-956.

[24] Li Y, Lei X, Morton A (2019) Performance evaluation of non- homogeneous hospitals: the case of Hong Kong hospitals. *Health Care Manag Sci* 22(2):215–228.

[25] Lin, S., Rouse, P., Wang, Y. M., Lin, L., & Zheng, Z. Q. (2023). Performance measurement of nonhomogeneous Hong Kong hospitals using directional distance functions.*Health Care Management Science* , 1-14.

[26] Moghaddas, Z., Oukil, A., & Vaez-Ghasemi, M. (2022). Global multi-period performance evaluation–New model and productivity index.*RAIRO-Operations Research* , 56(3), 1503-1521.

[27] Mohsenirad, S., & Triantis, K. (2025). Testing for heterogeneity in data envelopment analysis.*Annals of Operations Research* , 351(2), 1537–1558. https://doi.org/10.1007/s10479-024-06460-0

[28] Narayanan, E., binti Ismail, W. R., & bin Mustafa, Z. (2022). A data-envelopment analysis-based systematic review of the literature on innovation performance.*Heliyon* , 8(12).

[29] Panwar, A., Olfati, M., Pant, M., & Snasel, V. (2022). A review on the 40 years of existence of data envelopment analysis models: Historic development and current trends. *Archives of Computational Methods in Engineering*, 29(7), 5397-5426.

[30] Singh, S., & Ranjan, P. (2018). Efficiency analysis of non-homogeneous parallel sub-unit systems for the performance measurement of higher education.*Annals of Operations Research* , 269, 641-666.

[31] Yan, Q., Zhao, F., Wang, X., Yang, G., Baležentis, T., & Streimikiene, D. (2019). The network data envelopment analysis models for non-homogenous decision making units based on the sun network structure.*Central European journal of operations research* , 27, 1221-1244.

[32] Zha, Y., Song, A., Xu, C., & Yang, H. (2013). Dealing with missing data based on data envelopment analysis and halo effect.*Applied Mathematical Modelling* , 37(9), 6135-6145.

[33] Zhu, J. (2022). DEA under big data: Data enabled analytics and network data envelopment analysis.*Annals of Operations Research* , 309(2), 761-783.

[34] 35- Zhu, W., Yu, Y., & Sun, P. (2018). Data envelopment analysis cross-like efficiency model for non-homogeneous decision-making units: The case of United States companies' low-carbon investment to attain corporate sustainability.*European Journal of Operational Research* , 269(1), 99-110.

[35] Zotti, G. (2025). Regional differences in Italian school efficiency: A conditional DEA approach. *Journal of Productivity Analysis*, 64(3), 457–491. https://doi.org/10.1007/s11123-025-00780-4

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