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Boundary Element Method for the Helmholtz Equation

A. Mesforush
Shahrood University

Z. Mehraban*
Shahrood University

Abstract. The boundary element methos is applied to the Helholtltz equation. To this end we discretized the Helmholtz equation over the boundary Ω and conclude a system of equations. By applying the boundary condition we get a new system which gives the solution of the equation.

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1. Introduction

Numerical approximation for solving a differential equation has a wide range of application in engineering and mathematics. Some of these methods are: finite difference methods, finite element methods and boundary element method. In these methods we usually discritize the equation and apply the governing equations on the suitable mesh. The most important thing in the numerical approximation is determination of the size of mesh such that the approximation solution has the suitable accuracy.

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*Corresponding author

The boundary element method is a powerful tool for numerical approximation for a differential equation. For the first time, in 1872, Betti presented this method in elasticity theory [11]. In 1903 Fredholm extended this method [12], but in 1977 the boundary element method appears in some publications of Banerjee, Butterfield, Brebbia and Dominguez [13, 14].
In this paper we want to find an approximate solution for the Helmholtz equation by the boundary element method. This paper organized in three sections: in the first section, we introduce some preliminaries and also describe the structure of the boundary element method. In the second section, we applied boundary element method for Laplace equation which is a special case of helmholtz equation. Finally, in the third section we approximat the boundary element solution of Helmholtz equation.

In boundary element method we use three important things.

1- *The Gauss divergence theorem:* If $\Omega \subset \mathbb{R}^d$ be a bounded domain and $w = (w_1, \cdots, w_d) \in C^1(\overline{\Omega})^d$ be a vector-valued function, then for $x \in \mathbb{R}^d$ we have [8]

$$\int_\Omega \nabla \cdot w \, dv = \int_\Gamma w \cdot n \, ds,$$

where $n = (n_1, \cdots, n_d)$ is the outward unit normal to boundary Γ.

2- *The second Green identity:* Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $v, w \in C^2(\overline{\Omega})$, then [8]

$$\int_\Omega (w\nabla^2 u - u\nabla^2 w) dv = \int_\Gamma (w\frac{\partial u}{\partial n} - u\frac{\partial w}{\partial n}) ds.$$

3- *The Dirac delta function:* The Dirac delta function defined as [1]

$$\delta(X - X_0) = \begin{cases} \infty, & X = X_0, \\ 0, & X \neq X_0, \end{cases}$$

and has two following properties:

$$\int_{-\infty}^{\infty} \delta(X - X_0) \, dX = 1, \; \int_{-\infty}^{\infty} f(X)\delta(X - X_0) \, dX = f(X_0).$$

The structure of the boundary element method can be decribed as follows.
Consider the following boundary element value problem

$$\begin{cases} \mathcal{A}u := Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du = f, & \text{in } \Omega, \\ u = \bar{u}, & \text{on } \Gamma_1, \\ q = \dfrac{\partial u}{\partial n} = \bar{q}, & \text{on } \Gamma_2, \end{cases} \tag{1}$$

where the coefficients A, B, C, D are constant and Γ_1, Γ_2 are boundaries of domain Ω, q is the derivative of potential function in direction of n and $\bar{u}$, $\bar{q}$ are given values of the flux and potential function on the boundary. In boundary element method we multiply the first equation of (1) by weight function $w \in C^2(\overline{\Omega})$ and integrate over Ω to get the following integral form

$$\int_\Omega (\mathcal{A}u) w dv = \int_\Omega f w dv. \tag{2}$$

by choosing the fundamental solution $\mathcal{A}w = -\delta(X - X_0)$ as the weight function and using the second Green identity, we can convert (2) to a continuous integral equation over the boundary. For applying the boundary element method we discretize the problem over the boundary of Ω. To this end we consider $\partial\Omega = \cup_{j=1}^{N} \partial\Omega_j$ and conclude the system of equations $HU = GQ$. If we apply the boundary conditions, we get a new system of equations as $AX = B$, which gives the flux and potential u on the boundary.
In this paper we want to apply the bounday element method for the Helmholtz equation. In (1) if we consider $A = 1, B = 0, C = 1, D = k^2$ and by considering $\nabla^2 u = u_{xx} + u_{yy}$, we get the Helmholtz equation as

$$\begin{cases} \mathcal{A}u := \nabla^2 u + k^2 u = 0, & \text{in } \Omega \subset \mathbb{R}^2, \\ u = \bar{u}, & \text{on } \Gamma_1, \\ q = \dfrac{\partial u}{\partial n} = \bar{q}, & \text{on } \Gamma_2, \end{cases} \tag{3}$$

where k is the wave number.

2. Boundary Element Method for the Laplace Problem

In this section we apply the boundary element method to the Laplace equation which is the special case of Helmholtz equation. In equation (3) if we set $k = 0$, the following laplace equation is resulted:

$$\begin{cases} \nabla^2 u = 0, & \text{in}\, \Omega \subset \mathbb{R}^2, \\ u = \bar{u}, & \text{on}\, \Gamma_1, \\ q = \dfrac{\partial u}{\partial n} = \bar{q}, & \text{on}\, \Gamma_2. \end{cases} \tag{4}$$

Consider weight function w such that it satisfies

$$\nabla^2 w + \delta(X - X_0) = 0, \tag{5}$$

where $X_0 = (\xi, \eta)$ is a singular point and δ is the Dirac delta function. Considering the polar form $\nabla^2 w = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial w}{\partial r})$, where $r = |X - X_0| = \sqrt{(x-\xi)^2 + (y-\eta)^2}$. In this case for $r > 0, \delta(X - X_0) = 0$, one obtain

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial w}{\partial r}) = 0,$$

where A and B are arbitrary constants. Since we look for a particular solution, we may set $B = 0$. Which gives after integrating twice

$$w = A \ln r + B.$$

By considering a neighborhood of X_0 and using the integral property for dirac delta function, we obtain $A = \frac{1}{2\pi}$. Hence, the fundamental solution for the Laplac equation becomes [1]

$$w = \frac{1}{2\pi} \ln\left(\frac{1}{r}\right).$$

Also note that

$$\frac{\partial w}{\partial n} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial n} = -\frac{1}{2\pi r}(\nabla r \cdot n) = -\frac{1}{2\pi r^2}(r_x n_x + r_y n_y),$$

where $r_x = x - \xi$ and $r_y = y - \eta$.
Let $X_0 \in \Omega$ and consider a neighborhood of X_0 with radius ε, namely Ω_ε and consider the new domain $\Omega - \Omega_\varepsilon$ with boundary $\partial\Omega \cup \partial\Omega_\varepsilon$. In this domain by letting $q = \dfrac{\partial u}{\partial n}$ and $q^* = \dfrac{\partial w}{\partial n}$ the Green formula can be written as

$$\lim_{\varepsilon\to 0}\int_{\Omega-\Omega_\varepsilon}(u\nabla^2 w - w\nabla^2 u)dv = \lim_{\varepsilon\to 0}\int_{\partial\Omega}(uq^* - wq)ds + \lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon}(uq^* - wq)ds,$$

by using (5) and the definition of the Dirac delta and $X_0 \notin \Omega - \Omega_\varepsilon$, one obtain

$$\lim_{\varepsilon\to 0}\int_{\Omega-\Omega_\varepsilon} u\nabla^2 w dv = \lim_{\varepsilon\to 0}\left(-\int_{\Omega-\Omega_\varepsilon} u\delta(X - X_0)dv\right) = 0,$$

using the Laplac equation (4), result in

$$\lim_{\varepsilon\to 0}\int_{\Omega-\Omega_\varepsilon}(u\nabla^2 w - w\nabla^2 u)dv = 0.$$

On the other hand, [9]

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon}(uq^* - wq)ds = \lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon}\left(u - u(X_0) + u(X_0)\right)q^* ds \\
&- \lim_{\varepsilon\to 0}\left(-\frac{\ln\varepsilon}{2\pi}\int_{\partial\Omega_\varepsilon}\frac{\partial u}{\partial n}ds\right) \\
&= \lim_{\varepsilon\to 0}\left(u(X_0)\int_0^{2\pi}\frac{1}{2\pi}d\theta\right) + \lim_{\varepsilon\to 0}\left(-\frac{\varepsilon\ln\varepsilon}{2\pi}\frac{\partial u(X_0)}{\partial n}\int_0^{2\pi}d\theta\right) \\
&= u(X_0).
\end{aligned}$$

Therefore, the boundary integral equation in Ω is:

$$u(X_0) = \int_{\partial\Omega}(wq - uq^*)ds,$$

which means that for given u, q on $\partial\Omega$, the values of u can be determinded for the interior point of the domain.

Let $X_0 \in \partial\Omega$ and consider a neighborhood of X_0, namely Ω^i_ε and consider the domain $\Omega-\Omega^i_\varepsilon$ with boundary $\partial\Omega^i_\varepsilon \cup (\partial\Omega-\partial\Omega_\varepsilon)$. In this domain the Green formula can be written as,

$$\lim_{\varepsilon\to 0}\int_{\Omega-\Omega^i_\varepsilon}(u\nabla^2 w - w\nabla^2 u)dv = \lim_{\varepsilon\to 0}\int_{(\partial\Omega-\partial\Omega_\varepsilon)}(uq^* - wq)ds$$
$$+ \lim_{\varepsilon\to 0}\int_{\partial\Omega^i_\varepsilon}(uq^* - wq)ds.$$

Since the governing equation is Laplace equation and the X_0 is out of the domain $\Omega - \Omega^i_\varepsilon$,we have:

$$\lim_{\varepsilon\to 0}\int_{\partial\Omega-\partial\Omega_\varepsilon}(u\nabla^2 w - w\nabla^2 u)dv = 0.$$

On the other hand

$$\lim_{\varepsilon\to 0}\int_{\partial\Omega^i_\varepsilon}(uq^* - wq)ds = \lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon}\left(u - u(X_0) + u(X_0)\right)q^* ds$$
$$- \lim_{\varepsilon\to 0}\left(-\frac{\ln\varepsilon}{2\pi}\int_{\partial\Omega_\varepsilon}\frac{\partial u}{\partial n}ds\right)$$
$$= \lim_{\varepsilon\to 0}\left(u(X_0)\int_{\partial\Omega^i_\varepsilon}\frac{1}{2\pi}d\theta\right) + \lim_{\varepsilon\to 0}\left(-\frac{\varepsilon\ln\varepsilon}{2\pi}\frac{\partial u(X_0)}{\partial n}\int_0^{2\pi}d\theta\right)$$
$$= C(X_0)u(X_0),$$

where $C(X_0)$ is a geometry-dependent parameter. It varies between zero and unity and equals $\frac{1}{2}$. If the point is on a smooth part of the boundary at edges, it is related to the angle of the joining surfaces and equals $C(X_0) = \dfrac{\text{internal angle}}{2\pi}$. Also

$$\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon}(uq^* - wq)ds = \lim_{\varepsilon\to 0}\int_{\partial\Omega}(uq^* - wq)ds.$$

So the boundary integral equation in $\partial\Omega$ is:

$$C(\alpha)u(X_0) + \int_{\partial\Omega}uq^* ds = \int_{\partial\Omega}wq ds.$$

To apply the boundary element method, the boundary is divided into small elements as $\partial\Omega = \cup_{j=1}^{N} \partial\Omega_j$ where N represents the number of elements that form the boundary. The mesh is generated only on the boundary and it is more flexible than in the case of finite element mesh. The integrals are then expressed for each element and summed up in order to evaluate the integral on the whole boundary. Considering constant element discretization and the source points situated on a smooth boundary, the boundary integral equation can be written as

$$\frac{1}{2}u(X_0) + \int_{\cup_{j=1}^{N} \partial\Omega_j} uq^* ds = \int_{\cup_{j=1}^{N} \partial\Omega_j} wqds.$$

where by definition

$$H_{ij} = \begin{cases} \widehat{H}_{ij}, & i \neq j, \\ \frac{1}{2} + \widehat{H}_{ij}, & i = j, \end{cases}$$

and

$$\widehat{H}_{ij} = \int_{\partial\Omega_j} \frac{\partial w}{\partial n} ds_j, \quad G_{ij} = \int_{\partial\Omega_j} w\, ds_j,$$

we get

$$\sum_{j=1}^{N} H_{ij} u_j = \sum_{j=1}^{N} G_{ij} q_j.$$

Here we know some of u_j's and some of q_j's, i.e.,there are N known quantities and N unknowns. Applying the boundary conditions to every midpoint of the elements, we obtain the equations in the matrix-vector form as

$$HU = GQ.$$

Rearranging this system with known quantities on one side and unknowns on the other, we get a linear system of equations $AX = B$ where X is unknown values.

Example 2.1. Consider the Laplace equation in the rectangle plate with mixed boundary conditions as [7]

$$\begin{cases} \nabla^2 T = 0, & 0 < x < 2, 0 < y < 1, \\ \dfrac{\partial u}{\partial n}(x,0) = 0, T(2,y) = 0, \\ \dfrac{\partial u}{\partial n}(x,1) = 0, T(0,y) = 1. \end{cases}$$

Using four constant element discretization with the collocation points at the center of the elements, H_{ij} and G_{ij} for $i \neq j$, one obtain:

$$\begin{aligned} H_{ij} &= \int_{\partial\Omega_j} q^* ds_j = -\frac{1}{2\pi}\int_{X_j}^{X_{j+1}} -\frac{1}{r^2}(r_x n_x + r_y n_y) dX \\ &= -\frac{l_e}{4\pi}\sum_{k=1}^{4}\frac{w_k}{r_k^2}(r_{xk} n_{xk} + r_{yk} n_{yk}), \end{aligned}$$

and

$$\begin{aligned} G_{ij} &= \int_{\partial\Omega_j} w ds_j = \frac{1}{2\pi}\int_{X_j}^{X_{j+1}} \ln\left(\frac{1}{r}\right) dX \\ &= \frac{l_e}{4\pi}\sum_{k=1}^{4} w_k \ln\left(\frac{1}{r_k}\right). \end{aligned}$$

also for $i = j$, we have $H_{ij} = \dfrac{1}{2}$, and

$$G_{ij} = \frac{1}{\pi}\int_0^{\frac{l_e}{2}} \ln\left(\frac{1}{r}\right) dX = \frac{l_e}{2\pi}\left(\ln\left(\frac{2}{l_e}\right) + 1\right),$$

where $l_e = X_{j+1} - X_j$ is the length of the element on which the integral is calculated, w_k represent the weight functions from the Gauss integration scheme and r_k is the distance from the i^{th} source point to each integration points on the j^{th} element. So

$$H = \begin{bmatrix} 0.5000 & -0.1250 & -0.2497 & -0.1250 \\ -0.2113 & 0.5000 & -0.2113 & -0.0780 \\ -0.2497 & -0.1250 & 0.5000 & -0.1250 \\ -0.2113 & -0.0780 & -0.2113 & 0.5000 \end{bmatrix},$$

$$G = \begin{bmatrix} 0.3183 & -0.0210 & -0.0421 & -0.0210 \\ -0.0176 & 0.2695 & -0.0176 & -0.1119 \\ -0.0421 & -0.0210 & 0.3183 & -0.0210 \\ -0.0176 & -0.1119 & -0.0176 & 0.2695 \end{bmatrix}.$$

From these equations unknow values in system $AX = B$ will be calculated as

$$T_1 = 0.4993,\ T_3 = 0.4993,\ q_2 = -0.7578,\ q_4 = 0.7575.$$

Consider these values of T in interior points, result in

$$P_1 = (0.75, 0.5), P_2 = (1, 0.5), P_3 = (1.5, 0.5),$$

as

$$T_{(i)} = \sum_{k=1}^{N} G_{ik} q_k - \sum_{k=1}^{N} H_{ik} T_k,$$

where

$$G_{ik} = \int_{\Gamma_k} w_i ds_k, H_{ik} = \int_{\Gamma_k} \frac{\partial w_i}{\partial n} ds_k,$$

So

$$T_{P_1} = 0.5943, T_{P_2} = 0.4869, T_{P_3} = 0.3118,$$

Therefore the exact values are

$$T^e_{P_1} = 0.625, T^e_{P_2} = 0.5, T^e_{P_3} = 0.25.$$

Compared with the exact values, the solution shows quite a good approximation for the internal points not only for the boundary points. Figure 1, 2

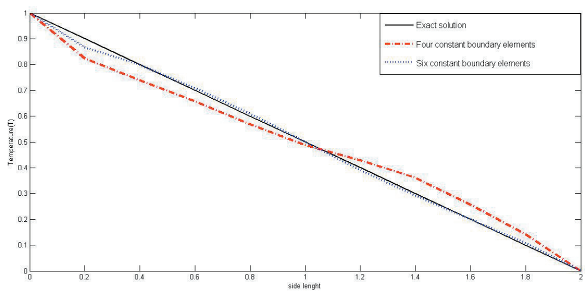

Figure 1. The effect of increasing the number of elements on approximate solution of the Laplace equation

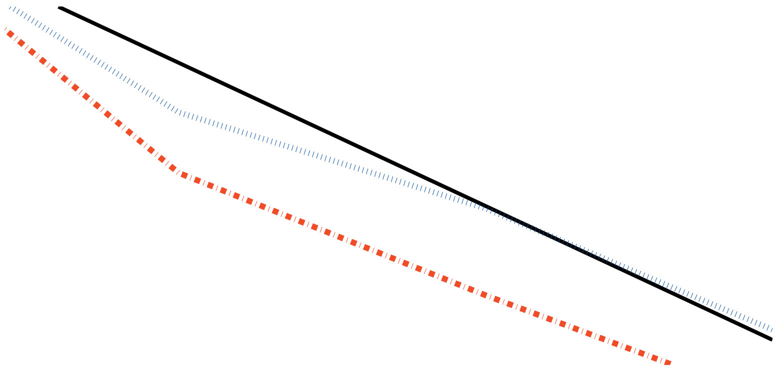

Figure 2. The effect of increasing the number of elements on approximate solution of the Laplace equation (zoom)

3. Boundary Element Method for the Helmholtz Equation

In this section we apply the boundary element method to the Helmholtz equation (3) for $k \neq 0$. Consider weight function w such that it satisfies

$$\nabla^2 w + k^2 w + \delta(X - X_0) = 0, \tag{6}$$

where $X_0 = (\xi, \eta)$ is a singular point. The solutions for the homogeneous case are the Hankel functions of the first and second kinds of order zero. So we write

$$w(r) = AH_0^{(2)}(kr),$$

where $r = \sqrt{(x-\xi)^2 + (y-\eta)^2}$ and $H_0^{(2)}$ is the Hankel function of the second kind of order zero and it is representable in terms of the 0th-order Bessel functions of the first and second kinds J_0, Y_0 as

$$H_0^{(2)}(z) = J_0(z) - iY_0(z) = \bar{H}_0^{(1)}(z).$$

By using the small-argument approximation to the Hankel function as [2]

$$H_0^{(2)}(kr) \approx 1 - i\frac{2}{\pi}\log(\frac{\gamma kr}{2}), \qquad r \to 0,$$

where $\gamma = 1.781$ and integrating (6) over a very small circle of radius ε centered at the origin, one obtain

$$A\int_\Omega [\nabla\cdot\nabla + k^2][1 - i\frac{2}{\pi}\log(\frac{\gamma kr}{2})]dv = -1,$$

using the divergence theorem, the first term is converted to a line integral as

$$-i\frac{2}{\pi}\int_0^{2\pi}\nabla\left(\log(\frac{\gamma kr}{2})\right)rd\phi = -4i,$$

the second term goes to zero, so

$$A = -\frac{i}{4},$$

Therefore, we get the following fundamental solution for helmholtz equation as [2]

$$w = \frac{1}{4i}H_0^{(2)}(kr).$$

Also we note that [10]

$$q^* = \frac{\partial w}{\partial n} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial n} = -\frac{1}{4i}kH_1^{(2)}(kr)(\nabla r\cdot n) = -\frac{n_x r_x + n_y r_y}{4ir}kH_1^{(2)}(kr),$$

where $r_x = x - \xi$ and $r_y = y - \eta$. To obtain the boundary integral equation, we integrate (3) over the domain Ω with a weighting function w to get,

$$\int_\Omega (\nabla^2 u + k^2 u)wdv = 0.$$

The second Green identity, implies that

$$\int_\Omega w\nabla^2 udv = \int_\Omega u\nabla^2 wdv + \int_{\partial\Omega}(uq^* - wq)ds,$$

so

$$\int_\Omega (\nabla^2 w + k^2 w)udv + \int_{\partial\Omega}(uq^* - wq)ds = 0. \tag{7}$$

By using the properties of Dirac delta gives

$$\int_\Omega (\nabla^2 w + k^2 w)udv = -u(X_0).$$

So the boundary integral equation for all the interior points is

$$u(X_0) + \int_{\partial\Omega} uq^* ds = \int_{\partial\Omega} wqds.$$

The boundary integral equation in general case is [3]

$$C(X_0)u(X_0) + \int_{\partial\Omega} uq^* ds = \int_{\partial\Omega} wqds,$$

where $C(X_0) = \dfrac{\alpha}{2\pi}$ is a geometry-dependent parameter where α is internal angle. Assume that the boundary is discretized and represented by N linear constant elements as $\partial\Omega = \cup_{j=1}^{N} \partial\Omega_j$ and the source points situated on a smooth boundary ($\alpha = \pi$), so the boundary integral equation can be written as

$$\frac{1}{2}u(X_0) + \int_{\cup_{j=1}^{N}\partial\Omega_j} u\frac{\partial w}{\partial n} ds = \int_{\cup_{j=1}^{N}\partial\Omega_j} w\frac{\partial u}{\partial n} ds.$$

Therefore

$$\sum_{j=1}^{N} H_{ij}u_j = \sum_{j=1}^{N} G_{ij}q_j,$$

where

$$H_{ij} = \begin{cases} \widehat{H}_{ij}, & i \neq j, \\ \dfrac{1}{2} + \widehat{H}_{ij}, & i = j, \end{cases}$$

and

$$\widehat{H}_{ij} = \int_{\partial\Omega_j} \frac{\partial w}{\partial n}\, ds_j, G_{ij} = \int_{\partial\Omega_j} w\, ds_j.$$

By applying these boundary conditions to every midpoint of elements, we obtain the equations in the matrix-vector form $HU = GQ$. Rearranging this system, a linear system of equations $AX = B$, where X is unknow values will be resulted.

Example 3.1. Consider the following Helmholtz equation with $k = \frac{\sqrt{2}\pi}{4}$ and the domain $0 < x < 1$, $0 < y < 1$, the boundary conditions in [4]

$$\begin{cases} \nabla^2 u + k^2 u = 0, & 0 < x < 1, 0 < y < 1, \\ u(0,y) = 0, & 0 < y < 1, \\ u(1,y) = \dfrac{\sqrt{2}}{2}\cos(\dfrac{\pi y}{4}), & 0 < y < 1, \\ \dfrac{\partial u}{\partial n}\Big|_{y=0} = 0, & 0 < x < 1, \\ \dfrac{\partial u}{\partial n}\Big|_{y=1} = -\dfrac{\pi\sqrt{2}}{8}\sin(\dfrac{\pi x}{4}), & 0 < x < 1. \end{cases}$$

The exact solution of this particular boundary value problem is

$$u(x,y) = \sin(\frac{\pi x}{4})\cos(\frac{\pi y}{4}).$$

In Table 1, a comparison between the exact solutions and the approximate solutions for differente values of elements of equation (6) of this element is given. Also in figure 3 and 4, these results are depicted.

Table 1. The effect of increasing the number of elements on approximate solution helmholtz equation for interior points

Interior point	*Forty elements*	*Eighty elements*	*Exact solution*
$(0.1, 0.1)$	0.0879	0.0876	0.0782
$(0.2, 0.2)$	0.1484	0.1490	0.1545
$(0.3, 0.3)$	0.2108	0.2121	0.2270
$(0.4, 0.4)$	0.2733	0.2756	0.2939
$(0.5, 0.5)$	0.3335	0.3375	0.3536
$(0.6, 0.6)$	0.3880	0.3944	0.4045
$(0.7, 0.7)$	0.4335	0.4418	0.4455
$(0.8, 0.8)$	0.4678	0.4751	0.4755
$(0.9, 0.9)$	0.4900	0.4934	0.4938

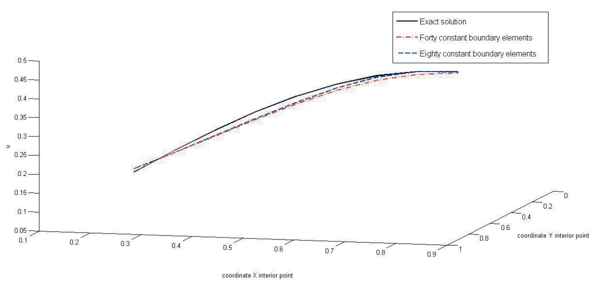

Figure 3. The effect of increasing the number of elements on approximate solution helmholtz equation

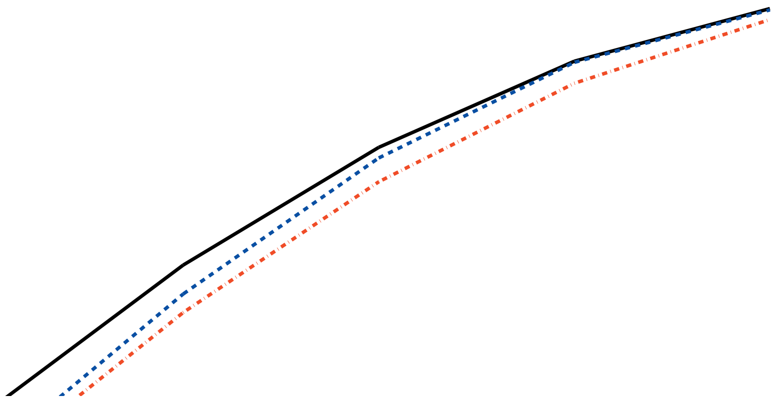

Figure 4. The effect of increasing the number of elements on approximate solution helmholtz equation (zoom)

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Ali Mesforush
Department of Mathematics
Assistant Professor of Mathematics
School of Mathematics
Shahrood University of Technology
Shahrood, Iran
E-mail: ali.mesforush@shahroodut.ac.ir

Zohreh Mehraban
Department of Mathematics
M.Sc Student of Mathematics
School of Mathematics
Shahrood University of Technology
Shahrood, Iran
E-mail: zohre.mehraban@ymail.com