Journal of Mathematical Extension
Vol. 10, No. 1, (2016), 53-63
ISSN: 1735-8299
URL: http://www.ijmex.com

Hyers-Ulam-Rassias Approximation on m-Lie Algebras

H. Azadi Kenary*
Yasouj University

Kh. Shafaat
Yasouj University

H. Keshavarz
Yasouj University

Abstract. Using the fixed point method, we establish the stability of $m-$Lie homomorphisms and Jordan $m-$Lie homomorphisms on $m-$Lie algebras associated to the following additive functional equation

$$2\mu f\left(\sum_{i=1}^{m} mx_i\right) = \sum_{i=1}^{m} f\left(\mu\left(mx_i + \sum_{j=1\ ,i\neq j}^{m} x_j\right)\right) + f\left(\sum_{i=1}^{m} \mu x_i\right)$$

where m is an integer greater than 2 and all $\mu \in \mathbb{T}_{\frac{1}{n_0}} := \left\{e^{i\theta}\ ;\ 0 \leq \theta \leq \frac{2\pi}{n_0}\right\}$.

AMS Subject Classification: 17A42; 39B82.
Keywords and Phrases: $m-$Lie algebra, homomorphism, Jordan homomorphism, stability, fixed point approach, functional equation

1. Introduction

Let n be a natural number greater or equal to 3. The notion of an $n-$Lie algebra was introduced by V.T. Filippov. The Lie product is taken

Received: January 2015; Accepted: July 2015
*Corresponding Author

between n elements of the algebra instead of two. This new bracket is $n-$linear, anti–symmetric and satisfies a generalization of the Jacobi identity.
An $n-$Lie algebra is a natural generalization of a Lie algebra. Namely:

A vector space V together with a multi–linear, antisymmetric $n-$ary operation $[\quad] : \Lambda^n V \to V$ is called an $n-$Lie algebra, $n \geq 3$, if the $n-$ary bracket is a derivation with respect to itself, i.e,

$$[[x_1, ..., x_n], x_{n+1}, ..., x_{2n-1}] = \sum_{i=1}^{n} [x_1, ...x_{i-1}[x_i, x_{n+1}, ..., x_{2n-1}], ..., x_n], \tag{1}$$

where $x_1, x_2, ..., x_{2n-1} \in V$. Equation (1) is called the generalized Jacobi identity. The meaning of this identity is similar to that of the usual Jacobi identity for a Lie algebra (which is a $2-$Lie algebra).
From now on, we only consider $n-$Lie algebras over the field of complex numbers. An $n-$Lie algebra A is a normed $n-$Lie algebra if there exists a norm $||\ ||$ on A such that $||[x_1, x_2, ..., x_n]|| \leq ||x_1||||x_2||...||x_n||$ for all $x_1, x_2, ..., x_n \in A$. A normed $n-$Lie algebra A is called a Banach $n-$Lie algebra, if $(A, ||\ ||)$ is a Banach space.

Let $(A, [\quad]_A)$ and $(B, [\quad]_B)$ be two Banach $n-$Lie algebras. A $\mathbb{C}-$linear mapping $H : (A, [\,]_A) \to (B, [\,]_B)$ is called an $n-$Lie homomorphism if $H([x_1 x_2 ... x_n]_A) = [H(x_1)H(x_2)...H(x_n)]_B$ for all $x_1, x_2, ..., x_n \in A$. A $\mathbb{C}-$linear mapping $H : (A, [\,]_A) \to (B, [\,]_B)$ is called a Jordan $n-$Lie homomorphism if $H([xx...x]_A) = [H(x)H(x)...H(x)]_B$ for all $x \in A$.

The study of stability problems had been formulated by Ulam [5] during a talk in 1940: Under what condition does there exist a homomorphism near an approximate homomorphism? In the following year, Hyers [3] was answered affirmatively the question of Ulam for Banach spaces, which states that if $\varepsilon > 0$ and $f : X \to Y$ is a map with X a normed space, Y a Banach space such that

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon \tag{2}$$

for all $x, y \in X$, then there exists a unique additive map $T : X \to Y$ such that $\|f(x) - T(x)\| \leq \varepsilon$ for all $x \in X$. A generalized version of the theorem of Hyers for approximately linear mappings was presented by Rassias [4] in 1978 by considering the case when inequality (2) is unbounded.

Due to that fact, the additive functional equation $f(x+y) = f(x)+f(y)$ is said to have the generalized Hyers–Ulam–Rassias stability property. A large list of references concerning the stability of functional equations can be found in [1, 2].
In this paper, by using the fixed point method, we establish the stability of $m-$Lie homomorphisms and Jordan $m-$Lie homomorphisms on $m-$Lie Banach algebras associated to the following generalized Jensen type functional equation

$$2\mu f\left(\sum_{i=1}^{m} mx_i\right) - \sum_{i=1}^{m} f\left(\mu\left(mx_i + \sum_{j=1\ ,i\neq j}^{m} x_j\right)\right) - f\left(\sum_{i=1}^{m} \mu x_i\right) = 0 \tag{3}$$

for all $\mu \in \mathbb{T}_{\frac{1}{n_o}} := \left\{e^{i\theta}\ ;\ 0 \leq \theta \leq \frac{2\pi}{n_o}\right\}$, where $m \geq 2$. Throughout this paper, assume that $(A, [\ ]_A), (B, [\ ]_B)$ are two m-Lie Banach algebras.

2. Main Results

Before proceeding to the main results, we recall a fundamental result in fixed point theory.

Theorem 2.1. [4] *Let (Ω, d) be a complete generalized metric space and $T : \Omega \to \Omega$ be a strictly contractive function with Lipschitz constant L. Then for each given $x \in \Omega$, either $d(T^m x, T^{m+1}x) = \infty$ for all $m \geq 0$, or other exists a natural number m_0 such that:*
(i) $d(T^m x, T^{m+1}x) < \infty$ for all $m \geq m_0$;
(ii) the sequence $\{T^m x\}$ is convergent to a fixed point y^* of T;
(iii) y^* is the unique fixed point of T in $\Lambda = \{y \in \Omega : d(T^{m_0}x, y) < \infty\}$;
(iv) $d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Lambda$.

Theorem 2.2. [2] *Let V and W be real vector spaces. A mapping $f : V \to W$ satisfies the following functional equation*

$$2f\left(\sum_{i=1}^{m} mx_i\right) = \sum_{i=1}^{m} f\left(mx_i + \sum_{j=1\ ,i\neq j}^{m} x_j\right) + f\left(\sum_{i=1}^{m} x_i\right),$$

if and only if f is additive.

We start our work with the main theorem of our paper.

Theorem 2.3. *Let $n_0 \in \mathbb{N}$ be a fixed positive integer. Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^m \to [0, \infty)$ such that*

$$\left\| 2\mu f\left(\sum_{i=1}^{m} mx_i\right) - \sum_{i=1}^{m} f\left(\mu\left(mx_i + \sum_{j=1\ ,i\neq j}^{m} x_j\right)\right) - f\left(\sum_{i=1}^{m} \mu x_i\right)\right\| \leq \varphi(x_1, x_2, \cdots, x_m) \quad (4)$$

for all $\mu \in \mathbb{T}_{\frac{1}{n_o}} := \left\{e^{i\theta}\ ;\ 0 \leq \theta \leq \frac{2\pi}{n_0}\right\}$ and all $x_1, \cdots, x_m \in A$, and that

$$\|f([x_1 x_2 \cdots x_n]_A) - [f(x_1)f(x_2)\cdots f(x_m)]_B\|_B \leq \varphi(x_1, x_2, \cdots, x_m) \quad (5)$$

for all $x_1, \cdots, x_m \in A$. If there exists an $L < 1$ such that

$$\varphi(x_1, x_2, \cdots, x_m) \leq mL\varphi\left(\frac{x_1}{m}, \frac{x_2}{m}, \cdots, \frac{x_m}{m}\right) \quad (6)$$

for all $x_1, \cdots, x_m \in A$, then there exists a unique $m-$Lie homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\| \leq \frac{\varphi(x, 0, 0, \cdots, 0)}{m - mL} \quad (7)$$

for all $x \in A$.

Proof. Let Ω be the set of all functions from A into B and let $d(g, h) := \inf\{C \in \mathbb{R}^+ : \|g(x) - h(x)\|_B \leq C\varphi(x, 0, \cdots, 0), \forall x \in A\}$. It is easy to show that (Ω, d) is a generalized complete metric space. Now we define the mapping $J : \Omega \to \Omega$ by $J(h)(x) = \frac{1}{m}h(mx)$ for all $x \in A$. Note that for all $g, h \in \Omega$, with $d(g, h) < C$ we have

$$\begin{aligned}
\|g(x) - h(x)\| &\leq C\phi(x, 0, \cdots, 0) \\
\left\|\frac{1}{m}g(mx) - \frac{1}{m}h(mx)\right\| &\leq \frac{C\varphi(mx, 0, \cdots, 0)}{m} \\
\left\|\frac{1}{m}g(mx) - \frac{1}{m}h(mx)\right\| &\leq LC\varphi(x, 0, \cdots, 0) \\
d(J(g), J(h)) &\leq L\ C.
\end{aligned}$$

for all $x \in A$. Hence we see that $d(J(g), J(h)) \leq Ld(g,h)$ for all $g, h \in \Omega$. It follows from (6) that

$$\lim_{k\to\infty} \frac{\varphi(m^k x_1, m^k x_2, \cdots, m^k x_m)}{m^k} \leq \lim_{k\to\infty} L^k \varphi(x_1, \cdots, x_m) = 0 \tag{8}$$

for all $x_1, \cdots, x_m \in A$. Putting $\mu = 1, x_1 = x$ and $x_j = 0$ $(j = 2, \cdots, m)$ in (4), we obtain

$$\left\| \frac{f(mx)}{m} - f(x) \right\| \leq \frac{\varphi(x, 0, \cdots, 0)}{m} \tag{9}$$

for all $x \in A$. Therefore,

$$d(f, J(f)) \leq \frac{1}{m} < \infty. \tag{10}$$

By Theorem 2.1, J has a unique fixed point in the set $X_1 := \{h \in \Omega : d(f,h) < \infty\}$. Let H be the fixed point of J. H is the unique mapping with $H(mx) = mH(x)$ for all $x \in A$, such that $\|f(x) - H(x)\|_B \leq C\varphi(x, 0, \cdots, 0)$ for all $x \in A$ and some $C \in (0, \infty)$. On the other hand we have $\lim_{k\to\infty} d(J^k(f), H) = 0$, and so

$$\lim_{k\to\infty} \frac{1}{m^k} f(m^k x) = H(x), \tag{11}$$

for all $x \in A$. Also by Theorem 2.1, we have

$$d(f, H) \leq \frac{d(f, J(f))}{1 - L}. \tag{12}$$

From (10) and (12), we have $d(f,H) \leq \frac{1}{m - mL}$. This implies the inequality (7). By (5), we have

$$\begin{aligned}
&\|H([x_1 x_2 \cdots x_m]_A) - [H(x_1)H(x_2)H(x_3)\cdots H(x_m)]_B\| \\
&= \lim_{k\to\infty} \left\| \frac{H([m^k x_1 m^k x_2 \cdots m^k x_m]_A)}{m^{mk}} \right. \\
&\left. - \frac{([H(m^k x_1)H(m^k x_2)H(m^k x_3)\cdots H(m^k x_m)]_B)}{m^{mk}} \right\| \\
&\leq \lim_{k\to\infty} \frac{\varphi(m^k x_1, m^k x_2, \cdots, m^k x_m)}{m^{mk}} = 0
\end{aligned}$$

for all $x_1, \cdots, x_m \in A$. Hence $H([x_1 x_2 \cdots x_m]_A) = [H(x_1)H(x_2)H(x_3)\cdots H(x_m)]_B$ for all $x_1, \cdots, x_m \in A$. On the other hand, it follows from (4), (8) and (11) that

$$\begin{aligned}&\left\| 2H\left(\sum_{i=1}^{m} m x_i\right) - \sum_{i=1}^{m} H\left(m x_i + \sum_{j=1\ ,i\neq j}^{m} x_j\right) - H\left(\sum_{i=1}^{m} x_i\right)\right\| \\ &= \lim_{k\to\infty} \frac{1}{m^k}\left\| 2f\left(\sum_{i=1}^{m} m^{k+1} x_i\right) - \sum_{i=1}^{m} f\left(m^{k+1} x_i + \sum_{j=1\ ,i\neq j}^{m} m^k x_j\right)\right. \\ &\qquad\qquad \left. - f\left(\sum_{i=1}^{m} m^k x_i\right)\right\| \\ &\leq \lim_{m\to\infty} \frac{\varphi(m^k x_1, m^k x_2, \cdots, m^k x_m)}{m^k} = 0\end{aligned}$$

for all $x_1, \cdots, x_m \in A$. Then

$$2H\left(\sum_{i=1}^{m} m x_i\right) = \sum_{i=1}^{m} H\left(m x_i + \sum_{j=1\ ,i\neq j}^{m} x_j\right) + H\left(\sum_{i=1}^{m} x_i\right) \qquad (13)$$

for all $x_1, \cdots, x_m \in A$. So by Theorem 2.2, H is additive. Letting $x_i = x$ for all $i = 1, 2, \cdots, m$ in (4), we obtain $\|\mu f(x) - f(\mu x)\|_B \leq \varphi(x, x, \cdots, x)$ for all $x \in A$. It follows that

$$\begin{aligned}\|H(\mu x) - \mu H(x)\| &= \lim_{k\to\infty} \frac{\|f(\mu m^k x) - \mu f(m^k x)\|}{m^k} \\ &\leq \lim_{k\to\infty} \frac{\varphi(m^k x, m^k x, \cdots, m^k x)}{m^k} = 0\end{aligned}$$

for all $\mu \in \mathbb{T}_{\frac{1}{n_o}} := \left\{e^{i\theta}\ ;\ 0 \leq \theta \leq \frac{2\pi}{n_0}\right\}$, and all $x \in A$. One can show that the mapping $H : A \to B$ is $\mathbb{C}$–linear. Hence, $H : A \to B$ is an $m-$Lie homomorphism satisfying (7), as desired. □

Corollary 2.4. *Let θ and p be non–negative real numbers such that*

$p < 1$. *Suppose that a function* $f : A \to B$ *satisfies*

$$\left\| 2\mu f\left(\sum_{i=1}^{m} mx_i\right) - \sum_{i=1}^{m} f\left(\mu\left(mx_i + \sum_{j=1\ ,i\neq j}^{m} x_j\right)\right) - f\left(\sum_{i=1}^{m} \mu x_i\right)\right\| \leq \theta\left(\sum_{i=1}^{m} \|x_i\|_A^p\right) \quad (14)$$

for all $\mu \in \mathbb{T}_{\frac{1}{n_o}}$ *and all* $x_1, \cdots, x_m \in A$ *and*

$$\|f([x_1 x_2 \cdots x_n]_A) - [f(x_1)f(x_2)\cdots f(x_m)]_B\| \leq \theta\left(\sum_{i=1}^{m} \|x_i\|_A^p\right) \quad (15)$$

for all $x_1, \cdots, x_n \in A$. *Then there exists a unique* $m-$*Lie homomorphism* $H : A \to B$ *such that*

$$\|f(x) - H(x)\| \leq \frac{\theta\|x\|_A^p}{m - m^p} \quad (16)$$

for all $x \in A$.

Proof. Put $\varphi(x_1, x_2, \cdots, x_m) := \theta\sum_{i=1}^{m}(\|x_i\|_A^p)$ for all $x_1, \cdots, x_n \in A$ in Theorem 2.3. Then (8) holds for $p < 1$, and (16) holds when $L = m^{p-1}$. $\square$

Similarly, we have the following and we will omit the proof.

Theorem 2.5. *Let* $n_0 \in \mathbb{N}$ *be a fixed positive integer. Let* $f : A \to B$ *be a mapping for which there exists a function* $\varphi : A^m \to [0, \infty)$ *satisfying (4) for all* $\mu \in \mathbb{T}_{\frac{1}{n_0}} := \left\{e^{i\theta}\ ;\ 0 \leq \theta \leq \frac{2\pi}{n_0}\right\}$ *and (5). If there exists an* $L < 1$ *such that*

$$\varphi\left(\frac{x_1}{m}, \frac{x_2}{m}, \cdots, \frac{x_m}{m}\right) \leq \frac{L\varphi(x_1, x_2, \cdots, x_m)}{m} \quad (17)$$

for all $x_1, \cdots, x_m \in A$, *then there exists a unique* $m-$*Lie homomorphism* $H : A \to B$ *such that*

$$\|f(x) - H(x)\| \leq \frac{L\varphi(x, 0, 0, \cdots, 0)}{m - mL} \quad (18)$$

for all $x \in A$.

Corollary 2.6. *Let θ and p be non–negative real numbers such that $p > 1$. Suppose that a function $f : A \to B$ satisfying (14) and (15). Then there exists a unique $m-$Lie homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\| \leq \frac{m\theta\|x\|_A^p}{m^{p+1} - m^2} \tag{19}$$

for all $x \in A$.

Proof. Put $\varphi(x_1, x_2, \cdots, x_m) := \theta \sum_{i=1}^{m}(\|x_i\|_A^p)$ for all $x_1, \cdots, x_n \in A$ in Theorem 2.5. Then (18) holds for $p < 1$, and (19) holds when $L = m^{(1-p)}$. $\square$

Theorem 2.7. *Let $n_0 \in \mathbb{N}$ be a fixed positive integer. Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^m \to [0, \infty)$ such that*

$$\left\|2\mu f\left(\sum_{i=1}^{m} mx_i\right) - \sum_{i=1}^{m} f\left(\mu\left(mx_i + \sum_{j=1\ ,i\neq j}^{m} x_j\right)\right) - f\left(\sum_{i=1}^{m} \mu x_i\right)\right\| \leq \varphi(x_1, x_2, \cdots, x_m) \tag{20}$$

for all $\mu \in \mathbb{T}_{\frac{1}{n_o}} := \left\{e^{i\theta}\ ;\ 0 \leq \theta \leq \frac{2\pi}{n_0}\right\}$ and all $x_1, \cdots, x_m \in A$, and that

$$\|f([xx\cdots x]_A) - [f(x)f(x)\cdots f(x)]_B\|_B \leq \varphi(x, x, \cdots, x) \tag{21}$$

for all $x \in A$. If there exists an $L < 1$ such that

$$\varphi(x_1, x_2, \cdots, x_m) \leq mL\varphi\left(\frac{x_1}{m}, \frac{x_2}{m}, \cdots, \frac{x_m}{m}\right) \tag{22}$$

for all $x_1, \cdots, x_m \in A$, then there exists a unique Jordan $m-$Lie homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\| \leq \frac{\varphi(x, 0, \cdots, 0)}{m - mL} \tag{23}$$

for all $x \in A$.

Proof. By the same reasoning as in the proof of Theorem 2.3, we can define the mapping $H(x) = \lim_{k\to\infty} \frac{1}{m^k} f(m^k x)$ for all $x \in A$. Moreover, we can show that H is $\mathbb{C}$–linear. It follows from (21) that

$$\begin{aligned}&\|H([xx\cdots x]_A)-[H(x)H(x)\cdots H(x)]_B\|\\&=\lim_{k\to\infty}\left\|\frac{H([m^kx\cdots m^kx]_A)}{m^{mk}}-\frac{[H(m^kx)H(m^kx)\cdots H(m^kx)]_B}{m^{mk}}\right\|\\&\leq\lim_{k\to\infty}\frac{1}{m^{mk}}\varphi(m^kx,m^kx,...,m^kx)=0\end{aligned}$$

for all $x\in A$. So $H([xx\cdots x]_A)=[H(x)H(x)\cdots H(x)]_B$ for all $x\in A$. Hence $H:A\to B$ is a Jordan $m-Lie$ homomorphism satisfying (23). □

Corollary 2.8. *Let θ and p be non–negative real numbers such that $p<1$. Suppose that a function $f:A\to B$ satisfies*

$$\left\|2\mu f\left(\sum_{i=1}^{m}mx_i\right)-\sum_{i=1}^{m}f\left(\mu\left(mx_i+\sum_{j=1\ ,i\neq j}^{m}x_j\right)\right)-f\left(\sum_{i=1}^{m}\mu x_i\right)\right\|\leq\theta\sum_{i=1}^{n}(\|x_i\|_A^p)$$

for all $\mu\in\mathbb{T}_{\frac{1}{n_o}}$ and all $x_1,\cdots,x_m\in A$ and $\|f([xx\cdots x]_A)-[f(x)f(x)\cdots f(x)]_B\|\leq n\theta(\|x\|_A^p)$ for all $x\in A$. Then there exists a unique Jordan $m-$Lie homomorphism $H:A\to B$ such that

$$\|f(x)-H(x)\|_B\leq\frac{\theta\|x\|_A^p}{m-m^p}\tag{24}$$

for all $x\in A$.

Proof. It follows from Theorem 2.7 by putting $\varphi(x_1,x_2,\cdots,x_m):=\theta\sum_{i=1}^{m}(\|x_i\|_A^p)$ for all $x_1,\cdots,x_m\in A$ and $L=m^{(p-1)}$. □

Similarly, we have the following and we will omit the proof.

Theorem 2.9. *Let $n_0\in\mathbb{N}$ be a fixed positive integer. Let $f:A\to B$ be a mapping for which there exists a function $\varphi:A^m\to[0,\infty)$ satisfying (4) for all $\mu\in\mathbb{T}_{\frac{1}{n_o}}:=\left\{e^{i\theta}\ ;\ 0\leq\theta\leq\frac{2\pi}{n_o}\right\}$ and (5). If there exists an $L<1$ such that $\varphi\left(\frac{x_1}{m},\frac{x_2}{m},\cdots,\frac{x_m}{m}\right)\leq\frac{L}{m}\varphi(x_1,x_2,\cdots,x_m)$ for all $x_1,\cdots,x_m\in$*

A, then there exists a unique Jordan $m-$Lie homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\| \leq \frac{L\varphi(x, 0, 0, \cdots, 0)}{m - mL} \tag{25}$$

for all $x \in A$.

Corollary 2.10. *Let θ and p be non–negative real numbers such that $p > 1$. Suppose that a function $f : A \to B$ satisfying (14) and (21). Then there exists a unique Jordan $m-$Lie homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\|_B \leq \frac{\theta\|x\|_A^p}{m^p - m} \tag{26}$$

for all $x \in A$.

Proof. Put $\varphi(x_1, x_2, \cdots, x_m) := \theta \sum_{i=1}^{m}(\|x_i\|_A^p)$ for all $x_1, \cdots, x_n \in A$ in Theorem 2.9. Then (25) holds for $p > 1$, and (26) holds when $L = m^{(1-p)}$. □

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Hassan Azadi Kenary
Department of Mathematics
Associate Professor of Mathematics
College of Sciences, Yasouj University
Yasouj, Iran
E-mail: azadi@yu.ac.ir

Khadijeh Shafaat
Department of Mathematics
PhD student of Mathematics
College of Sciences, Yasouj University
Yasouj, Iran
E-mail: kh.shafaat@yahoo.com

Hamid Reza Keshavarz
Department of Mathematics
PhD student of Mathematics
College of Sciences, Yasouj University
Yasouj, Iran
E-mail:h.keshavarz@gmail.com