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Original Research Paper

Application of a Modified Cubic B-spline Differential Quadrature Method to a Class of Nonlinear Hyperbolic PDEs with Nonlocal Boundary Conditions

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Abstract. This paper introduces an efficient numerical method for solving a class of second-order nonlinear hyperbolic PDEs with nonlocal boundary conditions. The proposed approach begins by transforming the governing equation into a system of first-order PDEs. Next, the Crank–Nicolson finite difference method is applied to derive the time semi-discrete formulation of the problem. Subsequently, a fully discrete scheme is developed using an improved cubic B-spline and its corresponding differential quadrature method. The method enables the simultaneous approximation of the solution and its time derivative. The stability and convergence of the time semi-discrete form are rigorously investigated using the energy method. Finally, the accuracy and performance of the proposed method are evaluated through numerical experiments on several test problems, with comparisons made against existing methods in the literature.

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1 Introduction

In recent years, studies have shown that hyperbolic PDEs, combined with integral conditions instead of traditional boundary conditions, can model a wide range of physical phenomena [14, 38, 40]. This class of equations has applications in various fields, including studies in thermoelasticity [13], the analysis of heterogeneity in chemical systems [12], food industry [27], some biological processes [18], and plasma physics phenomena [39].

In this research, we consider a nonlinear class of hyperbolic PDEs characterized by the following form:

$$\frac{\partial^2 u}{\partial t^2} + \lambda \frac{\partial u}{\partial t} = \mu \Delta u + f(x,t,u), \quad x \in \Omega = [a,b], \quad t \in (0,T], \tag{1}$$

subject to the initial conditions

$$u(x,0) = \phi_1(x), \quad x \in \Omega, \tag{2}$$

$$\frac{\partial u}{\partial t}(x,0) = \phi_2(x), \quad x \in \Omega, \tag{3}$$

and nonlocal boundary conditions

$$\alpha_1 u(a,t) + \beta_1 \frac{\partial u}{\partial x}(a,t) = g_1(t) + \int_a^b \psi_1(x) u(x,t) dx, \quad t \in (0,T], \tag{4}$$

$$\alpha_2 u(b,t) + \beta_2 \frac{\partial u}{\partial x}(b,t) = g_2(t) + \int_a^b \psi_2(x) u(x,t) dx, \quad t \in (0,T], \tag{5}$$

where f, ϕ_1, ϕ_2, g_1, g_2, ψ_1 and ψ_2 are known functions; λ, μ, α_1, α_2, β_1 and β_2 are real constants with $\lambda \geq 0$ and $\mu > 0$. Here, u represents the exact solution, which is an unknown function to be determined. Also, the nonlinear source function $f(x,t,u)$ satisfies the following Lipschitz condition with respect to the third variable:

$$|f(x,t,u) - f(x,t,v)| \leq \mathcal{L}|u - v|, \tag{6}$$

where $\mathcal{L} > 0$ is the Lipschitz constant.

In general, investigating the existence and uniqueness of solutions for this class of equations is challenging due to the nonlocal boundary conditions. However, for certain special cases, results have been established.

For instance in [9], the author proved the existence and uniqueness of a classical solution and derived its Fourier representation for the equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \Delta u + f(x,t), \quad x \in \Omega, \quad t \in (0,T], \\ u(x,0) = \phi_1(x), \quad x \in \Omega, \\ \dfrac{\partial u}{\partial t}(x,0) = \phi_2(x), \quad x \in \Omega, \\ u(a,t) = g_1(t), \quad t \in (0,T], \\ \displaystyle\int_a^b u(x,t)dx = g_2(t), \quad t \in (0,T]. \end{cases}$$

Furthermore, in [21], the existence and uniqueness of solution for equation

$$\begin{cases} \tau \dfrac{\partial^2 u}{\partial t^2} + \lambda \dfrac{\partial u}{\partial t} = \mu \Delta u + f(x,t,u), \quad x \in \Omega, \quad t \in (0,T], \\ u(x,0) = \phi_1(x), \quad x \in \Omega, \\ \dfrac{\partial u}{\partial t}(x,0) = \phi_2(x), \quad x \in \Omega, \\ \dfrac{\partial u}{\partial x}(a,t) = g_1(t), \quad t \in (0,T], \\ \displaystyle\int_a^b u(x,t)dx = g_2(t), \quad t \in (0,T], \end{cases} \tag{7}$$

was established using the Rothe time-discretization method for $\tau > 0$ and $\lambda \geq 1$.

So far, numerous numerical methods have been proposed to solve specific cases of the hyperbolic PDE (1) with initial and nonlocal boundary conditions (2)-(5), some of which we will discuss in the following section. These approaches can be broadly classified into spectral, meshfree, finite difference/element, method of lines, and semi-analytical methods.

For instance, in the category of spectral methods, Bhrawy et al. [11] developed an efficient spectral collocation method based on the Legendre polynomials for solving linear one- and two-dimensional wave equations with nonlocal boundary conditions. Their approach employs the Legendre-Gauss-Lobatto quadrature rule to handle the integral boundary conditions, reduces the problem to a system of ordinary differential equations (ODEs), and solves the resulting system using a fourth-order

implicit Runge-Kutta method. Bin Jebreen [23] proposed a wavelet-Galerkin approach that employs interpolating scaling functions for the numerical solution of one-dimensional hyperbolic PDEs subject to non-local conservation constraints. Other spectral-type techniques include the shifted Legendre tau method [38] and the Bernstein Ritz-Galerkin method [45].

In the class of meshfree methods, Dehghan and Shokri [16] implemented a radial basis function (RBF) collocation scheme for approximating solutions to linear one-dimensional hyperbolic PDEs with integral boundary conditions. Their methodology employs a θ-weighted scheme for temporal discretization and various RBF types for spatial discretization, reducing the problem to an algebraic system. Yao [44] designed a novel numerical iterative scheme based on the reproducing kernel method (RKM) for solving the nonlinear hyperbolic PDE (7), providing rigorous error bounds.

Among finite difference and finite element methods, Dehghan [14] constructed several finite difference schemes for hyperbolic PDEs with an integral condition, with the core innovation of transforming the non-local condition into an equivalent standard boundary condition. Khuri and Sayfy [29] derived a finite element method (FEM) based on cubic B-spline basis functions. Ramezani et al. [36] introduced a hybrid finite difference-spectral scheme that integrates FDM efficiency with Fourier spectral accuracy. Dehghan and Lakestani [15] developed an operational matrix method using cubic B-spline scaling functions. Abbas et al. [1] proposed a cubic trigonometric B-spline-based collocation scheme combined with FDM, and Rostamy et al. [37] presented a finite element approach.

For method of lines and semi-discrete approaches, Shakeri and Dehghan [40] proposed a method of lines combining central difference approximations for spatial discretization with an explicit Runge-Kutta $(4, 5)$ formula for temporal integration. This technique separates spatial and temporal discretizations, offering flexibility in choosing different schemes for each domain.

Regarding semi-analytical methods, Mohyud-Din et al. [31] introduced a homotopy perturbation method (HPM)-based semi-analytical scheme. Moosavi et al. [33] presented a reduced differential trans-

form method (RDTM) and proved its convergence with an error bound. Turkyilmazoglu [42] developed a power series expansion method.

The differential quadrature method (DQM) is a robust numerical approach for solving ODEs and PDEs [10, 41, 46]. DQM generalizes the principle of integral quadrature by approximating derivatives as weighted sums of function values at discrete nodal points within the domain. A critical aspect of this method lies in computing the weighting coefficients for these approximations. To determine these weighting coefficients, DQM employs diverse basis functions, including orthogonal polynomials [6, 24, 32], radial basis functions [25, 26], trigonometric basis functions [41, 17], and cubic B-splines [4, 5, 7, 8], to construct derivative operators with high accuracy.

In this paper, we propose a hybrid numerical method that combines the Crank–Nicolson finite difference scheme with an improved B-spline-based DQM, as introduced in [7], to solve the second-order nonlinear hyperbolic PDE (1) numerically. The overall framework consists of two main stages. First, we reformulate PDE (1) as a system of first-order PDEs and discretize it in time using the Crank–Nicolson finite difference scheme. Then, by applying the DQM from [7] to the resulting time-discrete system, we perform spatial discretization. To handle the integral boundary conditions, we employ the composite Simpson's rule for numerical integration. This process yields the fully discrete form of the problem, which is then solved iteratively at each time step by solving a system of linear algebraic equations. Moreover, the efficiency of the proposed method is significantly enhanced by employing the Richardson extrapolation technique to accelerate the convergence in the time domain. A key advantage of this approach is its ability to simultaneously approximate both the solution function and its first-order time derivative within a unified numerical framework.

To summarize, the main contributions of this work are as follows: (i) the development of a modified B-spline–based DQM specifically adapted for second-order nonlinear hyperbolic PDE (1) with nonlocal boundary conditions; (ii) the simultaneous numerical approximation of the unknown function u and its first-order time derivative $v = \frac{\partial u}{\partial t}$ within a unified framework; and (iii) a rigorous convergence analysis in the time direction accompanied by numerical verification of the theoretical

second-order accuracy. These features distinguish the proposed method from existing approaches and highlight its effectiveness for solving nonlocal hyperbolic problems.

The structure of the paper is outlined below:

- In Section 2, a semi-discrete scheme is developed using the Crank–Nicolson finite difference method, and its stability and convergence are analyzed via the energy method.
- Section 3 begins by introducing differential quadrature formulas constructed using modified cubic B-spline basis functions, followed by the derivation of a fully-discrete scheme for the problem using the DQM. Finally, the initial approximation values required by the method are determined based on the structure of the developed algorithm.
- Section 4 presents numerical experiments that illustrate the effectiveness of the proposed approach.
- The concluding remarks are provided in Section 5.

2 Time Semi-Discrete Scheme

For the time-discretization of the second-order nonlinear hyperbolic PDE (1), we first use the auxiliary function $v = \dfrac{\partial u}{\partial t}$ to convert (1) into the following system of PDEs:

$$\begin{cases} \dfrac{\partial u}{\partial t} = v, \\ \dfrac{\partial v}{\partial t} + \lambda v = \mu \Delta u + f(x,t,u). \end{cases} \tag{8}$$

Consider a uniform partition of the time domain $[0,T]$ into m subintervals $[t_{n-1}, t_n]$, defined by discrete points $t_n = n\Delta t$, where $\Delta t = \frac{T}{m}$ and $n = 0, 1, \ldots, m$. Now, applying the Crank-Nicolson finite difference

scheme to (8) at each time step t_n leads to the following semi-discrete formulation in the time direction:

$$\begin{cases} \dfrac{u^{n+1}-u^{n-1}}{2\Delta t} = \dfrac{1}{2}\Big[v^{n+1}+v^{n-1}\Big] + \mathcal{R}_1^n, \\ \dfrac{v^{n+1}-v^{n-1}}{2\Delta t} + \dfrac{\lambda}{2}\Big[v^{n+1}+v^{n-1}\Big] = \dfrac{\mu}{2}\Big[\Delta u^{n+1}+\Delta u^{n-1}\Big] \\ \qquad\qquad\qquad\qquad\qquad\qquad +f(x,t_n,u^n)+\mathcal{R}_2^n, \end{cases} \tag{9}$$

for $n = 1, 2, \ldots, m-1$ where the grid functions u^n and v^n are defined respectively as $u^n = u(x, t_n)$ and $v^n = v(x, t_n)$, and the local truncation errors $\mathcal{R}_1^n$ and $\mathcal{R}_2^n$ satisfy

$$|\mathcal{R}_1^n| \le \mathcal{C}_1\Delta t^2, \quad |\mathcal{R}_2^n| \le \mathcal{C}_2\Delta t^2,$$

for some positive constants $\mathcal{C}_1$ and $\mathcal{C}_2$. Omitting the local truncation errors $\mathcal{R}_1^n$ and $\mathcal{R}_2^n$ in (9), we obtain the following semi-discrete formulation:

$$\begin{cases} \dfrac{U^{n+1}-U^{n-1}}{2\Delta t} = \dfrac{1}{2}\Big[V^{n+1}+V^{n-1}\Big], \\ \dfrac{V^{n+1}-V^{n-1}}{2\Delta t} + \dfrac{\lambda}{2}\Big[V^{n+1}+V^{n-1}\Big] = \dfrac{\mu}{2}\Big[\Delta U^{n+1}+\Delta U^{n-1}\Big] \\ \qquad\qquad\qquad\qquad\qquad\qquad +f(x,t_n,U^n), \end{cases} \tag{10}$$

for $n = 1, 2, \ldots, m-1$ where $U^n = U(\mathbf{x}, t_n)$ and $V^n = V(\mathbf{x}, t_n)$ denote the numerical approximations of the exact solutions u^n and v^n, respectively, at time level n.

2.1 Analysis of stability and convergence

Here, we first define some functional spaces and present several useful lemmas. Then, we analyze the stability and convergence of the time-discrete scheme (10) using the energy method.

We define the following functional spaces:

$$H^1(\Omega) = \left\{ f \in L^2(\Omega), \nabla f \in L^2(\Omega) \right\},$$

$$H_0^1(\Omega) = \left\{ f \in H^1(\Omega), f\,|_{\partial\Omega} = 0 \right\}.$$

The inner product in $L^2(\Omega)$ is given by

$$(f,g) = \int_{\Omega} f g d\Omega,$$

with the associated norm defined as

$$\|f\| = (f,f)^{\frac{1}{2}}.$$

Lemma 2.1 (Grönwall-Type Discrete Inequality)**.** *[35] Consider two nonnegative real sequences $\{\varkappa_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$, and a nonnegative constant F. Suppose that for every integer $n \geq 1$, $\varkappa_n$ satisfies the following inequality*

$$\varkappa_n \leq F + \sum_{m=0}^{n-1} \gamma_m \varkappa_m.$$

Then, for all $n \geq 1$, we have

$$\varkappa_n \leq F \exp\left(\sum_{m=0}^{n-1} \gamma_m\right).$$

Lemma 2.2 (Poincaré Inequality)**.** *[35] Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded open set. Then, there exists a constant $\mathcal{C}_\Omega > 0$, depending on Ω, such that for all $f \in H_0^1(\Omega)$, the following inequality holds:*

$$\|f\| \leq \mathcal{C}_\Omega \|\nabla f\|.$$

Theorem 2.3. *Let $\widetilde{U}^n$ and $\widetilde{V}^n$ be the numerical approximations of U^n and V^n, respectively, where $U^n, \widetilde{U}^n \in H_0^1(\Omega)$ and $V^n, \widetilde{V}^n \in H_0^1(\Omega)$. Suppose that U^n and V^n satisfy equation (10). Then, for a sufficiently small time step size, the time semi-discrete scheme (10) is stable, and the following inequalities hold:*

$$\|\rho_1^m\| \leq \widetilde{\mathcal{C}}_4 \sqrt{\frac{3}{2\mu}\Big(\|\rho_2^0\|^2 + \|\rho_2^1\|^2 + 2\mu\Big[\|\nabla\rho_1^0\|^2 + \|\nabla\rho_1^1\|^2\Big]\Big)}\, \mathrm{e}^{2\mathcal{L}T\widetilde{\mathcal{C}}_3},$$

$$\|\rho_2^m\| \leq \sqrt{3\Big(\|\rho_2^0\|^2 + \|\rho_2^1\|^2 + 2\mu\Big[\|\nabla\rho_1^0\|^2 + \|\nabla\rho_1^1\|^2\Big]\Big)}\, \mathrm{e}^{2\mathcal{L}T\widetilde{\mathcal{C}}_3},$$

where $\rho_1^n = U^n - \widetilde{U}^n$ and $\rho_2^n = V^n - \widetilde{V}^n$, and $\widetilde{\mathcal{C}}_3, \widetilde{\mathcal{C}}_4$ are positive constants.

Proof. From the definitions of ρ_1^n and ρ_2^n, it is clear that $\rho_1^n \in H_0^1(\Omega)$ and $\rho_2^n \in H_0^1(\Omega)$. Using equation (10), the round-off error can be written in a time-discrete form as follows:

$$\begin{cases} \dfrac{\rho_1^{n+1}-\rho_1^{n-1}}{2\Delta t} = \dfrac{1}{2}\Big[\rho_2^{n+1}+\rho_2^{n-1}\Big], \\ \dfrac{\rho_2^{n+1}-\rho_2^{n-1}}{2\Delta t} + \dfrac{\lambda}{2}\Big[\rho_2^{n+1}+\rho_2^{n-1}\Big] = \dfrac{\mu}{2}\Big[\Delta\rho_1^{n+1}+\Delta\rho_1^{n-1}\Big] \\ \qquad\qquad\qquad\qquad +f(x,t_n,U^n)-f(x,t_n,\widetilde{U}^n), \end{cases} \tag{11}$$

for $n = 1, 2, \ldots, m-1$. From the first equation in relation (11), we have:

$$\rho_1^{n+1}-\rho_1^{n-1} = \Delta t\Big[\rho_2^{n+1}+\rho_2^{n-1}\Big]. \tag{12}$$

Additionally, taking the inner product of the second equation in (11) with $\rho_1^{n+1}-\rho_1^{n-1}$, we get the following expression:

$$\begin{aligned} & \Big(\frac{\rho_2^{n+1}-\rho_2^{n-1}}{2\Delta t}, \rho_1^{n+1}-\rho_1^{n-1}\Big) \\ = & -\frac{\lambda}{2}\Big(\rho_2^{n+1}+\rho_2^{n-1}, \rho_1^{n+1}-\rho_1^{n-1}\Big) \\ + & \frac{\mu}{2}\Big(\Delta\rho_1^{n+1}+\Delta\rho_1^{n-1}, \rho_1^{n+1}-\rho_1^{n-1}\Big) \\ + & \Big(f(x,t_n,U^n)-f(x,t_n,\widetilde{U}^n), \rho_1^{n+1}-\rho_1^{n-1}\Big). \end{aligned} \tag{13}$$

Using (12) and the divergence theorem, we derive the following:

$$\begin{aligned} \Big(\frac{\rho_2^{n+1}-\rho_2^{n-1}}{2\Delta t}, \rho_1^{n+1}-\rho_1^{n-1}\Big) &= \Big(\frac{\rho_2^{n+1}-\rho_2^{n-1}}{2\Delta t}, \Delta t\Big[\rho_2^{n+1}+\rho_2^{n-1}\Big]\Big) \\ &= \frac{1}{2}\Big[\big\|\rho_2^{n+1}\big\|^2 - \big\|\rho_2^{n-1}\big\|^2\Big], \end{aligned} \tag{14}$$

$$\begin{aligned} & -\frac{\lambda}{2}\Big(\rho_2^{n+1}+\rho_2^{n-1}, \rho_1^{n+1}-\rho_1^{n-1}\Big) \\ = & -\frac{\lambda}{2}\Big(\rho_2^{n+1}+\rho_2^{n-1}, \Delta t\Big[\rho_2^{n+1}+\rho_2^{n-1}\Big]\Big) \\ = & -\frac{\lambda\Delta t}{2}\big\|\rho_2^{n+1}+\rho_2^{n-1}\big\|^2, \end{aligned} \tag{15}$$

$$\begin{aligned} & \frac{\mu}{2}\Big(\Delta\rho_1^{n+1}+\Delta\rho_1^{n-1},\rho_1^{n+1}-\rho_1^{n-1}\Big) \\ = & -\frac{\mu}{2}\Big(\nabla\rho_1^{n+1}+\nabla\rho_1^{n-1},\nabla\rho_1^{n+1}-\nabla\rho_1^{n-1}\Big) \\ = & -\frac{\mu}{2}\Big[\big\|\nabla\rho_1^{n+1}\big\|^2-\big\|\nabla\rho_1^{n-1}\big\|^2\Big]. \end{aligned} \tag{16}$$

Also, by substituting relation (12) into the last term on the right-hand side of (13) and applying the inequality $2ab \le a^2+b^2$, the Cauchy-Schwarz inequality, and the Lipschitz condition (6), we obtain:

$$\begin{aligned} & \Big(f(x,t_n,U^n)-f(x,t_n,\widetilde{U}^n),\rho_1^{n+1}-\rho_1^{n-1}\Big) \\ = & \ \Delta t\Big(f(x,t_n,U^n)-f(x,t_n,\widetilde{U}^n),\rho_2^{n+1}+\rho_2^{n-1}\Big) \\ \le & \ \Delta t\Big\|f(x,t_n,U^n)-f(x,t_n,\widetilde{U}^n)\Big\|\,\big\|\rho_2^{n+1}+\rho_2^{n-1}\big\| \\ \le & \ \mathcal{L}\Delta t\,\|\rho_1^n\|\,\Big(\,\big\|\rho_2^{n+1}\big\|+\big\|\rho_2^{n-1}\big\|\,\Big) \\ \le & \ \frac{\mathcal{L}\Delta t}{2}\Big[2\|\rho_1^n\|^2+\big\|\rho_2^{n+1}\big\|^2+\big\|\rho_2^{n-1}\big\|^2\Big]. \end{aligned} \tag{17}$$

Now, combining relations (13)-(17), we have:

$$\begin{aligned} \frac{1}{2}\Big[\big\|\rho_2^{n+1}\big\|^2-\big\|\rho_2^{n-1}\big\|^2\Big]+\frac{\mu}{2}\Big[\big\|\nabla\rho_1^{n+1}\big\|^2-\big\|\nabla\rho_1^{n-1}\big\|^2\Big] \\ \le \frac{\mathcal{L}\Delta t}{2}\Big[2\|\rho_1^n\|^2+\big\|\rho_2^{n+1}\big\|^2+\big\|\rho_2^{n-1}\big\|^2\Big]. \end{aligned} \tag{18}$$

Summing (18) from $n=1$ to $m-1$ yields:

$$\begin{aligned} & \frac{1}{2}\sum_{n=1}^{m-1}\Big[\big\|\rho_2^{n+1}\big\|^2-\big\|\rho_2^{n-1}\big\|^2\Big] \\ + & \ \frac{\mu}{2}\sum_{n=1}^{m-1}\Big[\big\|\nabla\rho_1^{n+1}\big\|^2-\big\|\nabla\rho_1^{n-1}\big\|^2\Big] \\ \le & \ \mathcal{L}\Delta t\sum_{n=1}^{m-1}\|\rho_1^n\|^2+\frac{\mathcal{L}\Delta t}{2}\sum_{n=1}^{m-1}\Big[\big\|\rho_2^{n+1}\big\|^2+\big\|\rho_2^{n-1}\big\|^2\Big], \end{aligned}$$

and consequently

$$\begin{aligned}
&\frac{1}{2}\Big[\|\rho_2^m\|^2+\|\rho_2^{m-1}\|^2-\|\rho_2^1\|^2-\|\rho_2^0\|^2\Big]\\
+\;&\frac{\mu}{2}\Big[\|\nabla\rho_1^m\|^2+\|\nabla\rho_1^{m-1}\|^2-\|\nabla\rho_1^1\|^2-\|\nabla\rho_1^0\|^2\Big]\le\mathcal{L}\Delta t\sum_{n=1}^{m-1}\|\rho_1^n\|^2\\
+\;&\frac{\mathcal{L}\Delta t}{2}\Big[\|\rho_2^0\|^2+\|\rho_2^1\|^2+2\sum_{n=2}^{m-2}\|\rho_2^n\|^2+\|\rho_2^{m-1}\|^2+\|\rho_2^m\|^2\Big]. \qquad (19)
\end{aligned}$$

Rewriting inequality (19) leads to:

$$\begin{aligned}
&(1-\mathcal{L}\Delta t)\|\rho_2^m\|^2+\mu\|\nabla\rho_1^m\|^2\\
\le\;&(1+\mathcal{L}\Delta t)\Big[\|\rho_2^0\|^2+\|\rho_2^1\|^2\Big]-\mu\|\nabla\rho_1^{m-1}\|^2\\
+\;&\mu\Big[\|\nabla\rho_1^1\|^2+\|\nabla\rho_1^0\|^2\Big]+2\mathcal{L}\Delta t\sum_{n=2}^{m-2}\|\rho_2^n\|^2\\
+\;&(\mathcal{L}\Delta t-1)\|\rho_2^{m-1}\|^2+2\mathcal{L}\Delta t\sum_{n=1}^{m-1}\|\rho_1^n\|^2. \qquad (20)
\end{aligned}$$

For sufficiently small Δt, we have $1-\mathcal{L}\Delta t\ge\frac{1}{2}$, consequently $1+\mathcal{L}\Delta t\le\frac{3}{2}$ and $\mathcal{L}\Delta t-1\le 2\mathcal{L}\Delta t$. Using this fact, from inequality (20) we obtain:

$$\begin{aligned}
\|\rho_2^m\|^2+2\mu\|\nabla\rho_1^m\|^2\;\le\;&3\Big[\|\rho_2^0\|^2+\|\rho_2^1\|^2\Big]+2\mu\Big[\|\nabla\rho_1^1\|^2+\|\nabla\rho_1^0\|^2\Big]\\
&+\;4\mathcal{L}\Delta t\sum_{n=2}^{m-1}\|\rho_2^n\|^2+4\mathcal{L}\Delta t\sum_{n=1}^{m-1}\|\rho_1^n\|^2. \qquad (21)
\end{aligned}$$

Applying the Poincaré inequality established in Lemma 2.2, we conclude that:

$$\|\rho_1^n\|\le\widetilde{\mathcal{C}}_1\|\nabla\rho_1^n\|\Rightarrow\|\rho_1^n\|^2\le\widetilde{\mathcal{C}}_1^2\|\nabla\rho_1^n\|^2=2\mu\widetilde{\mathcal{C}}_2\|\nabla\rho_1^n\|^2, \qquad (22)$$

where $\widetilde{\mathcal{C}}_2=\frac{\widetilde{\mathcal{C}}_1^2}{2\mu}$. By substituting relation (22) into inequality (21), we derive the following result:

$$\begin{aligned}
\|\rho_2^m\|^2+2\mu\|\nabla\rho_1^m\|^2\;\le\;&3\Big[\|\rho_2^0\|^2+2\mu\|\nabla\rho_1^0\|^2\Big]+3\Big[\|\rho_2^1\|^2]+2\mu\|\nabla\rho_1^1\|^2\Big]\\
&+\;4\mathcal{L}\Delta t\sum_{n=0}^{m-1}\|\rho_2^n\|^2+4\mathcal{L}\Delta t\sum_{n=0}^{m-1}2\mu\widetilde{\mathcal{C}}_2\|\nabla\rho_1^n\|^2,
\end{aligned}$$

and therefore

$$\begin{aligned}\|\rho_2^m\|^2+2\mu\|\nabla\rho_1^m\|^2 &\leq 3\Big[\big\|\rho_2^0\big\|^2+2\mu\big\|\nabla\rho_1^0\big\|^2\Big]+3\Big[\big\|\rho_2^1\big\|^2+2\mu\big\|\nabla\rho_1^1\big\|^2\Big]\\ &+ 4\mathcal{L}\widetilde{\mathcal{C}}_3\Delta t\sum_{n=0}^{m-1}\Big[\|\rho_2^n\|^2+2\mu\|\nabla\rho_1^n\|^2\Big],\end{aligned}$$

where $\widetilde{\mathcal{C}}_3=\max\{1,\widetilde{\mathcal{C}}_2\}$. Applying the Grönwall inequality in Lemma 2.1, we obtain:

$$\|\rho_2^m\|^2+2\mu\|\nabla\rho_1^m\|^2\leq 3\Big(\big\|\rho_2^0\big\|^2+\big\|\rho_2^1\big\|^2+2\mu\Big[\big\|\nabla\rho_1^0\big\|^2+\big\|\nabla\rho_1^1\big\|^2\Big]\Big)\mathrm{e}^{4\mathcal{L}T\widetilde{\mathcal{C}}_3}.$$

Finally, we conclude

$$\|\rho_2^m\|\leq\sqrt{3\Big(\big\|\rho_2^0\big\|^2+\big\|\rho_2^1\big\|^2+2\mu\Big[\big\|\nabla\rho_1^0\big\|^2+\big\|\nabla\rho_1^1\big\|^2\Big]\Big)}\mathrm{e}^{2\mathcal{L}T\widetilde{\mathcal{C}}_3},$$

and

$$\begin{aligned}\|\rho_1^m\| &\leq \widetilde{\mathcal{C}}_4\|\nabla\rho_1^m\|\\ &\leq \widetilde{\mathcal{C}}_4\sqrt{\frac{3}{2\mu}\Big(\big\|\rho_2^0\big\|^2+\big\|\rho_2^1\big\|^2+2\mu\Big[\big\|\nabla\rho_1^0\big\|^2+\big\|\nabla\rho_1^1\big\|^2\Big]\Big)}\mathrm{e}^{2\mathcal{L}T\widetilde{\mathcal{C}}_3},\end{aligned}$$

which completes the proof. It is worth noting that, by the Poincaré inequality in Lemma 2.2, the constants $\widetilde{\mathcal{C}}_1$, $\widetilde{\mathcal{C}}_2$, and $\widetilde{\mathcal{C}}_3$ depend on the domain Ω. □

Theorem 2.4. *Let u^n, v^n, U^n, and V^n belong to $H_0^1(\Omega)$, where u^n and v^n satisfy relation (9), while U^n and V^n correspond to the solution of (10). Suppose further that the initial approximation errors for U^1 and V^1 are at least of order $\mathcal{O}(\Delta t^2)$. Then, the time-discrete scheme in (10) achieves a convergence rate of $\mathcal{O}(\Delta t^2)$.*

Proof. We start the proof by defining the error functions

$$e_1^n=u^n-U^n,\quad e_2^n=v^n-V^n,\quad n=0,1,\ldots,m.$$

Under the theorem's assumptions, we have $e_1^n,e_2^n\in H_0^1(\Omega)$. Moreover, the initial errors satisfy:

$$e_1^0=e_2^0=0,\quad e_1^1=e_2^1=\mathcal{O}(\Delta t^2). \tag{23}$$

Subtracting (10) from (9) for $n = 1, 2, \ldots, m-1$ yields:

$$\begin{cases} \dfrac{e_1^{n+1} - e_1^{n-1}}{2\Delta t} = \dfrac{1}{2}\Big[e_2^{n+1} + e_2^{n-1}\Big] + \mathcal{R}_1^n, \\ \dfrac{e_2^{n+1} - e_2^{n-1}}{2\Delta t} + \dfrac{\lambda}{2}\Big[e_2^{n+1} + e_2^{n-1}\Big] = \dfrac{\mu}{2}\Big[\Delta e_1^{n+1} + \Delta e_1^{n-1}\Big] \\ \qquad\qquad\qquad\qquad + f(x, t_n, u^n) - f(x, t_n, U^n) + \mathcal{R}_2^n. \end{cases} \tag{24}$$

The first equation in (24) gives:

$$e_1^{n+1} - e_1^{n-1} = \Delta t\Big[e_2^{n+1} + e_2^{n-1}\Big] + 2\Delta t\mathcal{R}_1^n. \tag{25}$$

As in Theorem 2.3, the inner product of the second equation in (24) with $e_1^{n+1} - e_1^{n-1}$ leads to:

$$\begin{aligned} & \Big(\frac{e_2^{n+1} - e_2^{n-1}}{2\Delta t}, e_1^{n+1} - e_1^{n-1}\Big) \\ = & -\frac{\lambda}{2}\Big(e_2^{n+1} + e_2^{n-1}, e_1^{n+1} - e_1^{n-1}\Big) \\ + & \frac{\mu}{2}\Big(\Delta e_1^{n+1} + \Delta e_1^{n-1}, e_1^{n+1} - e_1^{n-1}\Big) \\ + & \Big(f(x, t_n, u^n) - f(x, t_n, U^n), e_1^{n+1} - e_1^{n-1}\Big) \\ + & \Big(\mathcal{R}_2^n, e_1^{n+1} - e_1^{n-1}\Big). \end{aligned} \tag{26}$$

Using (25), we have

$$\begin{aligned} & \Big(\frac{e_2^{n+1} - e_2^{n-1}}{2\Delta t}, e_1^{n+1} - e_1^{n-1}\Big) \\ = & \Big(\frac{e_2^{n+1} - e_2^{n-1}}{2\Delta t}, \Delta t\Big[e_2^{n+1} + e_2^{n-1}\Big] + 2\Delta t\mathcal{R}_1^n\Big) \\ = & \frac{1}{2}\Big[\big\|e_2^{n+1}\big\|^2 - \big\|e_2^{n-1}\big\|^2\Big] + \Big(e_2^{n+1} - e_2^{n-1}, \mathcal{R}_1^n\Big), \end{aligned} \tag{27}$$

$$\begin{aligned} & -\frac{\lambda}{2}\Big(e_2^{n+1} + e_2^{n-1}, e_1^{n+1} - e_1^{n-1}\Big) \\ = & -\frac{\lambda}{2}\Big(e_2^{n+1} + e_2^{n-1}, \Delta t\Big[e_2^{n+1} + e_2^{n-1}\Big] + 2\Delta t\mathcal{R}_1^n\Big) \\ = & -\frac{\lambda\Delta t}{2}\big\|e_2^{n+1} + e_2^{n-1}\big\|^2 - \lambda\Delta t\Big(e_2^{n+1} + e_2^{n-1}, \mathcal{R}_1^n\Big), \end{aligned} \tag{28}$$

$$
\begin{aligned}
& \frac{\mu}{2}\Big(\Delta e_1^{n+1}+\Delta e_1^{n-1}, e_1^{n+1}-e_1^{n-1}\Big) \\
= & -\frac{\mu}{2}\Big(\nabla e_1^{n+1}+\nabla e_1^{n-1}, \nabla e_1^{n+1}-\nabla e_1^{n-1}\Big) \\
= & -\frac{\mu}{2}\Big[\big\|\nabla e_1^{n+1}\big\|^2-\big\|\nabla e_1^{n-1}\big\|^2\Big]. \qquad (29)
\end{aligned}
$$

Substituting relations (27)-(29) into (26), we arrive at:

$$
\begin{aligned}
& \frac{1}{2}\Big[\big\|e_2^{n+1}\big\|^2-\big\|e_2^{n-1}\big\|^2\Big]+\frac{\mu}{2}\Big[\big\|\nabla e_1^{n+1}\big\|^2-\big\|\nabla e_1^{n-1}\big\|^2\Big] \\
= & -\frac{\lambda\Delta t}{2}\big\|e_2^{n+1}+e_2^{n-1}\big\|^2 \\
- & \Delta t\Big(e_2^{n+1}-e_2^{n-1}, \mathcal{R}_1^n\Big)-\lambda\Delta t(e_2^{n+1}+e_2^{n-1}, \mathcal{R}_1^n) \\
+ & \Big(f(x,t_n,u^n)-f(x,t_n,U^n), e_1^{n+1}-e_1^{n-1}\Big) \\
+ & \Big(\mathcal{R}_2^n, e_1^{n+1}-e_1^{n-1}\Big). \qquad (30)
\end{aligned}
$$

Employing the inequality $2ab \le a^2+b^2$, the Cauchy-Schwarz inequality, and the Lipschitz condition (6), we obtain:

$$
\begin{aligned}
& \Big(f(x,t_n,u^n)-f(x,t_n,U^n), e_1^{n+1}-e_1^{n-1}\Big) \\
\le & \|f(x,t_n,u^n)-f(x,t_n,U^n)\|\,\big\|e_1^{n+1}-e_1^{n-1}\big\| \\
\le & \mathcal{L}\,\|e_1^n\|\,\big\|e_1^{n+1}-e_1^{n-1}\big\| \\
\le & \mathcal{L}\,\|e_1^n\|\,\Big[\Delta t\,\big\|e_2^{n+1}+e_2^{n-1}\big\|+2\Delta t\,\|\mathcal{R}_1^n\|\Big] \\
\le & \mathcal{L}\Delta t\Big[\,\|e_1^n\|\,\big\|e_2^{n+1}\big\|+\|e_1^n\|\,\big\|e_2^{n-1}\big\|\,\Big]+2\mathcal{L}\Delta t\,\|\mathcal{R}_1^n\|\,\|e_1^n\| \\
\le & \frac{\mathcal{L}\Delta t}{2}\Big[2\|e_1^n\|^2+\big\|e_2^{n+1}\big\|^2+\big\|e_2^{n-1}\big\|^2\Big]+2\mathcal{L}\Delta t\Big[\frac{1}{2}\|\mathcal{R}_1^n\|^2+\frac{1}{2}\|e_1^n\|^2\Big] \\
\le & 2\mathcal{L}\Delta t\|e_1^n\|^2+\frac{\mathcal{L}\Delta t}{2}\Big[\big\|e_2^{n+1}\big\|^2+\big\|e_2^{n-1}\big\|^2\Big]+\mathcal{L}\mathcal{C}_1^2\Delta t^5, \qquad (31)
\end{aligned}
$$

$$\begin{aligned}\left(\mathcal{R}_2^n, e_1^{n+1}-e_1^{n-1}\right) &\leq \|\mathcal{R}_2^n\|\left[\Delta t\left\|e_2^{n+1}+e_2^{n-1}\right\|+2\Delta t\,\|\mathcal{R}_1^n\|\right]\\ &\leq \frac{\Delta t}{2}\left[2\|\mathcal{R}_2^n\|^2+\left\|e_2^{n+1}\right\|^2+\left\|e_2^{n-1}\right\|^2\right]\\ &+ 2\Delta t\,\|\mathcal{R}_1^n\|\,\|\mathcal{R}_2^n\|\\ &\leq \frac{\Delta t}{2}\left[\left\|e_2^{n+1}\right\|^2+\left\|e_2^{n-1}\right\|^2\right]\\ &+ (\mathcal{C}_2^2+2\mathcal{C}_1\mathcal{C}_2)\Delta t^5. \end{aligned} \tag{32}$$

By applying Young's inequality and the Cauchy–Schwarz inequality, we obtain:

$$\begin{aligned}-\left(e_2^{n+1}-e_2^{n-1},\mathcal{R}_1^n\right) &\leq \|\mathcal{R}_1^n\|\left\|e_2^{n+1}-e_2^{n-1}\right\|\\ &\leq \frac{1}{2}\left[\|\mathcal{R}_1^n\|^2+\left\|e_2^{n+1}-e_2^{n-1}\right\|^2\right]\\ &\leq \frac{1}{2}\left[\|\mathcal{R}_1^n\|^2+(\|e_2^{n+1}\|+\|e_2^{n-1}\|)^2\right]\\ &\leq \frac{1}{2}\|\mathcal{R}_1^n\|^2+\|e_2^{n+1}\|^2+\|e_2^{n-1}\|^2. \end{aligned} \tag{33}$$

Similar to (33), we conclude that

$$-\left(e_2^{n+1}+e_2^{n-1},\mathcal{R}_1^n\right)\leq \frac{1}{2}\|\mathcal{R}_1^n\|^2+\|e_2^{n+1}\|^2+\|e_2^{n-1}\|^2. \tag{34}$$

Now, combining relations (30)-(32) and (33)-(34), we have:

$$\begin{aligned}&\frac{1}{2}\left[\left\|e_2^{n+1}\right\|^2-\left\|e_2^{n-1}\right\|^2\right]+\frac{\mu}{2}\left[\left\|\nabla e_1^{n+1}\right\|^2-\left\|\nabla e_1^{n-1}\right\|^2\right]\\ \leq\ & \delta(\Delta t)\left[\left\|e_2^{n+1}\right\|^2+\left\|e_2^{n-1}\right\|^2\right]+2\mathcal{L}\Delta t\|e_1^n\|^2+\theta(\Delta t), \end{aligned} \tag{35}$$

in which

$$\delta(\Delta t)=\left(\frac{3}{2}+\frac{\mathcal{L}}{2}+\lambda\right)\Delta t,$$

$$\theta(\Delta t)=\underbrace{\left[\left(1+\lambda+2\mathcal{L}\right)\frac{\mathcal{C}_1^2}{2}+\mathcal{C}_2^2+2\mathcal{C}_1\mathcal{C}_2\right]}_{\mathcal{C}_4}\Delta t^5=\mathcal{C}_4\Delta t^5.$$

By summing (35) from $n = 1$ to $m - 1$, we obtain:

$$\begin{aligned} & \frac{1}{2}\sum_{n=1}^{m-1}\left[\left\|e_2^{n+1}\right\|^2 - \left\|e_2^{n-1}\right\|^2\right] \\ + \quad & \frac{\mu}{2}\sum_{n=1}^{m-1}\left[\left\|\nabla e_1^{n+1}\right\|^2 - \left\|\nabla e_1^{n-1}\right\|^2\right] \\ \leq \quad & \delta(\Delta t)\sum_{n=1}^{m-1}\left[\left\|e_1^{n+1}\right\|^2 - \left\|e_1^{n-1}\right\|^2\right] \\ + \quad & 2\mathcal{L}\Delta t\sum_{n=1}^{m-1}\|e_1^n\|^2 + (m-1)\theta(\Delta t). \end{aligned} \tag{36}$$

Similar to Theorem 2.3, by computing the summations in (36) and rearranging terms, we obtain:

$$\begin{aligned} & (1-2\delta(\Delta t))\,\|e_2^m\| + \mu\|\nabla e_1^m\|^2 \\ \leq \quad & (1+2\delta(\Delta t))\left[\left\|e_2^1\right\|^2 + \left\|e_2^0\right\|^2\right] - \mu\left\|\nabla e_1^{m-1}\right\|^2 \\ + \quad & \mu\left[\left\|\nabla e_1^1\right\|^2 + \left\|\nabla e_1^0\right\|^2\right] + 4\delta(\Delta t)\sum_{n=2}^{m-2}\|e_2^n\|^2 \\ + \quad & (2\delta(\Delta t)-1)\left\|e_2^{m-1}\right\|^2 + 4\mathcal{L}\Delta t\sum_{n=1}^{m-1}\|e_1^n\|^2 + 2\mathcal{C}_4 T\Delta t^4. \end{aligned} \tag{37}$$

For sufficiently small Δt, the condition $1 - 2\delta(\Delta t) \geq \frac{1}{2}$ holds, which implies that $1 + 2\delta(\Delta t) \leq \frac{3}{2}$ and $2\delta(\Delta t) - 1 \leq 4\delta(\Delta t)$. Therefore from (37), one can get:

$$\begin{aligned} & \|e_2^m\|^2 + 2\mu\|\nabla e_1^m\|^2 \\ \leq \quad & 3\left[\left\|e_2^1\right\|^2 + \left\|e_2^0\right\|^2\right] + 2\mu\left[\left\|\nabla e_1^1\right\|^2 + \left\|\nabla e_1^0\right\|^2\right] \\ + \quad & 8\delta(\Delta t)\sum_{n=2}^{m-1}\|e_2^n\|^2 + 8\mathcal{L}\Delta t\sum_{n=1}^{m-1}\|e_1^n\|^2 + 4\mathcal{C}_4 T\Delta t^4. \end{aligned} \tag{38}$$

By using the Poincaré inequality, we derive:

$$\|e_1^n\| \leq \mathcal{C}_5\,\|\nabla e_1^n\| \Rightarrow \|e_1^n\|^2 \leq {\mathcal{C}_5}^2\|\nabla e_1^n\|^2 = 2\mu\mathcal{C}_6\|\nabla e_1^n\|^2, \tag{39}$$

where $\mathcal{C}_6 = \frac{{\mathcal{C}_5}^2}{2\mu}$. Since $8\mathcal{L}\Delta t \leq 16\delta(\Delta t)$, $e_2^0 = 0$ and $\nabla e_1^0 = 0$, it follows from (38) and (39) that

$$\begin{aligned}\|e_2^m\|^2 + 2\mu\|\nabla e_1^m\|^2 &\leq 3\Big[\big\|e_2^1\big\|^2 + 2\mu\big\|\nabla e_1^1\big\|^2\Big] \\ &+ 16\mathcal{C}_7\delta(\Delta t)\sum_{n=0}^{m-1}\Big[\|e_2^n\|^2 + 2\mu\|\nabla e_1^n\|^2\Big] + 4\mathcal{C}_4 T\Delta t^4,\end{aligned}$$

where $\mathcal{C}_7 = \max\{1, \mathcal{C}_6\}$. From (23) we conclude that $\|e_2^n\| \leq \mathcal{C}_8\Delta t^2$ and $\|\nabla e_1^n\| \leq \mathcal{C}_9\Delta t^2$. Therefore

$$\begin{aligned}&\|e_2^m\|^2 + 2\mu\|\nabla e_1^m\|^2 \\ \leq\ & (3\mathcal{C}_8^2 + 6\mu\mathcal{C}_9^2 + 4\mathcal{C}_4 T)\Delta t^4 + 16\mathcal{C}_7\delta(\Delta t)\sum_{n=0}^{m-1}\Big[\|e_2^n\|^2 + 2\mu\|\nabla e_1^n\|^2\Big].\end{aligned}$$

Finally, by utilizing Grönwall's inequality, we obtain

$$\|e_2^m\|^2 + 2\mu\|\nabla e_1^m\|^2 \leq (3\mathcal{C}_8^2 + 6\mu\mathcal{C}_9^2 + 4\mathcal{C}_4 T)\mathrm{e}^{16\mathcal{C}_7 m\delta(\Delta t)}\Delta t^4 \leq \mathcal{C}_{10}\Delta t^4,$$

where $\mathcal{C}_{10} = (3\mathcal{C}_8^2 + 6\mu\mathcal{C}_9^2 + 4\mathcal{C}_4 T)\mathrm{e}^{16\mathcal{C}_7 T(\mathcal{C}_1\mathcal{C}_3(1+\lambda T)T + \frac{1}{2}(\mathcal{L}+1))}$ and consequently,

$$\|e_2^m\| \leq \sqrt{\mathcal{C}_{10}}\Delta t^2,$$

and

$$\|e_1^m\| \leq \mathcal{C}_{11}\|\nabla e_1^m\| \leq \mathcal{C}_{11}\sqrt{\frac{\mathcal{C}_{10}}{2\mu}}\Delta t^2,$$

which completes the proof. Note that, by virtue of the Poincaré inequality in Lemma 2.2, the constants $\mathcal{C}_5$, $\mathcal{C}_6$, $\mathcal{C}_7$, and $\mathcal{C}_{11}$ depend on the domain Ω. □

3 Numerical Scheme

In this section, we aim to present the fully discrete formulation of the method based on the differential quadrature approach with an improved cubic B-spline basis. First, we provide a brief overview of the quadrature formulas, followed by a detailed description of the implementation of the proposed numerical method.

3.1 Differential quadrature formulas

Given a sequence of uniformly spaced points x_i for $i \in \mathbb{Z}$, the cubic B-spline, denoted by $\aleph_j(x)$, is defined as follows [34]:

$$\aleph_j(x) = \frac{1}{h^3} \sum_{i=0}^{4} \binom{4}{i} (-1)^i (x_{j+i-2} - x)_+^3, \quad j \in \mathbb{Z},$$

in which

$$(\xi - x)_+^3 = \begin{cases} (\xi - x)^3, & \xi \geq x, \\ 0, & \xi < x, \end{cases}$$

and $h = x_{i+1} - x_i$. The fundamental properties of this class of basis functions can be summarized as follows:

- $\text{Supp}(\aleph_j(x)) = [x_{j-2}, x_{j+2}]$,
- $\aleph_j(x) \geq 0, \quad \forall x \in \mathbb{R}$,
- $\sum_{j \in \mathbb{Z}} \aleph_j(x) = 1, \quad \forall x \in \mathbb{R}$,
- $\aleph_j^{(s)}(x_i) = \begin{cases} \delta_{i,j-1} + 4\delta_{i,j} + \delta_{i,j+1}, & s = 0, \\ 3h^{-1}(\delta_{i,j-1} - \delta_{i,j+1}), & s = 1, \\ 6h^{-2}(\delta_{i,j-1} - 2\delta_{i,j} + \delta_{i,j+1}), & s = 2. \end{cases}$

Babu et al. [7] developed new differential quadrature formulas by utilizing Bellman's initial idea (see [10, 41, 46]) for constructing quadrature formulas, replacing the polynomial basis functions with a new set of basis functions derived from cubic B-splines, as follows:

$$\begin{cases} \tilde{\aleph}_1(x) = \aleph_1(x) + 4\aleph_0(x), \\ \tilde{\aleph}_2(x) = \aleph_2(x) - \frac{7}{2}\aleph_0(x) + \frac{5}{8}\aleph_1(x), \\ \tilde{\aleph}_3(x) = \aleph_3(x) + \frac{88}{37}\aleph_0(x) - \frac{21}{37}\aleph_1(x) - \frac{4}{37}\aleph_2(x), \\ \tilde{\aleph}_4(x) = \aleph_4(x) - \aleph_0(x) + \frac{1}{4}\aleph_1(x) - \frac{1}{4}\aleph_3(x), \\ \tilde{\aleph}_j(x) = \aleph_j(x), \qquad j = 5, 6, \ldots, N-4, \\ \tilde{\aleph}_{N-3}(x) = \aleph_{N-3}(x) - \aleph_{N+1}(x) + \frac{1}{4}\aleph_N(x) - \frac{1}{4}\aleph_{N-2}(x), \\ \tilde{\aleph}_{N-2}(x) = \aleph_{N-2}(x) + \frac{88}{37}\aleph_{N+1}(x) - \frac{21}{37}\aleph_N(x) - \frac{4}{37}\aleph_{N-11}(x), \\ \tilde{\aleph}_{N-1}(x) = \aleph_{N-1}(x) - \frac{7}{2}\aleph_{N+1}(x) + \frac{5}{8}\aleph_N(x), \\ \tilde{\aleph}_N(x) = \aleph_N(x) + 4\aleph_{N+1}(x). \end{cases} \tag{40}$$

In the following, we briefly review these quadrature formulas for approximating the first- and second-order derivatives. To achieve this, we start by dividing the interval $[a, b]$ as follows:

$$x_i = a + (i-1)h, \quad i = 1, 2, \ldots, N, \tag{41}$$

where $N \in \mathbb{N}$ and $h = \frac{b-a}{N-1}$. An approximation of the first derivative of the function f at x_i can be expressed using the values $f(x_j)$ for $j = 1, 2, \ldots, N$, as follows:

$$f'(x_i) \approx \sum_{j=1}^{N} \vartheta_{ij}^{(l)} f(x_j), \quad i = 1, 2, \ldots, N. \tag{42}$$

By solving the system of algebraic equations

$$\tilde{\aleph}'_k(x_i) = \sum_{j=1}^{N} \vartheta_{ij}^{(1)} \tilde{\aleph}_k(x_j), \quad k = 1, 2, \ldots, N, \tag{43}$$

the coefficients $\vartheta_{ij}^{(l)}$, known as the weighting coefficients for the first-order derivative, can be determined. For computational simplicity, relation (43) can be rewritten in matrix form as follows:

$$\mathcal{A}\mathcal{V}_i = \mathcal{B}_i, \quad i = 1, 2, \ldots, N, \tag{44}$$

where

$$\mathcal{A} = \begin{pmatrix} 8 & 1 & & & & & & & & & \\ 0 & \frac{37}{8} & 1 & & & & & & & & \\ & 0 & \frac{144}{37} & 1 & & & & & & & \\ & & 0 & \frac{15}{4} & 1 & & & & & & \\ & & & 1 & 4 & 1 & & & & & \\ & & & & \ddots & \ddots & \ddots & & & & \\ & & & & & 1 & 4 & 1 & & & \\ & & & & & & 1 & \frac{15}{4} & 0 & & \\ & & & & & & & 1 & \frac{144}{37} & 0 & \\ & & & & & & & & 1 & \frac{37}{8} & 0 \\ & & & & & & & & & 1 & 8 \end{pmatrix},$$

$$\mathcal{V}_i = \begin{pmatrix} \vartheta_{i1}^{(1)} \\ \vartheta_{i2}^{(1)} \\ \vdots \\ \vartheta_{iN}^{(1)} \end{pmatrix}, \quad \mathcal{B}_i = \begin{pmatrix} \tilde{\aleph}'_1(x_i) \\ \tilde{\aleph}'_2(x_i) \\ \vdots \\ \tilde{\aleph}'_N(x_i) \end{pmatrix},$$

$$\begin{cases} \mathcal{B}_1 = \frac{1}{h}\left(-12, \frac{27}{2}, -\frac{276}{37}, 3, 0, 0, \ldots, 0\right)^T, \\ \mathcal{B}_2 = \frac{1}{h}\left(-3, -\frac{15}{8}, \frac{174}{37}, -\frac{3}{2}, 0, 0, \ldots, 0\right)^T, \\ \mathcal{B}_3 = \frac{1}{h}\left(0, -3, \frac{12}{37}, 3, 0, 0, \ldots, 0\right)^T, \\ \mathcal{B}_4 = \frac{1}{h}\left(0, 0, -3, \frac{3}{4}, 3, 0, \ldots, 0\right)^T, \\ \mathcal{B}_i = \frac{1}{h}\left(0, 0, \ldots, -3, 0, 3, \ldots, 0, 0\right)^T, \qquad i = 5, 6, \ldots, N-4, \\ \mathcal{B}_{N-3} = \frac{1}{h}\left(0, \ldots, 0, -3, -\frac{3}{4}, 3, 0, 0\right)^T, \\ \mathcal{B}_{N-2} = \frac{1}{h}\left(0, \ldots, 0, 0, -3, -\frac{12}{37}, 3, 0\right)^T, \\ \mathcal{B}_{N-1} = \frac{1}{h}\left(0, \ldots, 0, 0, \frac{3}{2}, -\frac{174}{37}, \frac{15}{8}, 3\right)^T, \\ \mathcal{B}_N = \frac{1}{h}\left(0, \ldots, 0, 0, -3, \frac{276}{37}, -\frac{27}{2}, 12\right)^T. \end{cases}$$

It is worth mentioning that matrix $\mathcal{A}$ in (44) is banded and invertible, so the linear system for the weighting coefficients is uniquely solvable. Now, by using the coefficients $\vartheta_{ij}^{(1)}$ and computing $\vartheta_{ij}^{(2)}$ as the weighting coefficients for the second-order derivative through the relation [41, 7]

$$\vartheta_{ij}^{(2)} = \begin{cases} 2\vartheta_{ij}^{(1)}\left(\vartheta_{ii}^{(1)} - (x_i - x_j)^{-1}\right), & i \neq j, \\ -\sum_{s=1, s\neq i}^{N} \vartheta_{is}^{(2)}, & i = j, \end{cases} \qquad i, j = 1, 2, \ldots, N,$$

the second-order derivative of the function f at x_i can be approximated as follows:

$$f''(x_i) \approx \sum_{j=1}^{N} \vartheta_{ij}^{(2)} f(x_j), \quad i = 1, 2, \ldots, N. \tag{45}$$

3.2 Fully-discrete scheme

To derive the fully-discrete form of nonlinear hyperbolic PDE (1), we first rewrite the time semi-discrete relation (10) as follows:

$$\begin{cases} U^{n+1} - \Delta t V^{n+1} = U^{n-1} + \Delta t V^{n-1}, \\ (1 + \lambda\Delta t)V^{n+1} - \mu\Delta t\Delta U^{n+1} \\ \qquad = (1 - \lambda\Delta t)V^{n-1} + \mu\Delta t\Delta U^{n-1} + 2\Delta t f(x, t_n, U^n). \end{cases} \tag{46}$$

By substituting the grid points x_i into (46), we obtain

$$\begin{cases} U_i^{n+1} - \Delta t V_i^{n+1} = U_i^{n-1} + \Delta t V_i^{n-1}, \ i = 2, 3, \ldots, N-1, \\ \\ (1 + \lambda \Delta t) V_i^{n+1} - \mu \Delta t \Delta U_i^{n+1} \\ \quad = (1 - \lambda \Delta t) V_i^{n-1} + \mu \Delta t \Delta U_i^{n-1} + 2\Delta t f(x_i, t_n, U_i^n), \ i = 1, 2, \ldots, N, \end{cases} \tag{47}$$

where $U_i^n = U(x_i, t_n)$ and $V_i^n = V(x_i, t_n)$ are the approximations of the exact values $u(x_i, t_n)$ and $v(x_i, t_n)$, respectively, and $n = 1, 2, \ldots, m-1$. Moreover, applying the differential quadrature formula given in (45) yields:

$$\begin{cases} \Delta U_i^{n+1} = \dfrac{\partial^2 U}{\partial x^2}(x_i, t_{n+1}) \approx \displaystyle\sum_{j=1}^{N} \vartheta_{ij}^{(2)} U_j^{n+1}, \\ \\ \Delta U_i^{n-1} = \dfrac{\partial^2 U}{\partial x^2}(x_i, t_{n-1}) \approx \displaystyle\sum_{j=1}^{N} \vartheta_{ij}^{(2)} U_j^{n-1}. \end{cases} \tag{48}$$

By substituting (48) into (47), the fully-discrete form of the PDE (1) is obtained as follows:

$$\begin{cases} U_i^{n+1} - \Delta t V_i^{n+1} = U_i^{n-1} + \Delta t V_i^{n-1}, \quad i = 2, 3, \ldots, N-1, \\ \\ (1 + \lambda \Delta t) V_i^{n+1} - \mu \Delta t \displaystyle\sum_{j=1}^{N} \vartheta_{ij}^{(2)} U_j^{n+1} = (1 - \lambda \Delta t) V_i^{n-1} \\ \quad + \mu \Delta t \displaystyle\sum_{j=1}^{N} \vartheta_{ij}^{(2)} U_j^{n-1} + 2\Delta t f(x_i, t_n, U_i^n), \quad i = 1, 2, \ldots, N. \end{cases} \tag{49}$$

On the other hand, by employing the differential quadrature formula (45) and the composite Simpson's integration rule (with an odd value of N), we have

$$\begin{aligned} \frac{\partial u}{\partial x}(x_l, t_n) &\approx \frac{1}{2}\left[\frac{\partial u}{\partial x}(x_l, t_{n+1}) + \frac{\partial u}{\partial x}(x_l, t_{n-1})\right] \\ &\approx \frac{1}{2} \sum_{j=1}^{N} \vartheta_{lj}^{(1)} \left[U_j^{n+1} + U_j^{n-1}\right], \quad l = 1, N, \end{aligned} \tag{50}$$

$$\int_a^b \psi_s(x)u(x,t_n)dx \approx \frac{1}{2}\int_a^b \psi_s(x)\Big[u(x,t_{n+1})+u(x,t_{n-1})\Big]dx \approx \frac{1}{2}\sum_{j=1}^{N} w_j\psi_s(x_j)\Big[U_j^{n+1}+U_j^{n-1}\Big], \tag{51}$$

for $s=1,2$ where

$$w_j=\begin{cases}\frac{h}{3}, & j=1,N,\\ \frac{4h}{3}, & j=2,4,\ldots,N-1,\\ \frac{2h}{3}, & j=3,5,\ldots,N-2.\end{cases}$$

Now, by substituting the approximations (50) and (51) into relations (4) and (5), the discrete form of the nonlocal boundary conditions is obtained as follows:

$$\frac{\alpha_1}{2}\Big[U_1^{n+1}+U_1^{n-1}\Big] + \frac{1}{2}\sum_{j=1}^{N}(\beta_1\vartheta_{1j}^{(1)}+w_j\psi_1(x_j))\Big[U_j^{n+1}+U_j^{n-1}\Big] = \frac{1}{2}\Big[g_1^{n+1}+g_1^{n-1}\Big], \tag{52}$$

$$\frac{\alpha_2}{2}\Big[U_N^{n+1}+U_N^{n-1}\Big] + \frac{1}{2}\sum_{j=1}^{N}(\beta_2\vartheta_{Nj}^{(1)}+w_j\psi_2(x_j))\Big[U_j^{n+1}+U_j^{n-1}\Big] = \frac{1}{2}\Big[g_2^{n+1}+g_2^{n-1}\Big]. \tag{53}$$

At each time level, equations (49), (52), and (53) yield a system of $2N$ linear algebraic equations involving $2N$ unknowns, namely $U_1^n, U_2^n, \ldots, U_N^n$ and $V_1^n, V_2^n, \ldots, V_N^n$ which can be determined by solving the resulting system.

3.3 Initial approximations

According to the proposed method and the convergence Theorem 2.4, it is observed that, in order to implement the scheme, the initial approximations U_i^0, V_i^0, U_i^1 and V_i^1 for $i=1,2,\ldots,N$ must be chosen such that they are at least of order $\mathcal{O}(\Delta t^2)$. Utilizing the initial conditions

(2) and (3), the values U_i^0 and V_i^0 can be computed exactly using the following relation:

$$\begin{cases} U_i^0 = u(x_i, 0) = \phi_1(x_i), \\ V_i^0 = \dfrac{\partial u}{\partial t}(x_i, 0) = \phi_2(x_i), \end{cases} \qquad i = 1, 2, \ldots, N.$$

To construct an accurate approximation for U_i^1, we employ the Taylor series expansion of $u(x, t + \Delta t)$ as follows:

$$u(x, t + \Delta t) = u(x, t) + \frac{\Delta t}{1!}\frac{\partial u}{\partial t}(x, t) + \frac{\Delta t^2}{2!}\frac{\partial^2 u}{\partial t^2}(x, t) + \mathcal{O}(\Delta t^3). \quad (54)$$

By utilizing the initial conditions (2) and (3) and evaluating the governing equation (1) at $(x_i, 0)$, we obtain:

$$\begin{cases} u(x_i, 0) = \phi_1(x_i), \\ \dfrac{\partial u}{\partial t}(x_i, 0) = \phi_2(x_i), \\ \dfrac{\partial^2 u}{\partial t^2}(x_i, 0) = \eta(x_i), \end{cases} \qquad i = 1, 2, \ldots, N, \quad (55)$$

where

$$\begin{aligned} \eta(x) &= -\lambda\frac{\partial u}{\partial t}(x, 0) + \mu\Delta u(x, 0) + f(x, 0, u(x, 0)) \\ &= -\lambda\phi_2(x) + \mu\phi_1''(x) + f(x, 0, \phi_1(x)). \end{aligned}$$

From (54) and (55), the approximation U_i^1 can be obtained by the following relation:

$$\begin{aligned} U_i^1 = U(x_i, t_1) &\approx u(x_i, 0) + \frac{\Delta t}{1!}\frac{\partial u}{\partial t}(x_i, 0) + \frac{\Delta t^2}{2!}\frac{\partial^2 u}{\partial t^2}(x_i, 0) \\ &= \phi_1(x_i) + \phi_2(x_i)\Delta t + \frac{1}{2}\eta(x_i)\Delta t^2, \quad i = 1, 2, \ldots, N. \end{aligned}$$

Similarly, under the assumption that $\lambda > 0$, the Taylor series expansion of $v(x, t + \Delta t)$ is employed to obtain an appropriate approximation for V_i^1, as follows:

$$v(x, t + \Delta t) = v(x, t) + \frac{\Delta t}{1!}\frac{\partial v}{\partial t}(x, t) + \frac{\Delta t^2}{2!}\frac{\partial^2 v}{\partial t^2}(x, t) + \mathcal{O}(\Delta t^3). \quad (56)$$

By applying the initial condition (3) and substituting $(x_i, 0)$ into the governing equation (1), we arrive at:

$$\begin{cases} v(x_i,0) = \phi_2(x_i), \\ \dfrac{\partial v}{\partial t}(x_i,0) = \eta(x_i), \end{cases} \qquad i = 1, 2, \ldots, N. \tag{57}$$

The value of $\dfrac{\partial^2 v}{\partial t^2}(x_i, 0)$ can be obtained by differentiating equation (1) with respect to time and then evaluating the result at the point $(x_i, 0)$, as follows:

$$\begin{aligned} \frac{\partial^2 v}{\partial t^2}(x_i,0) &= -\lambda \frac{\partial^2 u}{\partial t^2}(x_i,0) + \mu \frac{\partial^3 u}{\partial t \partial x^2}(x_i,0) + \frac{\partial f}{\partial t}(x_i,0,u(x_i,0)) \\ &+ \frac{\partial f}{\partial u}(x_i,0,u(x_i,0)) \frac{\partial u}{\partial t}(x_i,0) \\ &= -\lambda \eta(x_i) + \frac{\partial f}{\partial t}(x,0,\phi_1(x_i)) + \frac{\partial f}{\partial u}(x_i,0,\phi_1(x_i))\phi_2(x_i) \\ &+ \mu \frac{\partial^3 u}{\partial t \partial x^2}(x_i,0). \end{aligned} \tag{58}$$

To find $\dfrac{\partial^3 u}{\partial t \partial x^2}(x_i, 0)$, we take the second-order derivative of equation (1) with respect to x, which leads to:

$$\begin{aligned} \frac{\partial^3 u}{\partial t \partial x^2}(x_i,0) &= \frac{1}{\lambda}\Big[-\eta''(x_i) + \mu \phi_1^{(4)}(x_i) + \frac{\partial^2 f}{\partial x^2}(x_i,0,\phi_1(x_i)) \\ &+ 2\frac{\partial^2 f}{\partial x \partial u}(x_i,0,\phi_1(x_i))\phi_1'(x_i) + \frac{\partial^2 f}{\partial u^2}(x_i,0,\phi_1(x_i))(\phi_1'(x_i))^2 \\ &+ \frac{\partial f}{\partial u}(x_i,0,\phi_1(x_i))\phi_1''(x_i)\Big]. \end{aligned} \tag{59}$$

Finally, by substituting (57)-(59) into (56), the following approximation can be computed:

$$V_i^1 = V(x_i, t_1) \approx v(x_i, 0) + \frac{\Delta t}{1!}\frac{\partial v}{\partial t}(x_i, 0) + \frac{\Delta t^2}{2!}\frac{\partial^2 v}{\partial t^2}(x_i, 0), \;\; i = 1, 2, \ldots, N.$$

Remark 3.1. Since it is no longer possible to compute the value of $\dfrac{\partial^3 u}{\partial t \partial x^2}(x_i, 0)$ from relation (59) when $\lambda = 0$, we consider the following

second-order approximation in such a case:

$$V_i^1 = V(x_i, t_1) \approx v(x_i, 0) + \Delta t \frac{\partial v}{\partial t}(x_i, 0) = \phi_2(x_i) + \eta(x_i)\Delta t, \ i = 1, 2, \ldots, N.$$

4 Numerical Experiments

In this section, to evaluate the performance and efficiency of the proposed method, four numerical examples are considered, and the obtained results are compared with those of existing approaches. During the numerical experiments, the absolute errors and the experimental rate of convergence are calculated using the following formulas:

$$L_\infty^u = \max_{(i,n)\in\mathcal{S}} |u(x_i, t_n) - U_i^n|,$$
$$L_\infty^v = \max_{(i,n)\in\mathcal{S}} |v(x_i, t_n) - V_i^n|,$$
$$\text{C-order} = \frac{\ln(\text{E}_1) - \ln(\text{E}_2)}{\ln 2},$$

where

$$\mathcal{S} = \{(i, n) \,|\, i = 1, 2, \ldots, N, \ n = 0, 1, \ldots, m\},$$

and E_1 and E_2 represent the L_∞ errors corresponding to time steps of size Δt and $\frac{\Delta t}{2}$, respectively. The computational algorithm is implemented using `Maple` software, and all numerical simulations are performed on a Core(TM) i7 machine equipped with a 3.60 GHz processor and 8 GB of RAM. We emphasize that all numerical results reported in Tables 2, 4, 6, and 8 consistently exhibit an experimental convergence order of approximately 2. This empirical observation explicitly confirms the theoretical second-order convergence proven in Theorem 2.4 for all examples considered. Moreover, in order to enhance the temporal convergence rate of the proposed method, the approximate solutions can be improved using the Richardson extrapolation technique. This is done by executing the scheme with two different time step sizes, Δt and $\frac{\Delta t}{2}$, and applying the following relations:

$$\tilde{U}_i^n = \frac{1}{3}\left(4U_i^{2n,\frac{\Delta t}{2}} - U_i^{n,\Delta t}\right),$$
$$\tilde{V}_i^n = \frac{1}{3}\left(4V_i^{2n,\frac{\Delta t}{2}} - V_i^{n,\Delta t}\right).$$

Table 1: Comparison of the absolute error L_∞^u for PM, PM-REM, and BRGM in Example 4.1.

Methods	L_∞^u
BRGM ($n=1, m=2$) [45]	7.7000×10^{-4}
BRGM ($n=1, m=4$) [45]	7.0000×10^{-6}
PM ($h=0.1,\ \Delta t=0.0025$)	8.0175×10^{-6}
PM ($h=0.1,\ \Delta t=0.00125$)	2.0042×10^{-6}
PM ($h=0.1,\ \Delta t=0.000625$)	5.0106×10^{-7}
PM-REM ($h=0.1,\ \Delta t=0.0025$)	4.5359×10^{-8}
PM-REM ($h=0.1,\ \Delta t=0.00125$)	5.6226×10^{-9}

Example 4.1 ([45])**.**

Consider the problem (1)-(5) with

$$\begin{cases} a=0, \quad b=1, \quad \lambda=0, \quad \mu=1, \\ f(x,t,u)=\frac{2x(3x-2)(t^2-1)}{(t^2+1)^2}+6\ln\left(t^2+1\right), \\ \phi_1(x)=\phi_2(x)=0, \\ \alpha_1=1, \quad \alpha_2=\beta_1=\beta_2=0, \\ g_1(t)=g_2(t)=0, \\ \psi_1(x)=0, \quad \psi_2(x)=-1, \end{cases}$$

and the exact solution

$$u(x,t)=x(2-3x)\ln\left(t^2+1\right).$$

Table 1 provides a comparison between the results obtained by the proposed method (PM), its enhanced version incorporating Richardson extrapolation (PM-REM), and those achieved by the Bernstein Ritz-Galerkin method (BRGM) [45], evaluated at the final time $T=1$. Furthermore, Table 2 outlines the PM results along with the corresponding CPU times and estimated convergence rates. In addition, Figure 1 presents the exact and numerical solutions for both u and its time derivative $v=\dfrac{\partial u}{\partial t}$, along with the corresponding absolute errors for the final time $T=10$ to show the robustness of the PM-REM over a long time interval.

Example 4.2 ([38, 16, 15, 29, 1])**.**

Table 2: Numerical results for Example 4.1 with $h = 0.1$ and $T = 1$.

m	L_∞^u	C$_u$-order	L_∞^v	C$_v$-order	CPU time (s)
20	2.8944×10^{-3}	—	2.3032×10^{-2}	—	0.359
40	7.9373×10^{-4}	1.8666	5.9713×10^{-3}	1.9475	0.531
80	2.0021×10^{-4}	1.9871	1.5069×10^{-3}	1.9864	0.828
160	5.0122×10^{-5}	1.9980	3.8544×10^{-4}	1.9670	1.453
320	1.2528×10^{-5}	2.0003	9.6600×10^{-5}	1.9964	2.734
640	3.1317×10^{-6}	2.0001	2.4168×10^{-5}	1.9989	5.516

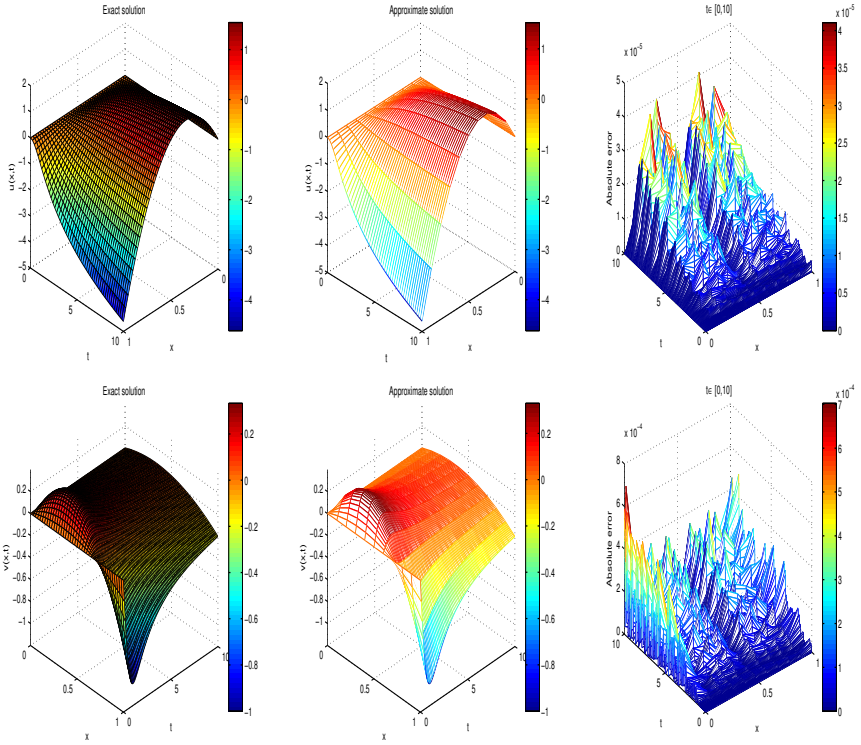


Figure 1: Exact and numerical solutions with absolute errors for Example 4.1 with $T = 10$, $h = 0.1$ and $\Delta t = 0.01$.

In this example, we consider the problem (1)-(5) with

$$\begin{cases} a=0, \quad b=1, \quad \lambda=0, \quad \mu=1, \\ f(x,t,u)=\left(\pi^2+\frac{1}{4}\right)\mathrm{e}^{-\frac{t}{2}}\sin(\pi x), \\ \phi_1(x)=\sin(\pi x), \quad \phi_2(x)=-\frac{1}{2}\sin(\pi x), \\ \alpha_1=1, \quad \alpha_2=0, \quad \beta_1=1, \quad \beta_2=0, \\ g_1(t)=\pi\mathrm{e}^{-\frac{t}{2}}, \quad g_2(t)=\frac{2}{\pi}\mathrm{e}^{-\frac{t}{2}}, \\ \psi_1(x)=0, \quad \psi_2(x)=-1, \end{cases}$$

and the exact solution

$$u(x,t)=\mathrm{e}^{-\frac{t}{2}}\sin(\pi x).$$

The numerical results obtained by the proposed method, both with and without Richardson extrapolation, are compared in Table 3 with those of the cubic B-spline-based finite element method (CBSFEM) [29], the cubic B-spline scaling functions method (CBSFM) [15], the shifted Legendre tau method (SLTM) [38], the collocation using RBFs (the thin plate spline (TPS), the multiquadric (MQ), and the compactly supported (CS) radial basis functions) [16] and the cubic trigonometric B-spline-based collocation scheme combined with FDM (CTBCM-FDM) [1]. This comparison demonstrates the favorable performance of the proposed approach relative to other numerical schemes. Table 4 also presents the results obtained by the PM, including the associated CPU times and the estimated rates of convergence. Figure 2 displays both the exact and numerical solutions computed via the PM-REM, as well as the absolute error with the final time $T=5$.

Example 4.3 ([44])**.**

Consider the problem (1)-(5) with

$$\begin{cases} a=0, \quad b=1, \quad \lambda=1, \quad \mu=1, \\ f(x,t,u)=u^2-\frac{3}{2}\left(1-3x^2\right)\left(\mathrm{e}^{-t}+1\right)+9\left(\mathrm{e}^{-t}-1\right)\left(t-\frac{1}{4}t^2\left(\mathrm{e}^{-t}-1\right)\left(1-3x^2\right)^2\right), \\ \phi_1(x)=\phi_2(x)=0, \\ \alpha_1=\alpha_2=0, \quad \beta_1=1, \quad \beta_2=0, \\ g_1(t)=g_2(t)=0, \\ \psi_1(x)=0, \quad \psi_2(x)=-1, \end{cases}$$

Table 3: Comparison of the absolute error L_∞^u for different methods in Example 4.2.

Methods	L_∞^u
CBSFEM [29]	2.3000×10^{-2}
CBSFM [15]	1.5000×10^{-2}
SLTM ($m = n = 4$) [38]	5.0000×10^{-3}
SLTM ($m = n = 5$) [38]	3.6000×10^{-3}
CS-RBF ($h = 0.01,\ \Delta t = 0.0001$) [16]	5.0746×10^{-2}
TPS-RBF ($h = 0.01,\ \Delta t = 0.0001$) [16]	6.8404×10^{-3}
MQ-RBF ($h = 0.01,\ \Delta t = 0.0001$) [16]	2.3794×10^{-3}
CTBCM-FDM ($h = 0.01,\ \Delta t = 0.0001$) [1]	8.0628×10^{-4}
PM ($h = 0.01,\ \Delta t = 0.01$)	4.6614×10^{-5}
PM ($h = 0.01,\ \Delta t = 0.005$)	1.0989×10^{-5}
PM-REM ($h = 0.01,\ \Delta t = 0.01$)	3.6802×10^{-6}

Table 4: Numerical results for Example 4.2 with $h = 0.0125$ and $T = 1$.

m	L_∞^u	C$_u$-order	L_∞^v	C$_v$-order	CPU time (s)
16	1.9689×10^{-3}	—	5.8764×10^{-2}	—	12.453
32	4.4541×10^{-4}	2.1442	1.0862×10^{-2}	2.4356	26.719
64	1.1217×10^{-4}	1.9894	1.7523×10^{-3}	2.6320	53.015
128	2.6538×10^{-5}	2.0795	4.1155×10^{-4}	2.0901	100.141

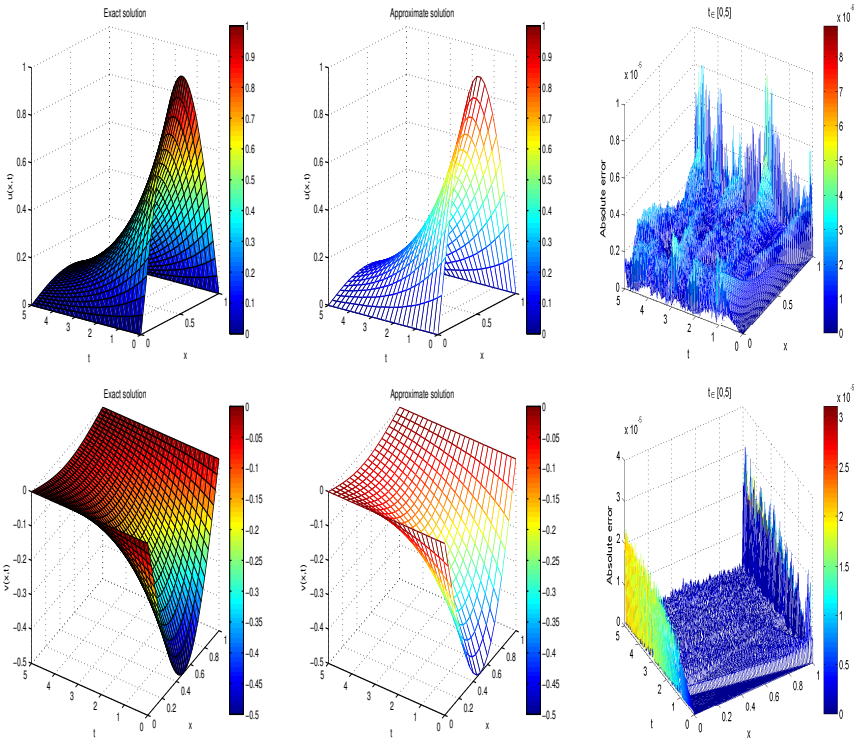


Figure 2: Exact and numerical solutions with absolute errors for Example 4.2 with $T = 5$, $h = 0.01$ and $\Delta t = 0.0125$.

Table 5: Comparison of the absolute error L_∞^u for PM, PM-REM, and RKM in Example 4.3.

Methods	L_∞^u
RKM ($N = 255$) [44]	3.3901×10^{-4}
PM ($N = 11,\ \Delta t = 0.005$)	5.8139×10^{-5}
PM ($N = 11,\ \Delta t = 0.0025$)	1.4518×10^{-5}
PM ($N = 11,\ \Delta t = 0.00125$)	3.6287×10^{-6}
PM-REM ($N = 11,\ \Delta t = 0.0025$)	8.5507×10^{-8}

Table 6: Numerical results for Example 4.3 with $h = 0.1$ and $T = 1$.

m	L_∞^u	C_u-order	L_∞^v	C_v-order	CPU time (s)
20	5.8509×10^{-3}	—	2.6403×10^{-2}	—	0.312
40	1.4857×10^{-3}	1.9775	7.6016×10^{-3}	1.7963	0.501
80	3.6572×10^{-4}	2.0223	1.9511×10^{-3}	1.9620	0.813
160	9.0879×10^{-5}	2.0087	4.9241×10^{-4}	1.9864	1.609
320	2.2691×10^{-5}	2.0018	1.2335×10^{-4}	1.9971	2.906
640	5.6701×10^{-6}	2.0007	3.0868×10^{-5}	1.9986	5.953

and the exact solution

$$u(x,t) = \frac{3}{2}t(\mathrm{e}^{-t} - 1)(1 - 3x^2).$$

Table 5 provides a comparison between the results obtained by the PM, PM-REM, and those obtained by the reproducing kernel method (RKM) [44] for $T = 1$. The results for the PM, together with the corresponding CPU times and the computational order of convergence, are listed in Table 6. Figure 3 shows the exact and approximate solutions obtained using PM-REM, along with the absolute error for the final time $T = 5$.

Example 4.4 ([44])**.**

Consider the problem (1)-(5) with

$$\begin{cases} a = 0, \quad b = 1, \quad \lambda = 1, \quad \mu = 1, \\ f(x,t,u) = \sin(u) + p(x,t), \\ \phi_1(x) = \phi_2(x) = 0, \\ \alpha_1 = \alpha_2 = 0, \quad \beta_1 = 1, \quad \beta_2 = 0, \\ g_1(t) = g_2(t) = 0, \\ \psi_1(x) = 0, \quad \psi_2(x) = -1. \end{cases}$$

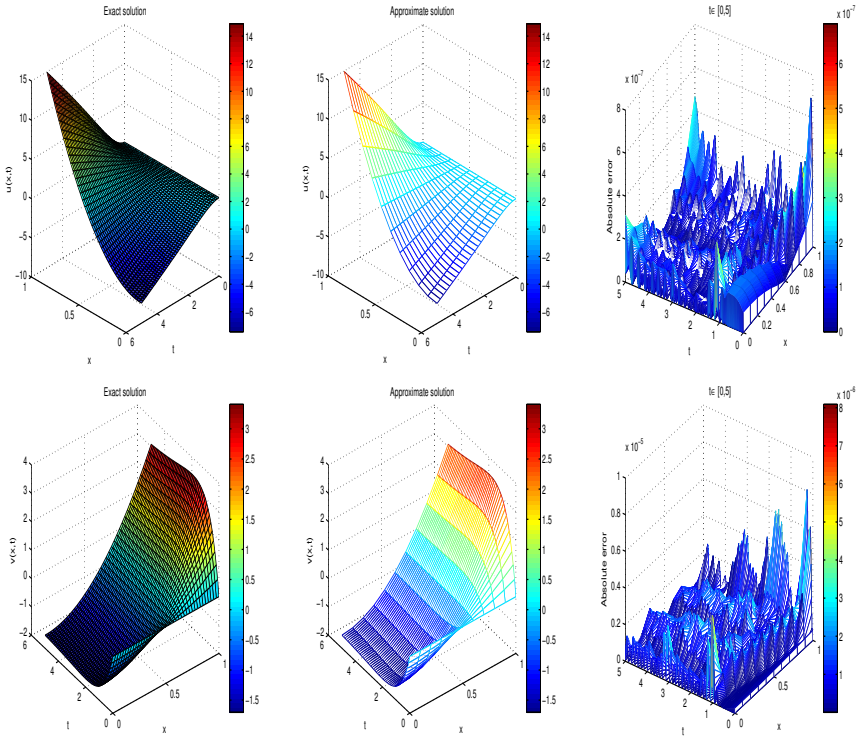


Figure 3: Exact and numerical solutions with absolute errors for Example 4.3 with $T = 5$, $h = 0.1$ and $\Delta t = 0.005$.

Table 7: Comparison of the absolute error L_∞^u for PM, PM-REM, and RKM in Example 4.4.

Methods	L_∞^u
RKM ($N = 255$) [44]	4.3313×10^{-4}
PM ($N = 51,\ \Delta t = 0.005$)	2.9015×10^{-5}
PM ($N = 51,\ \Delta t = 0.0025$)	8.2357×10^{-6}
PM ($N = 81,\ \Delta t = 0.005$)	2.7999×10^{-5}
PM ($N = 81,\ \Delta t = 0.0025$)	7.2224×10^{-6}
PM-REM ($N = 81,\ \Delta t = 0.005$)	4.0299×10^{-7}

Table 8: Numerical results for Example 4.4 with $h = 0.1$ and $T = 1$.

m	L_∞^u	C_u-order	L_∞^v	C_v-order	CPU time (s)
10	1.0982×10^{-2}	—	4.6061×10^{-2}	—	7.797
20	2.7805×10^{-3}	1.9817	1.2169×10^{-2}	1.9204	14.437
40	6.9347×10^{-4}	2.0034	2.8601×10^{-3}	2.0891	33.203
80	1.7345×10^{-4}	1.9993	7.1985×10^{-4}	1.9903	62.453

The exact solution of this problem is given by

$$u(x,t) = 2t^2 \left(\frac{2e^x - 2x}{2e - 3} - 1 \right).$$

The source term $p(x,t)$ is obtained by substituting the above exact solution into the governing equation, namely,

$$p(x,t) = \frac{\partial^2 u}{\partial t^2} + \lambda \frac{\partial u}{\partial t} - \mu \frac{\partial^2 u}{\partial x^2} - \sin(u).$$

Table 7 presents the numerical outcomes obtained by the PM and PM-REM, as well as those produced by the reproducing kernel method (RKM) [44], for the final time $T = 1$. Additional details for the PM, including the corresponding CPU times and the computational order of convergence, are summarized in Table 8. Figure 4 illustrates the exact and approximate solutions via PM-REM, together with the corresponding absolute error for $T = 5$.

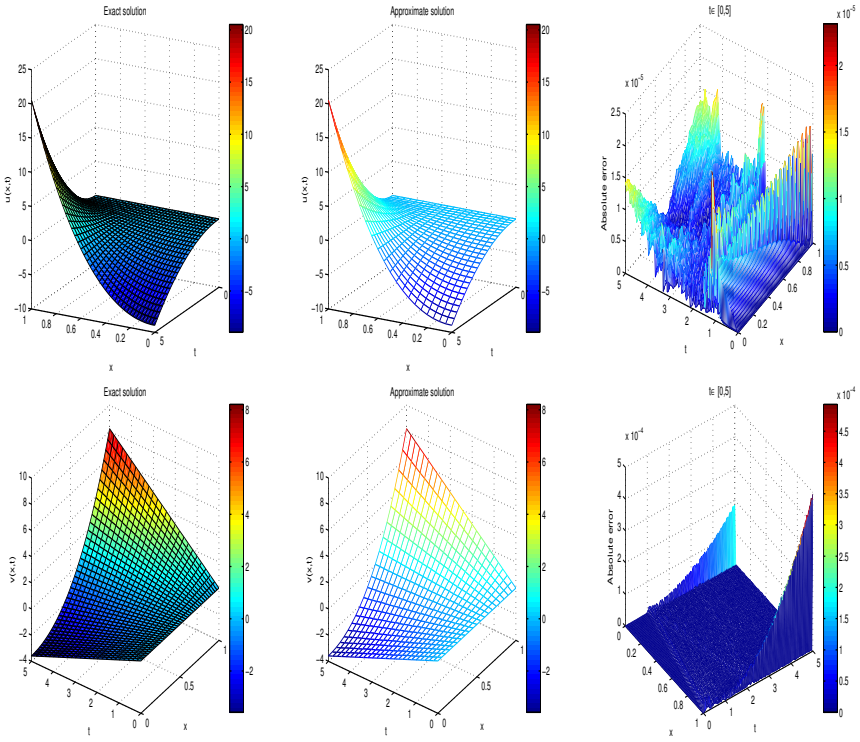


Figure 4: Exact and numerical solutions with absolute errors for Example 4.4 with $T = 5$, $h = 0.0125$ and $\Delta t = 0.025$.

5 Conclusion

In this paper, a numerical method was proposed to approximate the solution of hyperbolic PDEs with nonlocal boundary conditions. The method combined the Crank–Nicolson finite difference scheme in the time direction with a modified cubic B-spline-based DQM in the spatial direction. By transforming the original equation into a system of first-order differential equations, a fully-discrete formulation of the method was derived. This transformation enables both the solution function and its first-order time derivative to be numerically obtained by solving a system of linear algebraic equations at each time step. The convergence and stability of the method in the temporal direction were examined in detail. Finally, the efficiency and capability of the proposed method were evaluated through several numerical examples, including both linear and nonlinear problems over different time intervals. The results confirmed the favorable accuracy of the proposed scheme.

For future work, we intend to extend the present approach to solve partial integro-differential equations with nonlocal boundary conditions (e.g., multi-singular fractional integro-differential equations). Moreover, the proposed technique can be adapted for time-fractional hyperbolic PDEs. In such an extension, the classical first-order time derivative would be replaced by a fractional derivative (e.g., in the Caputo or Riemann-Liouville sense). This modification would require revisiting the stability and convergence analysis, as well as adjusting the temporal discretization scheme to handle the nonlocal nature of fractional derivatives. Further extensions include fractal KdV equation [2], LWR traffic model [20], tumor growth with piecewise operators [22], and nonlinear Volterra integro-differential equations of fractional order [28, 3, 19]. Another promising direction is to generalize the existing fractional models for Mumps-induced hearing loss [30] and COVID-19 transmission [43] to higher-dimensional cases, and then solve them using the methodology proposed in this paper. Investigating these aspects remains an interesting direction for future research.

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