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Original Research Paper

Solutions of Pexiderized Functional Equation on Restricted Domain

S. Izadi
Shiraz University of Technology

S. Jahedi*
Shiraz University of Technology

M. Dehghanian
Sirjan University of Technology

Abstract. The aim of this paper is to investigate the solutions of the Pexider-quadratic functional equation under additional conditions that leads to continuous additive or derivation functions.

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1 Preliminaries

One of the attractive topics in mathematical analysis is finding the solution to a functional equation, i.e., a function that satisfies the given equation.

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*Corresponding Author

A function $A:\mathbb{R}\to\mathbb{R}$ is called additive if the equation

$$A(x+y)=A(x)+A(y)$$

holds for all $x,y\in\mathbb{R}$.

A function $B:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is called bi-additive if B is additive in each variable. A bi-additive function B is called symmetric if $B(x,y)=B(y,x)$ for all $x,y\in\mathbb{R}$.

Note that the additive function $A:\mathbb{R}\to\mathbb{R}$ is $\mathbb{Q}$-homogeneous, i.e.,

$$A(qx)=qA(x) \tag{1}$$

for all $x\in\mathbb{R}$ and $q\in\mathbb{Q}$ (see [12, Theorem 5.2.1]).

The existence of discontinuous additive functions was an open problem for many years. Researchers could neither show that all additive functions are continuous, nor give an example to a discontinuous additive function. In 1905 G. Hamel [11] succeded in proving that there exist discontinuous additive functions.

Theorem 1.1. *[15] Let $m\in\mathbb{Z}$, and assume that $A:\mathbb{R}\to\mathbb{R}$ is an additive function. If the function A satisfies*

$$A(x^m)=x^{m-1}A\left(x\right),\qquad x\in\mathbb{R}\backslash\{0\},$$

then $A(x)=A(1)x$ for every $x\in\mathbb{R}$.

A function $\rho:\mathbb{R}\to\mathbb{R}$ is called quadratic if the equation

$$\rho(x+y)+\rho(x-y)=2\rho(x)+2\rho(y)$$

holds for all $x,y\in\mathbb{R}$.

In [2], Aczél et al. have been proved that a function $\rho:\mathbb{R}\to\mathbb{R}$ is quadratic if and only if, there is a symmetric bi-additive function $B:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ such that $\rho(x)=B(x,x)$ for all $x\in\mathbb{R}$. This B is unique.

Recently, some mathematicians have studied the solution of quadratic functional equation on $\mathbb{R}$ under certain additional conditions (see [5, 6, 10]).

In 1965, Aczél [1] showed that a quadratic function $\rho:\mathbb{R}\to\mathbb{R}$ can be associated with a symmetric and bi-additive function $B:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ given by the following formula

$$B(x,y)=\frac{1}{2}[\rho(x+y)-\rho(x)-\rho(y)],\qquad x,y\in\mathbb{R}. \tag{2}$$

So, by using the $\mathbb{Q}$-homogeneity of additive functions, we have

$$B(px, qy) = pqB(x, y), \qquad \rho(qx) = B(qx, qx) = q^2\rho(x)$$

for all $x, y \in \mathbb{R}$ and $p, q \in \mathbb{Q}$. Also, using (2) and induction on n, one can show that

$$\rho\left(\sum_{i=0}^{n} \omega_i\right) = \sum_{i=0}^{n} \rho(\omega_i) + 2 \sum_{0 \le j < k \le n} B(\omega_j, \omega_k)$$

for all $n \in \mathbb{N}$ and $\omega_0, \cdots, \omega_n \in \mathbb{R}$.

Recall that an additive function $\sigma : \mathbb{R} \to \mathbb{R}$ is called derivation if $\sigma(xy) = x\sigma(y) + y\sigma(x)$ is fulfilled for all $x, y \in \mathbb{R}$. Thus, every derivation σ satisfies $\sigma(x^2) = 2x\sigma(x)$ for all $x \in \mathbb{R}$. Moreover, there exist nontrivial derivations on $\mathbb{R}$ (see [12, Theorem 14.2.2]). Also, both $\sigma(x^2)$ and $(\sigma(x))^2$ are quadratic functions [3].

Lemma 1.2. *[13, 14] Let A be an additive function.*

(i) The equation

$$A(x^2) = 2xA(x) \tag{3}$$

holds for all $x \in \mathbb{R}\backslash\{0\}$ if and only if A is a derivation.

(ii) The equation

$$A(x^{-1}) = -x^{-2}A(x) \tag{4}$$

holds for all $x \in \mathbb{R}\backslash\{0\}$ if and only if A is a derivation.

Theorem 1.3. *[15] Let $m, n \in \mathbb{Z}$, and let $\alpha \neq 1$ be a real number such that $m = \alpha n \neq 0$. The additive function $A : \mathbb{R} \to \mathbb{R}$ satisfy the condition*

$$A(x^m) = \alpha x^{m-n} A(x^n)$$

for all $x \in \mathbb{R}$ if and only if A is a derivation.

In 1968, A. Nishiyama and S. Horinouchi [15] showed in the following theorem under what conditions the solutions of an additive functional equation are continuous.

Theorem 1.4. *[15] Assume that $A : \mathbb{R} \to \mathbb{R}$ is an additive function such that*

$$A\left(x^m\right) = \alpha x^{m-n} A(x^n)$$

hold for every $x \in \mathbb{R} \backslash \{0\}$, wherever $\alpha \in \mathbb{R}$ is constant and $m, n \in \mathbb{Z}$ with $m \neq \alpha n$. If $\alpha = 1$, then

$$A(x) = A(1)x$$

for every $x \in \mathbb{R}$. If $\alpha \neq 1$, then $A(x) = 0$ for every $x \in \mathbb{R}$.

Let the unit circle denoted by

$$S^1 = \left\{(x, z) \in \mathbb{R}^2 : x^2 + z^2 = 1\right\}.$$

Below are the theorems proved by Boros and Erdei [4], which will be used in the proof of the main results.

Theorem 1.5. *Let $\lambda \in \mathbb{R}$ and $A : \mathbb{R} \to \mathbb{R}$ be an additive function such that*

$$xA(x) + zA(z) = \lambda \tag{5}$$

holds for all $(x, z) \in S^1$. Then $\mathcal{F}(x) = A(x) - \lambda x$ is derivation.

Theorem 1.6. *Let $A : \mathbb{R} \to \mathbb{R}$ be an additive function such that*

$$xA(z) - zA(x) = 0 \tag{6}$$

holds for all $(x, z) \in S^1$. Then A is linear.

We also need the following Lemma:

Lemma 1.7. *[5] Let $m \in \mathbb{N}$ and $\mathbb{K}$ be a field. Assume that S is a set, $W \subset \mathbb{K}$ contains at least $m+1$ elements, and the functions $\Delta_j : S \to \mathbb{K}$, $j = 0, 1, ..., m$, satisfy*

$$\sum_{j=0}^{m} \Delta_j(x) t^j = 0$$

for all $x \in S$ and $t \in W$. Then $\Delta_j(x) = 0$ for all $x \in S$ and $0 \leq j \leq m$.

Numerous authors have conducted research on functional equations, including additive, quadratic, Drygas and Pexider equations, as well as their generalized form ([3, 4, 5, 6, 7]). In this paper, motivated by [4, 5], we characterize the solutions of the following Pexider functional equation

$$f_1(x+z) + f_2(x-z) = f_3(x) + f_4(z), \qquad x, z \in \mathbb{R}, \tag{7}$$

under additional conditions that leads to continuous additive or derivation functions, where $f_j : \mathbb{R} \to \mathbb{R}$, for $j = 1, 2, 3, 4$, are functions. The general solutions of (7), which we will use in the proof of main results, ere obtained by Ebanks et al. in [9, Theorem 4] as follows.

Theorem 1.8. *The general solutions $f_j : \mathbb{R} \to \mathbb{R}$ for $j = 1, 2, 3, 4$ of (7) are given by*

$$\begin{cases} f_1(x) = \frac{1}{2}B(x,x) - \frac{1}{2}(A_1 - A_2)(x) + c_1 \\ f_2(x) = \frac{1}{2}B(x,x) - \frac{1}{2}(A_1 + A_2)(x) + c_2 \\ f_3(x) = B(x,x) - A_1(x) + c_3 \\ f_4(x) = B(x,x) + A_2(x) + c_4 \end{cases}$$

for every $x \in \mathbb{R}$, where $A_1, A_2 : \mathbb{R} \to \mathbb{R}$ are additive functions and $B : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a symmetric bi-additive function and $c_1 + c_2 = c_3 + c_4$.

2 Main Results

First, we discuss the conditions under which the functions f_j's become derivations.

Theorem 2.1. *Let $m, n \in \mathbb{Z}$, and let $\alpha \neq 1$ be a real number such that $m = \alpha n \neq 0$. Let $f_j : \mathbb{R} \to \mathbb{R}$ for $j = 1, 2, 3, 4$ satisfy the equation (7). Then $f_j(0) = 0$ and the conditions*

$$f_1(x^m) = \alpha x^{m-n} f_1(x^n), \tag{8}$$

$$f_2(x^m) = \alpha x^{m-n} f_2(x^n) \tag{9}$$

hold for all $x \in \mathbb{R}$ if and only if f_j, $(j = 1, 2, 3, 4)$, are derivation on $\mathbb{R}$.

Proof. If f_j, $j = 1, 2, 3, 4$, are derivation functions, then by applying induction and using Lemma 1.2, it follows that

$$f_j\left(x^{\kappa}\right) = \kappa x^{\kappa-1} f_j(x)$$

where $\kappa \in \mathbb{Z}$. So, the equations (8) and (9) are verified.

To prove the converse of the theorem, we consider the following cases:

Case 1. $m > 0$, $n > 0$.

Replace x by tx in (8), where $t \in \mathbb{Q}$, apply Theorem 1.8 together ith the assumption $f_j(0) = c_j = 0$.

Then we obtain

$$t^n x^n \left[t^{2m} B\left(x^m, x^m\right) - t^m (A_1 - A_2)\left(x^m\right)\right] = t^m \alpha x^m \left[t^{2n} B\left(x^n, x^n\right) - t^n (A_1 - A_2)\left(x^n\right)\right]$$

for all $x \in \mathbb{R}$, where $A_1, A_2 : \mathbb{R} \to \mathbb{R}$ are additive functions and $B : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the following identity symmetric bi-additive function. So,

$$\begin{aligned} &t^{2m+n} x^n B\left(x^m, x^m\right) - t^{m+2n} \alpha x^m B\left(x^n, x^n\right) \\ &+ t^{m+n}\left[\alpha x^m (A_1 - A_2)\left(x^n\right) - x^n (A_1 - A_2)\left(x^m\right)\right] = 0. \end{aligned}$$

By Lemma 1.7, considering the coefficient of t^{2m+n}, we conclude that $x^n B\left(x^m, x^m\right) = 0$ and so $B(x, x) = 0$ for every $x \in \mathbb{R}$.

Thus, by Theorem 1.3

$$f_1(x) = -\frac{1}{2}(A_1 - A_2)(x)$$

is a derivation.

Similarly,

$$f_2(x) = -\frac{1}{2}(A_1 + A_2)(x)$$

is a derivation. Hence,

$$f_3(x) = -A_1(x) = f_1(x) + f_2(x), \qquad f_4(x) = A_2(x) = f_1(x) - f_2(x)$$

are derivations.

Case 2. $m < 0$, $n < 0$.

Replace x by x^{-1} in (8) and (9), we get

$$\begin{aligned} f_1\left(x^{-m}\right) &= \alpha x^{-m-(-n)} f_1(x^{-n}), \\ f_2\left(x^{-m}\right) &= \alpha x^{-m-(-n)} f_2(x^{-n}), \end{aligned}$$

where $0 \neq -m = \alpha(-n)$, $\alpha \neq 1$, $-m > 0$ and $-n > 0$. By applying Case 1, we gain the desired result.

Case 3. $m < 0$, $n > 0$.

Substitute x^m and then x^n in place of x in (8), to obtain

$$f_1(x^{m^2}) = \alpha x^{m(m-n)} f_1(x^{nm}),$$
$$f_1(x^{nm}) = \alpha x^{n(m-n)} f_1(x^{n^2})$$

for all $x \in \mathbb{R}$. From the resulting equations, we arrive at

$$f_1(x^{m^2}) = \alpha^2 x^{m^2-n^2} f_1(x^{n^2})$$

and similarly,

$$f_2(x^{m^2}) = \alpha^2 x^{m^2-n^2} f_2(x^{n^2})$$

whence $m^2 = \alpha^2 n^2$ and $m^2 \neq 0$. If $\alpha^2 \neq 1$, the result follows by Case 1.

If $\alpha^2 = 1$, then $\alpha = -1$ (since $\alpha \neq 1$) and hence $m = (-1)n$. Therefore, equations (8) and (9) become

$$f_1\left(x^{-n}\right) = -x^{-2n} f_1(x^n),$$
$$f_2\left(x^{-n}\right) = -x^{-2n} f_2(x^n).$$

For arbitrary $\vartheta > 0$, set $\vartheta = x^n$ with $x \in \mathbb{R}\backslash\{0\}$, so

$$f_1\left(\vartheta^{-1}\right) = -\vartheta^{-2} f_1(\vartheta),$$
$$f_2\left(\vartheta^{-1}\right) = -\vartheta^{-2} f_2(\vartheta).$$

Also, these equations hold for $\vartheta < 0$, since f_1 and f_2 are odd functions. Thus, according to Lemma 1.2, the result follows.

Case 4. $m > 0$, $n < 0$.

In this case, equations (8) and (9), reduce to the same form as in case 3

$$f_1\left(x^n\right) = \frac{1}{\alpha} x^{n-m} f_1(x^m),$$
$$f_2\left(x^n\right) = \frac{1}{\alpha} x^{n-m} f_2(x^m).$$

This completes the proof. $\square$

Example 2.2. Let $j = 1, 2, 3, 4$. Define $f_j : \mathbb{R} \to \mathbb{R}$ by

$$\begin{cases} f_1(x) = \sigma(x) \\ f_2(x) = x \\ f_3(x) = x + \sigma(x) \\ f_4(x) = \sigma(x) - x \end{cases}$$

for all $x \in \mathbb{R}$, where σ is nontrivial derivation on $\mathbb{R}$. Then, f_j satisfying the equation (7) and $f_j(0) = 0$. However, f_2 does not satisfy condition (9).

Theorem 2.3. *Let $j = 1, 2, 3, 4$. Assume that the functions $f_j : \mathbb{R} \to \mathbb{R}$ satisfy the equation (7), $f_j(0) = 0$ and the conditions*

$$f_1(x^m) = \alpha x^{m-n} f_1(x^n), \tag{10}$$

$$f_2(x^m) = \alpha x^{m-n} f_2(x^n) \tag{11}$$

hold for every $x \in \mathbb{R}\backslash\{0\}$, where $\alpha \in \mathbb{R}$ is constant and $m, n \in \mathbb{Z}$ with $m \neq \alpha n$. If $\alpha = 1$, then

$$\begin{cases} f_1(x) = \lambda_1 x \\ f_2(x) = \lambda_2 x \\ f_3(x) = (\lambda_1 + \lambda_2)x \\ f_4(x) = (\lambda_1 - \lambda_2)x \end{cases}$$

for all $x \in \mathbb{R}$, where $\lambda_1 = f_1(1)$ and $\lambda_2 = f_2(1)$. If $\alpha \neq 1$, then $f_j(x) = 0$ for every $x \in \mathbb{R}$.

Proof. Let $\alpha = 1$ and $m \neq \alpha n$.

If $m = 0$ or $n = 0$, from (10), (11) and $f_j(0) = 0$ for $j = 1, 2, 3, 4$, then $f_1(x) = xf_1(1)$ and $f_2(x) = xf_2(1)$ for all $x \in \mathbb{R}$. Therefore, by Theorem 1.8, $f_3(x) = f_1(x) + f_2(x) = x(f_1 + f_2)(1)$ and $f_4(x) = f_1(x) - f_2(x) = x(f_1 - f_2)(1)$ for all $x \in \mathbb{R}$.

Now, suppose that $m \neq 0$ and $n \neq 0$. By a similar methods in the proof of Theorem 2.1, it can be shown that

$$\begin{cases} f_1(x) = -\frac{1}{2}(A_1 - A_2)(x) \\ f_2(x) = -\frac{1}{2}(A_1 + A_2)(x) \\ f_3(x) = -A_1(x) = f_1(x) + f_2(x) \\ f_4(x) = A_2(x) = f_1(x) - f_2(x) \end{cases}$$

are additive functions. Hence, by Theorem 1.4, the result is verified in this case.

Let $\alpha \neq 1$ and $m \neq \alpha n$ and take $x = 1$ in (10) and (11). Then $f_1(1) = f_2(1) = 0$, since $\alpha \neq 1$.

If $m = 0$ or $n = 0$, then $f_1(x) = \alpha x f_1(1)$ and $f_2(x) = \alpha x f_2(1)$ for all $x \in \mathbb{R}$. Thus $f_j(x) = 0, 1 \leq j \leq 4$, for every $x \in \mathbb{R}$.

In the case $m \neq 0$ and $n \neq 0$, by Theorem 1.4, the proof is complete. □

In the sequel, we find the solution of the system (7) on the restricted domain S^1.

Theorem 2.4. *Let $\lambda_1, \lambda_2 \in \mathbb{R}$. Suppose that $f_j : \mathbb{R} \to \mathbb{R}$ for $j = 1, 2, 3, 4$ satisfy equation (7), with $f_j(0) = 0$ and assume that for all $(x, z) \in S^1$*

$$x f_1(x) + z f_1(z) = \lambda_1, \tag{12}$$

$$x f_2(x) + z f_2(z) = \lambda_2. \tag{13}$$

Then

$$\begin{cases} \mathcal{F}_1(x) = f_1(x) - \lambda_1 x \\ \mathcal{F}_2(x) = f_2(x) - \lambda_2 x \\ \mathcal{F}_3(x) = f_3(x) - (\lambda_1 + \lambda_2)x \\ \mathcal{F}_4(x) = f_4(x) - (\lambda_1 - \lambda_2)x \end{cases}$$

are derivations.

Proof. Using Theorem 1.8, (12) and (13), we have

$$\frac{1}{2}[xB(x,x) - x(A_1 - A_2)(x) + zB(z,z) - z(A_1 - A_2)(z)] = \lambda_1, \tag{14}$$

$$\frac{1}{2}[xB(x,x) - x(A_1 + A_2)(x) + zB(z,z) - z(A_1 + A_2)(z)] = \lambda_2 \tag{15}$$

for all $(x,z) \in S^1$, where $A_1, A_2 : \mathbb{R} \to \mathbb{R}$ are additive functions and $B : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a symmetric bi-additive function.

Subtracting (15) from (14), we obtain

$$xA_2(x) + zA_2(z) = \lambda_1 - \lambda_2, \qquad (x,z) \in S^1$$

Substitute $-x$ for x and $-z$ for z in (14), we obtain

$$\frac{1}{2}[-xB(x,x) - x(A_1 - A_2)(x) - zB(z,z) - z(A_1 - A_2)(z)] = \lambda_1, \quad (16)$$

for all $(x,z) \in S^1$. Adding (15) and (16), we see that

$$xA_1(x) + zA_1(z) = -(\lambda_1 + \lambda_2) \quad (17)$$

for all $(x,z) \in S^1$. Thus by Theorem 1.5, $A_1(x) + (\lambda_1 + \lambda_2)x$ and $A_2(x) - (\lambda_1 - \lambda_2)x$ are derivations.

Adding (14) and (15) and applying (17), we get

$$xB(x,x) + zB(z,z) = 0$$

for all $(x,z) \in S^1$.

Now, set $z = \sqrt{1-x^2}$ in the above equation. Then

$$xB(x,x) + \sqrt{1-x^2}B\left(\sqrt{1-x^2}, \sqrt{1-x^2}\right) = 0 \quad (18)$$

for all $x \in \mathbb{R}$.

Replacing x with $-x$ in (18), we get

$$-xB(x,x) + \sqrt{1-x^2}B\left(\sqrt{1-x^2}, \sqrt{1-x^2}\right) = 0 \quad (19)$$

for all $x \in \mathbb{R}$.

Subtracting (19) from (18), we obtain $xB(x,x) = 0$ for all $x \in \mathbb{R}$. Hence, $B(x,x) = 0$ for all $x \in \mathbb{R}$. Therefore by Theorem 1.8,

$$\begin{cases} \mathcal{F}_1(x) = -\frac{1}{2}(A_1(x) + (\lambda_1 + \lambda_2)x - A_2(x) + (\lambda_1 - \lambda_2)x) = f_1(x) - \lambda_1 x \\ \mathcal{F}_2(x) = -\frac{1}{2}(A_1(x) + (\lambda_1 + \lambda_2)x + A_2(x) - (\lambda_1 - \lambda_2)x) = f_2(x) - \lambda_2 x \\ \mathcal{F}_3(x) = -A_1(x) - (\lambda_1 + \lambda_2)x = f_3(x) - (\lambda_1 + \lambda_2)x \\ \mathcal{F}_4(x) = A_2(x) - (\lambda_1 - \lambda_2)x = f_4(x) - (\lambda_1 - \lambda_2)x \end{cases}$$

are derivations. □

In Theorem 2.4, by taking $\lambda_1 = \lambda_2 = 0$ we get the following result.

Corollary 2.5. *Assume that* $f_j : \mathbb{R} \to \mathbb{R}$ *for* $j = 1, 2, 3, 4$*, satisfy equation* (7)*,* $f_j(0) = 0$ *and*

$$\begin{aligned} xf_1(x) + zf_1(z) &= 0, \\ xf_2(x) + zf_2(z) &= 0 \end{aligned}$$

hold for all $(x, z) \in S^1$*. Then* f_j*,* $j = 1, 2, 3, 4$*, are derivations.*

Theorem 2.6. *If* $f_j : \mathbb{R} \to \mathbb{R}$*,* $j = 1, 2, 3, 4$*, satisfy the Pexider equation* (7)*,* $f_j(0) = 0$ *and*

$$xf_1(z) - zf_1(x) = 0, \tag{20}$$
$$xf_2(z) - zf_2(x) = 0 \tag{21}$$

hold for all $(x, z) \in S^1$*, then* f_j*,* $j = 1, 2, 3, 4$*, are linear.*

Proof. Since $f_j(0) = 0$ for $j = 1, 2, 3, 4$, then by Theorem 1.8, $c_j = 0$. Conditions (20) and (21) yields

$$xB(z, z) - x(A_1 - A_2)(z) - zB(x, x) + z(A_1 - A_2)(x) = 0, \tag{22}$$
$$xB(z, z) - x(A_1 + A_2)(z) - zB(x, x) + z(A_1 + A_2)(x) = 0 \tag{23}$$

for all $(x, z) \in S^1$, where $A_1, A_2 : \mathbb{R} \to \mathbb{R}$ are additive functions and $B : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a symmetric bi-additive function.

Subtracting (23) from (22), we get

$$xA_2(z) - zA_2(x) = 0$$

for all $(x, z) \in S^1$. Thus by Theorem 1.6, A_2 is linear.

Now, substitute $(-x, -z)$ for (x, z) in (22), we obtain

$$-xB(z, z) - x(A_1 - A_2)(z) + zB(x, x) + z(A_1 - A_2)(x) = 0, \tag{24}$$

for all $(x, z) \in S^1$.

Adding (23) with (24), we obtain

$$xA_1(z) - zA_1(x) = 0, \qquad (x, z) \in S^1.$$

Therefore, by Theorem 1.6, we conclude that A_1 is linear.

Adding (22) to (23) we get

$$xB(z,z) - zB(x,x) = xA_1(z) - zA_1(x) = 0$$

for all $(x,z) \in S^1$. Hence

$$xB\left(\sqrt{1-x^2}, \sqrt{1-x^2}\right) = \sqrt{1-x^2}B(x,x) \tag{25}$$

for all $x \in \mathbb{R}$.

Substituting $-x$ in place of x in (25), we have

$$-xB\left(\sqrt{1-x^2}, \sqrt{1-x^2}\right) = \sqrt{1-x^2}B(x,x) \tag{26}$$

for all $x \in \mathbb{R}$.

From (25) and (26), we get $B(x,x) = 0$ for all $x \in \mathbb{R}$. So $f_j : \mathbb{R} \to \mathbb{R}$ for $j = 1,2,3,4$ are linear. □

Conclusion

We obtain the additive solutions of the Pexider functional equation (7) under conditional equations that leads to continuous additive or derivation functions.

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Sadegh Izadi
PhD Candidate of Pure Mathematics
Department of Mathematics
Shiraz University of Technology
Shiraz, Iran
E-mail: s.izadi@sutech.ac.ir

Sedigheh Jahedi
Associate Professor of Pure Mathematics
Department of Mathematics
Shiraz University of Technology
Shiraz, Iran
E-mail: jahedi@sutech.ac.ir

Mehdi Dehghanian
Associate Professor of Pure Mathematics
Department of Mathematics
Sirjan University of Technology
Sirjan, Iran
E-mail: mdehghanian@sirjantech.ac.ir