Journal of Mathematical Extension
Vol. 9, No. 3, (2015), 73-85
ISSN: 1735-8299
URL: http://www.ijmex.com

More on Energy and Randić Energy of Specific Graphs

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Abstract. Let G be a simple graph of order n. The energy $E(G)$ of G is the sum of the absolute values of the eigenvalues of G. The Randić matrix of G, denoted by $R(G)$, is defined as the $n\times n$ matrix whose (i,j)-entry is $(d_i d_j)^{\frac{-1}{2}}$ if v_i and v_j are adjacent and 0 for another cases. The Randić energy $RE(G)$ of G is the sum of absolute values of the eigenvalues of $R(G)$. In this paper we compute the energy and the Randić energy for certain graphs. We also propose a conjecture on the Randić energy.

AMS Subject Classification: 15A18; 05C50
Keywords and Phrases: 3-Regular graphs, energy, Randić energy, characteristic polynomial, Petersen graph

1. Introduction

In this paper we are concerned with simple finite graphs, without directed, multiple, or weighted edges, and without self-loops. Let $A(G)$ be adjacency matrix of G and $\lambda_1, \lambda_2, \ldots, \lambda_n$ its eigenvalues. These are said to be the eigenvalues of the graph G and to form its spectrum [5]. The

Received: January 2015; Accepted: March 2015
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energy $E(G)$ of the graph G is defined as the sum of the absolute values of its eigenvalues

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

Details and more information on graph energy can be found in [6, 7, 9, 13].

The Randić matrix $R(G) = (r_{ij})_{n\times n}$ is defined as [2, 3, 10]

$$r_{ij} = \begin{cases} \frac{1}{\sqrt{d_i d_j}} & \text{if } v_i \sim v_j \\ 0 & \text{otherwise.} \end{cases}$$

Denote the eigenvalues of the Randić matrix $R(G)$ by $\rho_1, \rho_2, \ldots, \rho_n$ and label them in non-increasing order. The Randić energy [2, 3, 10] of G is defined as

$$RE(G) = \sum_{i=1}^{n} |\rho_i|.$$

Two graphs G and H are said to be *Randić energy equivalent*, or simply $\mathcal{RE}$-equivalent, written $G \sim H$, if $RE(G) = RE(H)$. It is evident that the relation $\sim$ of being $\mathcal{RE}$-equivalence is an equivalence relation on the family $\mathcal{G}$ of graphs, and thus $\mathcal{G}$ is partitioned into equivalence classes, called the *$\mathcal{RE}$-equivalence classes*. Given $G \in \mathcal{G}$, let

$$[G] = \{H \in \mathcal{G} : H \sim G\}.$$

We call $[G]$ the equivalence class determined by G. A graph G is said to be *Randić energy unique*, or simply *$\mathcal{RE}$-unique*, if $[G] = \{G\}$.

Similarly, we can define $\mathcal{E}$-equivalence for energy and *$\mathcal{E}$-unique* for a graph.

A graph G is called *k-regular* if all vertices have the same degree k. One of the famous graphs is the Petersen graph which is a symmetric non-planar 3-regular graph. In the study of energy and the Randić energy, it is interesting to investigate the characteristic polynomial and the energy of this graph. We denote the Petersen graph by P.

In this paper, we study the energy and Randić energy of specific graphs. In the next section, we study energy and Randić energy of 2-regular and

3-regular graphs. We study cubic graphs of order 10 and list all characteristic polynomial, energy and Randić energy of them. As a result, we show that Petersen graph is not $\mathcal{RE}$-unique ($\mathcal{E}$-unique) but can be determined by its Randić energy (energy) and its eigenvalues. In the last section we consider some another families of graphs and study their Randić characteristic polynomials.

2. Energy of 2-Regular and 3-Regular Graphs

The energy and Randić energy of regular graphs have not been widely studied. In this section we consider 2-regular and 3-regular graphs. The following theorem gives a relationship between the Randić energy and energy of k-regular graphs.

Lemma 2.1. ([11]) *If the graph G is k-regular then $RE(G) = \frac{1}{k}E(G)$.*

Also we have the following easy lemma:

Lemma 2.2. *Let $G = G_1 \cup G_2 \cup \ldots \cup G_m$. Then*

(i) $E(G) = E(G_1) + E(G_2) + \ldots + E(G_m)$.

(ii) $RE(G) = RE(G_1) + RE(G_2) + \ldots + RE(G_m)$.

Randić characteristic polynomial of the cycle graph C_n can be determined by the following theorem:

Lemma 2.3. ([1]) *For $n \geqslant 3$, the Randić characteristic polynomial of the cycle graph C_n is*

$$RP(C_n, \lambda) = \lambda\Lambda_{n-1} - \frac{1}{2}\Lambda_{n-2} - (\frac{1}{2})^{n-1},$$

where for every $k \geqslant 3$, $\Lambda_k = \lambda\Lambda_{k-1} - \frac{1}{4}\Lambda_{k-2}$ with $\Lambda_1 = \lambda$ and $\Lambda_2 = \lambda^2 - \frac{1}{4}$.

By Lemma 2.3, we can find all the eigenvalues of Randić matrix of cycle graphs. So we can compute the Randić energy of cycles. Also every cycle is 2-regular. By Lemma 2.1, we have $E(C_n) = 2RE(C_n)$. Hence we can compute energy of cycle graphs too. Every 2-regular graph is a

disjoint union of cycles. Therefore by Lemma 2.2, we can find energy and Randić energy of 2-regular graphs.
Let to consider the characteristic polynomial of 3-regular graphs of order 10. Also we shall compute energy and Randić energy of this class of graphs. There are exactly 21 cubic graphs of order 10 given in Figure 1 (see [12]). We denote these 21 graphs by $G_1, G_2, ..., G_{21}$.
We show that Petersen graph is not $\mathcal{RE}$-unique ($\mathcal{E}$-unique) but can be determined by its Randić energy (energy) and its eigenvalues. There are just two non-connected cubic graphs of order 10. The following theorem gives us characteristic polynomial of 3-regular graphs of order 10. We denote the characteristic polynomial of the graph G by $P(G, \lambda)$.
Using Maple we computed the characteristic polynomials of 3-regular graphs of order 10 in Table 1.

Table 1: Characteristic polynomial $P(G_i, \lambda)$, for $1 \leqslant i \leqslant 21$.

G_i	$P(G_i, \lambda)$
G_1	$\lambda^{10} - 15\lambda^8 - 8\lambda^7 + 71\lambda^6 + 64\lambda^5 - 101\lambda^4 - 104\lambda^3 + 44\lambda^2 + 48\lambda$
G_2	$\lambda^{10} - 15\lambda^8 - 4\lambda^7 + 71\lambda^6 + 28\lambda^5 - 121\lambda^4 - 48\lambda^3 + 64\lambda^2 + 24\lambda$
G_3	$\lambda^{10} - 15\lambda^8 - 6\lambda^7 + 69\lambda^6 + 48\lambda^5 - 96\lambda^4 - 76\lambda^3 + 30\lambda^2 + 26\lambda + 3$
G_4	$\lambda^{10} - 14\lambda^8 - 4\lambda^7 + 53\lambda^6 + 34\lambda^5 - 48\lambda^4 - 50\lambda^3 - 12\lambda^2$
G_5	$\lambda^{10} - 15\lambda^8 - 8\lambda^7 + 71\lambda^6 + 68\lambda^5 - 93\lambda^4 - 132\lambda^3 - 36\lambda^2$
G_6	$\lambda^{10} - 15\lambda^8 + 65\lambda^6 - 105\lambda^4 + 55\lambda^2 - 9$
G_7	$\lambda^{10} - 15\lambda^8 + 69\lambda^6 - 12\lambda^5 - 117\lambda^4 + 36\lambda^3 + 59\lambda^2 - 12\lambda - 9$
G_8	$\lambda^{10} - 15\lambda^8 + 71\lambda^6 - 16\lambda^5 - 133\lambda^4 + 64\lambda^3 + 76\lambda^2 - 48\lambda$
G_9	$\lambda^{10} - 15\lambda^8 - 2\lambda^7 + 71\lambda^6 + 8\lambda^5 - 132\lambda^4 - 2\lambda^3 + 91\lambda^2 - 8\lambda - 12$
G_{10}	$\lambda^{10} - 15\lambda^8 + 65\lambda^6 - 4\lambda^5 - 85\lambda^4 - 20\lambda^3 + 35\lambda^2 + 20\lambda + 3$
G_{11}	$\lambda^{10} - 15\lambda^8 - 4\lambda^7 + 69\lambda^6 + 32\lambda^5 - 105\lambda^4 - 64\lambda^3 + 23\lambda^2 + 20\lambda + 3$
G_{12}	$\lambda^{10} - 15\lambda^8 - 4\lambda^7 + 75\lambda^6 + 24\lambda^5 - 157\lambda^4 - 36\lambda^3 + 144\lambda^2 + 16\lambda - 48$
G_{13}	$\lambda^{10} - 15\lambda^8 - 2\lambda^7 + 67\lambda^6 + 12\lambda^5 - 96\lambda^4 - 22\lambda^3 + 35\lambda^2 + 12\lambda$
G_{14}	$\lambda^{10} - 15\lambda^8 - 6\lambda^7 + 75\lambda^6 + 48\lambda^5 - 144\lambda^4 - 114\lambda^3 + 75\lambda^2 + 68\lambda + 12$
G_{15}	$\lambda^{10} - 15\lambda^8 - 2\lambda^7 + 69\lambda^6 + 12\lambda^5 - 116\lambda^4 - 24\lambda^3 + 54\lambda^2 + 26\lambda + 3$
G_{16}	$\lambda^{10} - 15\lambda^8 + 63\lambda^6 - 85\lambda^4 + 36\lambda^2$
G_{17}	$\lambda^{10} - 15\lambda^8 + 75\lambda^6 - 24\lambda^5 - 165\lambda^4 + 120\lambda^3 + 120\lambda^2 - 160\lambda + 48$
G_{18}	$\lambda^{10} - 15\lambda^8 - 8\lambda^7 + 63\lambda^6 + 64\lambda^5 - 37\lambda^4 - 56\lambda^3 - 12\lambda^2$
G_{19}	$\lambda^{10} - 15\lambda^8 - 4\lambda^7 + 73\lambda^6 + 28\lambda^5 - 141\lambda^4 - 52\lambda^3 + 99\lambda^2 + 16\lambda - 21$
G_{20}	$\lambda^{10} - 15\lambda^8 - 12\lambda^7 + 63\lambda^6 + 96\lambda^5 - 13\lambda^4 - 84\lambda^3 - 36\lambda^2$
G_{21}	$\lambda^{10} - 15\lambda^8 - 8\lambda^7 + 51\lambda^6 + 72\lambda^5 + 27\lambda^4$

By computing the roots of characteristic polynomial of cubic graphs of order

10, we can have the energy of these graphs. We compute them to four decimal places. So we have table 2.

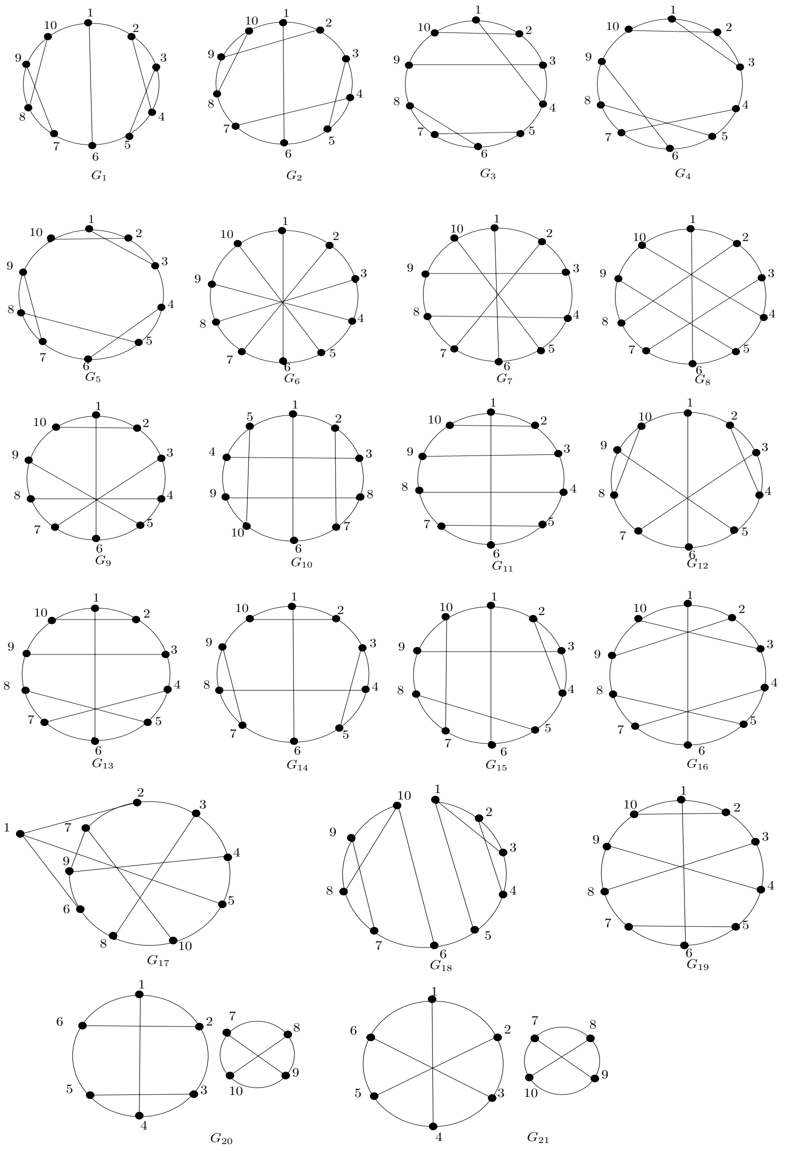

Figure 1. Cubic graphs of order 10.

Table 2: Energy and Randić energy of cubic graphs of order 10.

G_i	$E(G_i)$	$RE(G_i)$	G_i	$E(G_i)$	$RE(G_i)$	G_i	$E(G_i)$	$RE(G_i)$
G_1	15.1231	5.0410	G_8	15.1231	5.0410	G_{15}	14.7943	4.9314
G_2	14.8596	4.9532	G_9	15.3164	5.1054	G_{16}	14.0000	4.6666
G_3	14.8212	4.9404	G_{10}	14.4721	4.8240	G_{17}	16.0000	5.3333
G_4	13.5143	4.5047	G_{11}	14.7020	4.9006	G_{18}	13.5569	4.5189
G_5	14.2925	4.7641	G_{12}	16.0000	5.3333	G_{19}	15.5791	5.1930
G_6	14.9443	4.9814	G_{13}	14.3780	4.7926	G_{20}	14.0000	4.6666
G_7	15.0777	5.0259	G_{14}	15.0895	5.0298	G_{21}	12.0000	4.0000

Theorem 2.1. *Six cubic graphs of order* 10 *are not* $\mathcal{E}$*-unique (*$\mathcal{RE}$*-unique). If two cubic graphs of order* 10 *have equal energy (Randić energy), then their eigenvalues are different in exactly* 3 *values.*

Proof. Using Table 2, we see that $[G_1] = \{G_1, G_8\}$, $[G_{12}] = \{G_{12}, G_{17}\}$ and $[G_{16}] = \{G_{16}, G_{20}\}$. Now, it suffices to find the eigenvalues of G_1, G_8, G_{12}, G_{16} , G_{17} and G_{20}. By Table 1 we have:

$$
\begin{aligned}
P(G_1,\lambda) &= \lambda^{10} - 15\lambda^8 - 8\lambda^7 + 71\lambda^6 + 64\lambda^5 - 101\lambda^4 - 104\lambda^3 + 44\lambda^2 + 48\lambda \\
&= \lambda(\lambda-3)(\lambda+2)^2(\lambda-1)^2(\lambda+1)^2(\lambda - \frac{1-\sqrt{17}}{2})(\lambda - \frac{1+\sqrt{17}}{2}),
\end{aligned}
$$

$$
\begin{aligned}
P(G_8,\lambda) &= \lambda^{10} - 15\lambda^8 + 71\lambda^6 - 16\lambda^5 - 133\lambda^4 + 64\lambda^3 + 76\lambda^2 - 48\lambda \\
&= \lambda(\lambda-3)(\lambda+2)^2(\lambda-1)^3(\lambda+1)(\lambda - \frac{-1+\sqrt{17}}{2})(\lambda - \frac{-1-\sqrt{17}}{2}).
\end{aligned}
$$

Also

$$
\begin{aligned}
P(G_{12},\lambda) &= \lambda^{10} - 15\lambda^8 - 4\lambda^7 + 75\lambda^6 + 24\lambda^5 - 157\lambda^4 - 36\lambda^3 + 144\lambda^2 + 16\lambda - 48 \\
&= (\lambda-3)(\lambda-2)(\lambda+2)^3(\lambda-1)^3(\lambda+1)^2,
\end{aligned}
$$

$$
\begin{aligned}
P(G_{17},\lambda) &= \lambda^{10} - 15\lambda^8 + 75\lambda^6 - 24\lambda^5 - 165\lambda^4 + 120\lambda^3 + 120\lambda^2 - 160\lambda + 48 \\
&= (\lambda-3)(\lambda+2)^4(\lambda-1)^5,
\end{aligned}
$$

and

$$
\begin{aligned}
P(G_{16},\lambda) &= \lambda^{10} - 15\lambda^8 + 63\lambda^6 - 85\lambda^4 + 36\lambda^2 \\
&= \lambda^2(\lambda-3)(\lambda+3)(\lambda-2)(\lambda+2)(\lambda-1)^2(\lambda+1)^2,
\end{aligned}
$$

$$\begin{aligned} P(G_{20},\lambda) &= \lambda^{10} - 15\lambda^8 - 12\lambda^7 + 63\lambda^6 + 96\lambda^5 - 13\lambda^4 - 84\lambda^3 - 36\lambda^2 \\ &= \lambda^2(\lambda-3)^2(\lambda+2)^2(\lambda-1)(\lambda+1)^3. \end{aligned}$$

So we have the result. □

Now we consider Petersen graph P. We have shown this graph in Figure 2.

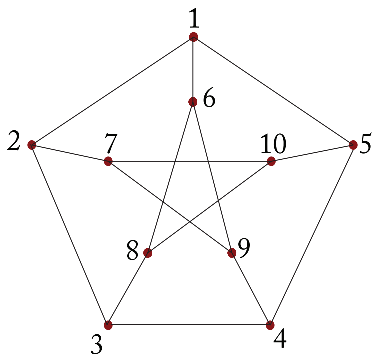

Figure 2. Petersen graph

Theorem 2.2. *Let $\mathcal{G}$ be the family of 3-regular graphs of order 10. For the Petersen graph P, we have the following properties:*

(i) *P is not $\mathcal{E}$-unique ($\mathcal{RE}$-unique) in $\mathcal{G}$.*

(ii) *P has the maximum energy (Randić energy) in $\mathcal{G}$.*

(iii) *P can be identify by its energy (Randić energy) and its eigenvalues in $\mathcal{G}$.*

Proof.

(i) The adjacency matrix of P is

$$A(P) = \begin{pmatrix} 0&1&0&0&1&1&0&0&0&0 \\ 1&0&1&0&0&0&1&0&0&0 \\ 0&1&0&1&0&0&0&1&0&0 \\ 0&0&1&0&1&0&0&0&1&0 \\ 1&0&0&1&0&0&0&0&0&1 \\ 1&0&0&0&0&0&0&1&1&0 \\ 0&1&0&0&0&0&0&0&1&1 \\ 0&0&1&0&0&1&0&0&0&1 \\ 0&0&0&1&0&1&1&0&0&0 \\ 0&0&0&0&1&0&1&1&0&0 \end{pmatrix}.$$

So $det(\lambda I - A(P)) = (\lambda-3)(\lambda+2)^4(\lambda-1)^5$. Therefore we have:

$$\lambda_1 = 3 \quad , \quad \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = -2 \quad , \quad \lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = \lambda_{10} = 1,$$

and so we have $E(P) = 16$. By Table 2, we have $P \in \{G_{12}, G_{17}\}$.
Hence P is not $\mathcal{E}$-unique (and $\mathcal{RE}$-unique) in $\mathcal{G}$.

(ii) It follows from Part (i) and Table 2.

(iii) It follows from Part (i) and Theorem 2.1. So G_{17} is the Petersen graph. □

The following result gives a relationship between energy and permanent of adjacency matrix of two connected graphs in the family of cubic graphs of order 10 whose have the same $\mathcal{E}$-equivalence class.

Theorem 2.3. *If two connected cubic graphs of order* 10 *have the same energy, then their adjacency matrices have the same permanent.*

Proof. By Table 2, it suffices to find $per(A(G_1))$, $per(A(G_8))$, $per(A(G_{12}))$ and $per(A(G_{17}))$. For graph G_1, we have

$$A(G_1) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

By Ryser's method, we have $per(A(G_1)) = 72$. Similarly we have:

$$per(A(G_8)) = 72 \quad , \quad per(A(G_{12})) = 60 \quad , \quad per(A(G_{17})) = 60.$$

So we have the result. □

Remark 2.4. *The converse of Theorem 2.3, is not true. Because per* $(A(G_7)) = per(A(G_{11})) = 85$*, but as we see in Table 2,* $E(G_7) \neq E(G_{11})$*.*

Corollary 2.5. *If two connected cubic graphs of order* 10 *have the same Randić energy, then their adjacency matrices have the same permanent.*

Proof. It follows from Lemma 2.1, Table 2 and Theorem 2.3. □

3. Randić Characteristic Polynomial of a Kind of Dutch-Windmill Graphs

We recall that a complex number ζ is called an algebraic number (resp. an algebraic integer) if it is a root of some monic polynomial with rational (resp. integer) coefficients (see [14]). Since the Randić characteristic polynomial $P(G, \lambda)$ is a monic polynomial in λ with integer coefficients, its roots are, by definition, algebraic integers. This naturally raises the questions: Which algebraic integers can occur as zeros of Randić characteristic polynomials? Which real numbers can occur as Randić energy of graphs? We are interested to numbers which are occur as Randić energy. Clearly those lying in $(-\infty, 2)$ are forbidden set, because we know that if graph G possesses at least one edge, then $RE(G) \geqslant 2$ (see [4]). We think that the Randić energy of graphs are dense in $[2, \infty)$. In this section we would like to study some further results of this kind.
Let n be any positive integer and D_m^n be Dutch Windmill graph with $(m-1)n+1$ vertices and mn edges. In other words, the graph D_m^n is a graph that can be constructed by coalescence n copies of the cycle graph C_m of length m with a common vertex. We recall that D_3^n is friendship graphs. Figure 3 shows some examples of this kind of Dutch Windmill graphs. In this section we shall investigate the Randić characteristic polynomial of Dutch Windmill graphs.

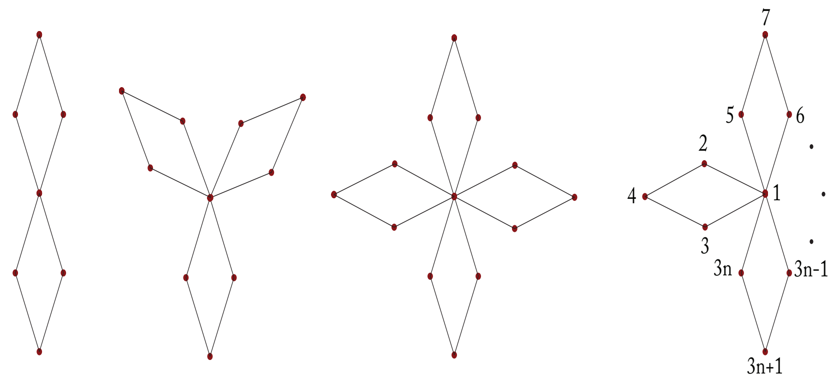

Figure 3. Dutch Windmill Graph D_4^2, D_4^3, D_4^4 and D_4^n, respectively

By Lemma 2.3, we know that

$$RP(C_m, \lambda) = \lambda\Lambda_{m-1} - \frac{1}{2}\Lambda_{m-2} - (\frac{1}{2})^{m-1},$$

where for every $k \geqslant 3$, $\Lambda_k = \lambda\Lambda_{k-1} - \frac{1}{4}\Lambda_{k-2}$ with $\Lambda_1 = \lambda$ and $\Lambda_2 = \lambda^2 - \frac{1}{4}$.
We show that the Randić characteristic polynomial of Dutch Windmill graphs can compute by the Randić characteristic of the cycle which constructed it.

Theorem 3.1. *For $m \geqslant 3$, the Randić characteristic polynomial of the Dutch Windmill graph D_m^n is*

$$RP(D_m^n, \lambda) = \Lambda_{m-1}^{n-1}.RP(C_m, \lambda),$$

where for every $k \geqslant 3$, $\Lambda_k = \lambda\Lambda_{k-1} - \frac{1}{4}\Lambda_{k-2}$ with $\Lambda_1 = \lambda$ and $\Lambda_2 = \lambda^2 - \frac{1}{4}$.

Proof. For every $k \geqslant 3$, consider

$$B_k := \begin{pmatrix} \lambda & \frac{-1}{2} & 0 & 0 & \dots & 0 & 0 & 0 \\ \frac{-1}{2} & \lambda & \frac{-1}{2} & 0 & \dots & 0 & 0 & 0 \\ 0 & \frac{-1}{2} & \lambda & \frac{-1}{2} & \dots & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{2} & \lambda & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \lambda & \frac{-1}{2} & 0 \\ 0 & 0 & 0 & 0 & \dots & \frac{-1}{2} & \lambda & \frac{-1}{2} \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{-1}{2} & \lambda \end{pmatrix}_{k \times k},$$

and let $\Lambda_k = det(B_k)$. It is easy to see that $\Lambda_k = \lambda\Lambda_{k-1} - \frac{1}{4}\Lambda_{k-2}$.
Suppose that $RP(D_m^n, \lambda) = det(\lambda I - R(D_m^n))$. We have

$$RP(D_m^n, \lambda) = det \left(\begin{array}{c|cccc} \lambda & A & A & \dots & A \\ \hline A^t & B_{m-1} & 0 & \dots & 0 \\ A^t & 0 & B_{m-1} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A^t & 0 & 0 & \dots & B_{m-1} \end{array}\right),$$

where $A = \begin{pmatrix} \frac{-1}{2\sqrt{n}} & 0 & 0 & \dots & 0 & \frac{-1}{2\sqrt{n}} \end{pmatrix}_{1 \times (m-1)}$. So

$det(\lambda I - R((D_m^n)) =$
$\lambda\Lambda_{m-1}^n + \left(\frac{-1}{4}\Lambda_{m-2} + 2((-1)^{m+1}(\frac{-1}{2})^m) + (-1)^{2m+1}(\frac{1}{4})\Lambda_{m-2}\right)\Lambda_{m-1}^{n-1}.$

Therefore

$$det(\lambda I - R(D_m^n)) = \lambda\Lambda_{m-1}^n + \left(\frac{-1}{2}\Lambda_{m-2} - (\frac{1}{2})^{m-1}\right)\Lambda_{m-1}^{n-1}.$$

Hence

$$det(\lambda I - R(D_m^n)) = \Lambda_{m-1}^{n-1}\left(\lambda\Lambda_{m-1} - \frac{1}{2}\Lambda_{m-2} - (\frac{1}{2})^{m-1}\right) = \Lambda_{m-1}^{n-1}RP(C_m, \lambda). \quad \square$$

In [1] we have presented two families of graphs such that their Randić energy are $n+1$ and $2+(n-1)\sqrt{2}$. Here we recall the following results:

Theorem 3.2.([1])

(i) *The Randić energy of friendship graph F_n is $RE(F_n) = n+1$.*

(ii) *The Randić energy of Dutch-Windmill graph D_4^n is $RE(D_4^n) = 2+(n-1)\sqrt{2}$.*

(iii) *For every $m, n \geqslant 2$, the Randić energy of $K_{m,n} - e$ is $RE(K_{m,n} - e) = 2 + \frac{2}{\sqrt{mn}}$.*

We can use Theorem 3.2 to obtain $RE(D_5^n)$. Here using the definition of Randić characteristic polynomial, we prove the following result:

Theorem 3.3. *The Randić energy of D_5^n is*

$$RE(D_5^n) = 1 + n\sqrt{5}.$$

Proof. The Randić matrix of D_5^n is

$$R(D_5^n) = \begin{pmatrix} 0 & \frac{1}{2\sqrt{n}} & \frac{1}{2\sqrt{n}} & 0 & 0 & \cdots & \frac{1}{2\sqrt{n}} & \frac{1}{2\sqrt{n}} & 0 & 0 \\ \frac{1}{2\sqrt{n}} & 0 & 0 & \frac{1}{2} & 0 & \dots & 0 & 0 & 0 & 0 \\ \frac{1}{2\sqrt{n}} & 0 & 0 & 0 & \frac{1}{2} & \dots & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{2\sqrt{n}} & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2\sqrt{n}} & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \dots & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}_{(4n+1)\times(4n+1)}.$$

Let $A = \begin{pmatrix} \lambda & 0 & \frac{-1}{2} & 0 \\ 0 & \lambda & 0 & \frac{-1}{2} \\ \frac{-1}{2} & 0 & \lambda & \frac{-1}{2} \\ 0 & \frac{-1}{2} & \frac{-1}{2} & \lambda \end{pmatrix}$ and $C = \begin{pmatrix} \frac{-1}{2\sqrt{n}} & 0 & \frac{-1}{2} & 0 \\ \frac{-1}{2\sqrt{n}}0 & \lambda & 0 & \frac{-1}{2} \\ 0 & 0 & \lambda & \frac{-1}{2} \\ 0 & \frac{-1}{2} & \frac{-1}{2} & \lambda \end{pmatrix}$. Then

$$det(\lambda I - R(D_5^n)) = \lambda det(A)^n + \sqrt{n} det(C) det(A)^{n-1}.$$

So

$$det(\lambda I - R(D_5^n)) = det(A)^{n-1}(\lambda - 1)(\lambda - (\frac{\sqrt{5}}{4} - \frac{1}{4}))^2(\lambda + (\frac{\sqrt{5}}{4} + \frac{1}{4}))^2.$$

Hence

$$RE(D_5^n) = 1 + n\sqrt{5}. \quad \square$$

Part (iii) of Theorem 3.2 implies that the Randić energy of graphs are dense in $[2, 3)$. Motivated by this notation, Theorems 3,2 and 3.3, we think that the Randić energy of graphs are dense in $[2, \infty)$. We close this paper by the following conjecture:

Conjecture 3.4. Randić energy of graphs are dense in $[2, \infty)$.

Acknowledgments
The authors would like to express their gratitude to the referee for helpful comments.

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