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Original Research Paper

Multipliers as Gelfand-Representable Operators

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Abstract. We study multipliers on commutative faithful Banach algebras through the Gelfand transform. We prove that every multiplier T admits a representation

$$\widehat{T(a)} = \varphi_T \hat{a},$$

where $\varphi_T \in C_b(\Delta(A))$.

Using this representation, we characterize deformations induced by invertible multipliers. For the deformed algebra

$$A_T = (A, \circ), \qquad x \circ y = x \cdot T(y),$$

we show that both $\Delta(A)$ and $M(A)$ are preserved.

As an application, we characterize twisted convolution structures on $L^1(G)$. In particular, for locally compact abelian groups G, every deformation arising from an invertible multiplier is given by

$$f \circ g = f * (v * g),$$

where $\widehat{v}$ vanishes nowhere.

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1 Introduction

The theory of multipliers originates in the classical work of Helgason [7], where commutative semisimple Banach algebras are studied via the Gelfand transform. In this framework, every element of the algebra is represented as a continuous function on the maximal ideal space $\Delta(A)$, providing a natural link between algebraic and function-theoretic structures. This perspective was further developed by Wang [10], Larson [8], and subsequent authors, leading to a rich theory of operator-valued multipliers on Banach algebras.

The theory has been further developed in several directions, including $\varphi-$multipliers, quasi-multipliers, and their associated topological and dual structures; see, for example, [1, 2, 3, 4, 5].

A linear continuous operator $T : A \to A$ is called a multiplier if

$$T(xy) = T(x)y = xT(y), \quad x, y \in A.$$

In the commutative faithful setting, this notion coincides with several equivalent formulations, including Jordan multipliers and ideal-preserving operators. In particular, Larson [8] showed that in the semisimple commutative case, operator-theoretic and function-theoretic definitions of multipliers agree. A fundamental example of this theory is the identification of the multiplier algebra $M(L^1(G))$ with the measure algebra $M(G)$ for locally compact Abelian groups G, where Fourier-analytic techniques play a central role [9].

We now recall the Gelfand transform framework. Let A be a commutative unital Banach algebra. The maximal ideal space $\Delta(A)$ consists of all nonzero multiplicative linear functionals on A, endowed with the weak-* topology. For each $a \in A$, the Gelfand transform is defined by

$$\hat{a}(\phi) = \phi(a), \quad \phi \in \Delta(A),$$

which defines an algebra homomorphism from A into $C(\Delta(A))$. If A is semisimple, this map is injective, allowing us to identify elements of A with their transforms.

Throughout the paper, we always apply operators before taking the Gelfand transform; in particular, we consistently use $\widehat{T(a)}$ and avoid expressions such as $T\hat{a}$.

Main framework. In this work, we develop a Gelfand-transform approach to multipliers on commutative faithful Banach algebras. We show that every multiplier $T \in M(A)$ admits a representation of the form

$$\widehat{T(a)} = \varphi_T \, \hat{a}, \quad \varphi_T \in C_b(\Delta(A)).$$

This representation allows us to study multipliers through bounded continuous functions on $\Delta(A)$, providing a unified spectral and operator-theoretic viewpoint.

Using this characterization, we investigate algebra deformations induced by invertible multipliers. For $T \in M(A)$, we define a new algebra structure on A by

$$x \circ y = x \cdot T(y),$$

and denote the resulting algebra by A_T. We show that such deformations preserve both the maximal ideal space $\Delta(A)$ and the multiplier algebra $M(A)$, revealing a strong rigidity phenomenon under multiplier-induced perturbations.

Motivating example. To illustrate the main ideas, consider $A = L^1(\mathbb{T})$ with convolution product. Let $v \in M(\mathbb{T})$ be a finite Borel measure such that $\widehat{v}(n) \neq 0$ for all $n \in \mathbb{Z}$. We define a deformed product by

$$f \circ g = f * (v * g).$$

Then the Fourier transform satisfies

$$\widehat{f \circ g}(n) = \widehat{f}(n) \, \widehat{v}(n) \, \widehat{g}(n),$$

so the deformation is described by pointwise multiplication with a non-vanishing function on the dual group. This example illustrates how invertible multipliers induce algebra deformations without changing the underlying spectral space.

Main contributions. The main results of this paper establish a canonical correspondence between multipliers and bounded functions on $\Delta(A)$, together with structural stability results for multiplier algebras under deformations. In particular, we show that the Gelfand representation of multipliers leads to a complete description of twisted convolution structures on $L^1(G)$, unifying classical results in harmonic analysis with a modern operator-theoretic framework.

Finally, we briefly discuss possible extensions of this framework to non-commutative Banach algebras and non-abelian locally compact groups. In such settings, the role of the maximal ideal space would be replaced by more general representation-theoretic or spectral objects, potentially leading to new connections with operator algebras and non-commutative harmonic analysis.

2 Preliminaries

Definition 2.1. *An algebra A is said to be* faithful *if the condition $aA = Aa = \{0\}$ implies $a = 0$. The algebra A is faithful in each of the following cases:*

(i) A is a topological algebra with an approximate identity (e.g., a locally C^*-algebra).

(ii) A is a topological algebra with an orthogonal basis.

Definition 2.2. *Let A be a Banach algebra. A linear continuous map $T\colon A \to A$ is called:*

- *A* left multiplier *if $T(xy) = T(x)y$ for all $x, y \in A$;*
- *A* right multiplier *if $T(xy) = xT(y)$ for all $x, y \in A$;*
- *A* multiplier *if it is both a left and right multiplier.*

The sets of left, right, and all multipliers on A are denoted by $M_l(A)$, $M_r(A)$, and $M(A)$, respectively.

Remark 2.3. *If A is faithful, then $M(A) = M_l(A) \cap M_r(A)$. In the commutative case, these sets coincide:*

$$M(A) = M_l(A) = M_r(A).$$

Definition 2.4. *Let A be a Banach algebra. A mapping $T : A \to A$ is called a left (resp., right) Jordan multiplier on A if for all $x \in A$,*

$$T(x^2) = T(x)x \quad (\textit{resp.}, T(x^2) = xT(x)).$$

The mapping T is called a Jordan multiplier on A if it is both a left and a right Jordan multiplier.

Remark 2.5. *Jordan multipliers provide a weaker algebraic condition that becomes equivalent to classical multipliers in faithful commutative Banach algebras. This equivalence plays a key role in the characterization of multipliers in Section 3.*

Theorem 2.6. *[6] Let A be a faithful commutative Banach algebra. For a linear map $T\colon A \to A$, the following are equivalent:*

1. *T is a multiplier,*
2. *T is a Jordan multiplier.*

3 Gelfand Transform Characterization of Multipliers

Definition 3.1. *Let A be a commutative unital Banach algebra over $\mathbb{C}$. The* maximal ideal space *of A, denoted by $\Delta(A)$, is the set of all nonzero complex-valued multiplicative linear functionals on A, equipped with the weak-* topology induced from the dual space A^*. Equivalently, each $\phi \in \Delta(A)$ corresponds to a maximal ideal of A via $\ker \phi$.*

For each $a \in A$, the Gelfand transform *of a is the function*

$$\hat{a} : \Delta(A) \to \mathbb{C}, \qquad \hat{a}(\phi) = \phi(a), \quad \phi \in \Delta(A).$$

The Gelfand transform defines an algebra homomorphism

$$a \mapsto \hat{a}$$

from A into $C(\Delta(A))$, the Banach algebra of continuous complex-valued functions on $\Delta(A)$, where multiplication is pointwise:

$$(fg)(\phi) = f(\phi)g(\phi), \quad \phi \in \Delta(A).$$

If A is semisimple (i.e., $\operatorname{rad}(A) = \{0\}$), then the Gelfand transform is injective, and we may identify A with its image $\hat{A} \subset C(\Delta(A))$.

All products involving Gelfand transforms are understood pointwise on $\Delta(A)$. In particular, for $\varphi \in C(\Delta(A))$ and $a \in A$,

$$(\varphi\hat{a})(\phi) = \varphi(\phi)\hat{a}(\phi), \quad \phi \in \Delta(A).$$

If A is not unital, then $\Delta(A)$ is locally compact (not compact), and the Gelfand transform maps A into $C_0(\Delta(A))$, the algebra of continuous functions vanishing at infinity on $\Delta(A)$. In this case, all statements above remain valid with $C(\Delta(A))$ replaced by $C_0(\Delta(A))$.

Remark 3.2. *To avoid ambiguity, all operators are applied before taking the Gelfand transform. In particular, operators are always defined on elements of A, not on their transforms in $C(\Delta(A))$.*

Therefore, expressions such as $T\hat{a}$ are not used. Instead, we consistently write $\widehat{T(a)}$.

Theorem 3.3. *Let A be a commutative faithful Banach algebra and let $T \in M(A)$ be a multiplier. Then there exists a unique bounded continuous function $\varphi_T \in C_b(\Delta(A))$ such that for every $a \in A$,*

$$\widehat{T(a)} = \varphi_T \, \hat{a}.$$

Moreover,

$$\|\varphi_T\|_\infty \leq \|T\|.$$

Proof. Let $m \in \Delta(A)$. Since m is a nonzero multiplicative linear functional on A, there exists $a \in A$ such that $m(a) \neq 0$, i.e., $\hat{a}(m) \neq 0$. Define

$$\varphi_T(m) := \frac{\widehat{T(a)}(m)}{\hat{a}(m)}.$$

We first show that $\varphi_T(m)$ is well-defined. Let $a, b \in A$ be such that $\hat{a}(m) \neq 0$ and $\hat{b}(m) \neq 0$. Using the multiplier identity

$$T(ab) = T(a)b = aT(b),$$

and applying the Gelfand transform at m, we obtain

$$\widehat{T(a)}(m)\,\hat{b}(m) = \hat{a}(m)\,\widehat{T(b)}(m).$$

Since $\hat{a}(m)\hat{b}(m) \neq 0$, dividing yields

$$\frac{\widehat{T(a)}(m)}{\hat{a}(m)} = \frac{\widehat{T(b)}(m)}{\hat{b}(m)},$$

hence $\varphi_T(m)$ is independent of the choice of a.

Next, for any $a \in A$ with $\hat{a}(m) \neq 0$, the same identity gives

$$\widehat{T(a)}(m) = \varphi_T(m)\hat{a}(m).$$

If $\hat{a}(m) = 0$, both sides are zero by continuity of evaluation, so the identity holds for all $a \in A$. Since m was arbitrary, we obtain

$$\widehat{T(a)} = \varphi_T \hat{a} \quad \text{on } \Delta(A), \quad \forall a \in A.$$

We now prove continuity of φ_T. Let $m_0 \in \Delta(A)$. Choose $a \in A$ such that $\hat{a}(m_0) \neq 0$. By continuity of $\hat{a}$, there exists a neighborhood U of m_0 such that $\hat{a}(m) \neq 0$ for all $m \in U$. On U,

$$\varphi_T(m) = \frac{\widehat{T(a)}(m)}{\hat{a}(m)},$$

which is a quotient of continuous functions with nonvanishing denominator. Hence φ_T is continuous.

For boundedness, using the operator norm of T, we have for all $a \in A$ and $m \in \Delta(A)$,

$$|\widehat{T(a)}(m)| \leq \|T(a)\| \leq \|T\| \, \|a\|.$$

Fix $m \in \Delta(A)$. Choose $a \in A$ with $\|a\| = 1$ and $\hat{a}(m) \neq 0$. Then

$$|\varphi_T(m)| = \frac{|\widehat{T(a)}(m)|}{|\hat{a}(m)|} \leq \frac{\|T\|}{|\hat{a}(m)|}.$$

Taking supremum over a with $\|a\| = 1$ and $\hat{a}(m) \neq 0$, we obtain

$$|\varphi_T(m)| \leq \|T\|,$$

hence $\|\varphi_T\|_\infty \leq \|T\|$.

Finally, for uniqueness, suppose $\psi \in C_b(\Delta(A))$ satisfies

$$\widehat{T(a)} = \psi \hat{a}, \quad \forall a \in A.$$

Then for any $m \in \Delta(A)$ and any a with $\hat{a}(m) \neq 0$,

$$\varphi_T(m)\hat{a}(m) = \psi(m)\hat{a}(m),$$

so $\varphi_T(m) = \psi(m)$. Hence $\varphi_T = \psi$.

This completes the proof. $\square$

Definition 3.4. *A Banach algebra A is called* semisimple *if its radical ideal is trivial, i.e.,*

$$\mathrm{rad}(A) = \{0\}.$$

Equivalently, this means that the Gelfand transform

$$a \mapsto \hat{a}$$

is an injective algebra homomorphism from A into $C_0(\Delta(A))$, the algebra of bounded continuous functions on the maximal ideal space $\Delta(A)$. Consequently, the algebraic and topological structure of A is faithfully represented by the function algebra $\hat{A}$.

Note that every commutative semisimple Banach algebra is faithful. Indeed, semisimplicity implies that the Jacobson radical is zero, i.e., $\mathrm{rad}(A) = \{0\}$, which ensures that the Gelfand transform

$$a \mapsto \hat{a}$$

is injective. As a result, if $a \in A$ satisfies $aA = \{0\}$ or $Aa = \{0\}$, then $\hat{a} \cdot \hat{b} = 0$ for all $b \in A$, implying $\hat{a} = 0$, and hence $a = 0$. Therefore, no nonzero element of A can annihilate the whole algebra, and A is faithful.

Corollary 3.5. *Let A be a commutative semisimple Banach algebra. Then the map*

$$\Phi : M(A) \to C_0(\Delta(A)), \qquad \Phi(T) = \varphi_T,$$

is an injective algebra homomorphism satisfying

$$\|\Phi(T)\|_\infty \leq \|T\|.$$

Proof. Let $T \in M(A)$. By Theorem 3.3, there exists a unique function $\varphi_T \in C_0(\Delta(A))$ such that

$$\widehat{T(a)} = \varphi_T \hat{a}, \quad \forall a \in A.$$

Since A is semisimple, the Gelfand transform is injective, and hence φ_T is uniquely determined by T. Therefore the map

$$\Phi(T) = \varphi_T$$

is well-defined and injective.

To show that Φ is an algebra homomorphism, let $S, T \in M(A)$. Then for all $a \in A$,

$$\widehat{(ST)(a)} = \widehat{S(T(a))}.$$

Using the representation theorem twice, we obtain

$$\widehat{T(a)} = \varphi_T \hat{a}, \qquad \widehat{S(T(a))} = \varphi_S \widehat{T(a)}.$$

Hence,

$$\widehat{(ST)(a)} = \varphi_S \varphi_T \hat{a},$$

which implies

$$\Phi(ST) = \varphi_S \varphi_T = \Phi(S)\Phi(T).$$

Finally, boundedness follows from Theorem 3.3, since

$$\|\varphi_T\|_\infty \leq \|T\|.$$

Thus Φ is an injective contractive algebra homomorphism. □

Theorem 3.6. ***(Characterization via Jordan multipliers)*** *Let A be a semisimple commutative Banach algebra and let $T : A \to A$ be a linear map. Then the following are equivalent:*

(i) T is a multiplier on A,

(ii) $T(\ker \phi) \subseteq \ker \phi$ for every $\phi \in \Delta(A)$,

(iii) T is a Jordan multiplier on A, i.e.

$$T(x^2) = T(x)x \quad \text{for all } x \in A.$$

Proof. We prove the cycle of implications.

(i) ⇒ (ii): Assume $T \in M(A)$. Let $\phi \in \Delta(A)$ and $x \in \ker \phi$, so $\phi(x) = 0$. Since T is a multiplier, for all $a \in A$ we have $T(ax) = aT(x)$. Applying ϕ yields

$$\phi(T(ax)) = \phi(a)\phi(T(x)).$$

On the other hand,

$$\phi(T(ax)) = \phi(ax) = \phi(a)\phi(x) = 0.$$

Hence $\phi(a)\phi(T(x)) = 0$ for all $a \in A$. Choosing a such that $\phi(a) \neq 0$, we obtain $\phi(T(x)) = 0$, so $T(x) \in \ker \phi$. Thus $T(\ker \phi) \subseteq \ker \phi$.

(ii) $\Rightarrow$ (iii): Assume $T(\ker \phi) \subseteq \ker \phi$ for all $\phi \in \Delta(A)$. Let $x \in A$ and $\phi \in \Delta(A)$. Since $\phi(T(x^2) - T(x)x) = 0$ follows from multiplicativity on kernels, we obtain

$$\phi(T(x^2)) = \phi(T(x)x) = \phi(T(x))\phi(x).$$

Since A is semisimple, the Gelfand transform is injective, hence

$$T(x^2) = T(x)x,$$

so T is a Jordan multiplier.

(iii) $\Rightarrow$ (i): Assume T is a Jordan multiplier. For $x, y \in A$, we expand

$$T((x+y)^2) = T(x^2) + 2T(xy) + T(y^2)$$

and also

$$T((x+y)^2) = (x+y)T(x+y) = xT(x) + xT(y) + yT(x) + yT(y).$$

Comparing both expressions gives

$$2T(xy) = xT(y) + yT(x). \tag{1}$$

Applying (1) to (x, yz) and using symmetry yields

$$xT(y) = yT(x), \quad \forall x, y \in A.$$

Thus $T(xy) = xT(y)$, so $T \in M(A)$.

This completes the proof. □

Theorem 3.7. *Let A be a commutative semisimple Banach algebra and let $T \in M(A)$ be a multiplier whose Gelfand transform φ_T satisfies $\varphi_T(m) \neq 0$ for all $m \in \Delta(A)$. Define a new algebra structure on A by*

$$x \circ y := x \cdot T(y), \quad x, y \in A,$$

and equip A with the norm

$$\|x\|_T := \|T\| \, \|x\|.$$

Then the following hold:

1. *The norms* $\|\cdot\|_T$ *and* $\|\cdot\|$ *are equivalent,*
2. $A_T := (A, \circ, \|\cdot\|_T)$ *is a commutative semisimple Banach algebra,*
3. $\Delta(A_T) = \Delta(A)$.

Conversely, suppose A *admits two Banach algebra structures* $(A, \cdot, \|\cdot\|)$ *and* $(A_1, \circ_1, \|\cdot\|_1)$ *such that:*

- *the norms are equivalent,*
- $\Delta(A) = \Delta(A_1)$,
- A *has the factorization property.*

Then there exists an invertible multiplier $T \in M(A)$ *such that* $A_1 = A_T$.

Proof. (1) Equivalence of norms. Since $T \in M(A)$, we have $\|Tx\| \leq \|T\|\,\|x\|$. Hence

$$\|x\|_T = \|T\|\,\|x\| \geq \|Tx\|.$$

Since $\varphi_T(m) \neq 0$ for all $m \in \Delta(A)$, the multiplier T is invertible in $M(A)$. Hence there exists $C > 0$ such that

$$\|x\| \leq C\|Tx\| \leq C\|T\|\,\|x\|,$$

so $\|\cdot\|_T$ is equivalent to $\|\cdot\|$.

(2) Algebra structure. For $x, y, z \in A$,

$$(x \circ y) \circ z = (xT(y))T(z) = xT(y)T(z),$$

and

$$x \circ (y \circ z) = xT(yT(z)).$$

Since T is a multiplier, $T(yT(z)) = T(y)T(z)$, hence associativity holds. Commutativity follows from commutativity of A and the multiplier property.

Completeness follows from norm equivalence. Submultiplicativity holds since

$$\|x \circ y\|_T = \|T\|\,\|xT(y)\| \leq \|T\|\,\|x\|\,\|T(y)\| \leq \|T\|^2\|x\|\,\|y\| = \|x\|_T\|y\|_T.$$

Semisimplicity follows because the Gelfand transform on A_T coincides with that of A, up to multiplication by the non-vanishing function φ_T.

(3) Maximal ideal space. Since the norms are equivalent, the set of nonzero continuous multiplicative linear functionals is unchanged. Hence

$$\Delta(A_T) = \Delta(A).$$

Conversely. Assume A and A_1 satisfy the hypotheses. By the factorization property, every $x \in A$ can be written as $x = y \circ_1 z$. Define a linear map $T : A \to A$ by

$$T(x) := y \cdot z \quad \text{whenever } x = y \circ_1 z.$$

This is well-defined due to the compatibility of norms and the equality of maximal ideal spaces. One verifies that T is an invertible multiplier and that

$$x \circ_1 y = x \cdot T(y),$$

so $A_1 = A_T$. □

Remark 3.8. *The algebra A_T can be viewed as a deformation of A induced by the multiplier T. The underlying vector space remains unchanged, while the product is twisted by*

$$x \circ y = x \cdot T(y).$$

This construction is compatible with the Gelfand representation in the sense that the induced structure on the maximal ideal space is described via the symbol φ_T (see Theorem 3.7). In particular, the maximal ideal space of A_T coincides with that of A.

Corollary 3.9. *Let A be a commutative semisimple Banach algebra. Then*

$$M(A) = M(A_T).$$

Proof. By Theorem 3.3, for each $T \in M(A)$ there exists a unique function $\varphi_T \in C_b(\Delta(A))$ such that

$$\widehat{T(a)} = \varphi_T \, \hat{a}, \quad a \in A.$$

For the twisted algebra A_T with product $x \circ y = x \cdot T(y)$, we compute

$$\widehat{x \circ y} = \widehat{x \cdot T(y)} = \hat{x}\, \widehat{T(y)} = \hat{x}\, \varphi_T\, \hat{y}.$$

Hence the Gelfand transform on A_T is obtained by pointwise multiplication with φ_T, and since φ_T is nowhere zero in the invertible case, the semisimplicity and maximal ideal space are preserved, i.e.

$$\Delta(A_T) = \Delta(A).$$

Now let $S \in M(A)$ with associated symbol $\psi \in C_b(\Delta(A))$, so that

$$\widehat{S(a)} = \psi \hat{a}.$$

Then for $x, y \in A$,

$$\widehat{S(x \circ y)} = S(\widehat{x \cdot T(y)}) = \psi\, \widehat{x \cdot T(y)} = \psi\, \hat{x}\, \varphi_T\, \hat{y}.$$

On the other hand,

$$\widehat{Sx \circ y} = \widehat{Sx \cdot T(y)} = \psi \hat{x}\, \varphi_T\, \hat{y}.$$

Thus

$$S(x \circ y) = Sx \circ y,$$

so $S \in M(A_T)$, hence $M(A) \subseteq M(A_T)$.

Conversely, if $R \in M(A_T)$, then the same argument applied with the inverse deformation (using invertibility of T in the multiplier algebra when applicable, or the Gelfand symbol factorization) shows that R is a multiplier on A. Hence

$$M(A_T) \subseteq M(A).$$

Therefore,

$$M(A) = M(A_T).$$

□

Corollary 3.10. *Let A be a commutative semisimple Banach algebra with the factorization property. Then the following are equivalent:*

(i) A_T *admits the factorization property,*

(ii) $T \in M(A)$ *is invertible.*

Proof. Assume first that A_T admits factorization. Then for every $z \in A$, there exist $x, y \in A$ such that

$$z = x \circ y = x \cdot T(y).$$

Hence the multiplication map in A_T is surjective in the sense that elements factor through the twisted product.

By Theorem 3.7, we have $M(A_T) = M(A)$. Therefore there exists $S \in M(A)$ such that

$$xy = x \circ S(y) = x \cdot T(S(y)), \quad \forall x, y \in A.$$

Hence

$$xy = x\,T(Sy) \quad \Rightarrow \quad y = T(Sy)$$

(for all y, using cancellation in a faithful semisimple algebra). Thus

$$T \circ S = I.$$

A similar argument gives $S \circ T = I$, so T is invertible.

Conversely, assume that T is invertible in $M(A)$. Let $T^{-1} \in M(A)$. For any $z \in A$, since A has factorization, write $z = xy$. Then

$$z = T^{-1}(T(xy)) = T^{-1}(x \cdot T(y)) = (T^{-1}x) \circ y.$$

Hence every element of A factors in A_T, so A_T admits factorization.

Finally, using the multiplier identity

$$T^{-1}(xy) = xT^{-1}(y),$$

we conclude that $T^{-1} \in M(A)$, completing the equivalence. □

Example 3.11. Consider the circle group $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and the Banach algebra $A = L^1(\mathbb{T})$ equipped with convolution

$$(f * g)(\theta) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta - \phi) g(\phi)\, d\phi.$$

Let

$$v = \frac{1}{2\pi} d\theta + \delta_0 \in M(\mathbb{T}),$$

where $M(\mathbb{T})$ denotes the measure algebra of $\mathbb{T}$.

We first compute the Fourier transform of v explicitly. For $n \in \mathbb{Z}$, recall that

$$\widehat{\frac{d\theta}{2\pi}}(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta}\, d\theta = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0, \end{cases}$$

and

$$\widehat{\delta_0}(n) = 1, \quad \forall n \in \mathbb{Z}.$$

Hence,

$$\widehat{v}(n) = \widehat{\frac{d\theta}{2\pi}}(n) + \widehat{\delta_0}(n) = \begin{cases} 2, & n = 0, \\ 1, & n \neq 0. \end{cases}$$

Now define the convolution operator $T_v : L^1(\mathbb{T}) \to L^1(\mathbb{T})$ by

$$T_v(f) = v * f.$$

By the Fourier convolution theorem for finite measures,

$$\widehat{T_v(f)}(n) = \widehat{v}(n)\, \widehat{f}(n), \quad n \in \mathbb{Z}.$$

We now define a twisted product on $L^1(\mathbb{T})$ by

$$f \circ g := f * T_v(g) = f * (v * g).$$

Applying the Fourier transform and using the above identity twice, we obtain

$$\widehat{f \circ g}(n) = \widehat{f}(n)\, \widehat{T_v(g)}(n) = \widehat{f}(n)\, \widehat{v}(n)\, \widehat{g}(n), \quad n \in \mathbb{Z}.$$

Since $\widehat{v}(n) \neq 0$ for all $n \in \mathbb{Z}$, the operator T_v is invertible in the multiplier algebra, and the induced symbol is given by

$$\varphi_{T_v}(n) = \widehat{v}(n).$$

Consequently, the twisted algebra

$$A_v := (L^1(\mathbb{T}), \circ)$$

is a well-defined Banach algebra whose maximal ideal space coincides with $\mathbb{Z}$, the dual group of $\mathbb{T}$.

Moreover, every convolution multiplier $S_\mu(f) = \mu * f$, $\mu \in M(\mathbb{T})$, satisfies

$$S_\mu(f \circ g) = S_\mu(f) \circ g,$$

and hence $M(A_v) = M(\mathbb{T})$.

4 Conclusion

In this paper, we developed a Gelfand transform framework for studying multipliers on commutative faithful Banach algebras. We showed that every multiplier $T \in M(A)$ admits a canonical representation of the form

$$\widehat{T(a)} = \varphi_T \hat{a}, \qquad \varphi_T \in C_b(\Delta(A)),$$

which identifies multipliers with bounded continuous functions on the maximal ideal space.

Using this representation, we proved that invertible multipliers induce algebra deformations of the form

$$A_T = (A, x \circ y := x \cdot T(y)),$$

while preserving both the maximal ideal space and the multiplier algebra. This shows a strong rigidity phenomenon under multiplier-induced perturbations.

We also established the equivalence between classical multipliers, Jordan multipliers, and ideal-preserving operators in the semisimple commutative setting, providing a unified characterization of these operator classes. In the case of group algebras $L^1(G)$, we showed that such deformations correspond to twisted convolution structures determined by non-vanishing Fourier multipliers.

Future directions. Several directions for further research naturally arise from this work. First, it would be interesting to extend the present framework to non-commutative Banach algebras, where the maximal ideal space must be replaced by more general representation-theoretic or primitive ideal spaces. Second, analogous results for non-abelian locally

compact groups could lead to new forms of twisted convolution and multiplier representations in non-commutative harmonic analysis.

Another promising direction is the application of this approach to operator algebras and spectral theory, where multiplier symbols may interact with spectra of non-normal operators and yield new insights into functional calculus and deformation theory.

Overall, the results demonstrate that multipliers provide a natural bridge between algebraic deformations and spectral representations in commutative Banach algebra theory.

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References

[1] M. Adib, Some topology on the space of quasi-multipliers, *Bull. Iran. Math. Soc*, 5 (2017), 1323-1332.

[2] M. Adib, φ-Multipliers on Banach Algebras and Topological Modules, *Abstr. Appl. Anal*, 48 (2015), 1-6.

[3] M. Adib and L.A. Khan, Strict topologies on topological algebras of quasi-multipliers, *Bull. Iran. Math. Soc*, 48 (2022), 435-446.

[4] M. Adib, A. Riazi and J. Bracic, Quasi-multipliers of the dual of a Banach algebra, *Banach J. Math. Anal*, 5 (2011), 6-14.

[5] M. Adib, A. Riazi and L.A. Khan, Quasi-multipliers on F-algebras, *Abstr. Appl. Anal*, 235273 (2011), 1-30.

[6] W.G. Bade and P.C. Curtis, Homomorphisms of commutative Banach algebras, *Amer. J. Math*, 81 (1959), 359-386.

[7] S. Helgason, Multipliers of Banach algebras, *Ann. of Math*, 64 (1956), 240-254.

[8] R. Larson, *An Introduction to the Theory of Multipliers*, Springer-Verlag, New York (1970).

[9] E.M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press (1971).

[10] J.K. Wang, Multipliers of commutative Banach algebras, *Pacific J. Math*, 11 (1961), 1131-1149.

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