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Some Counterexamples of (ϕ, ψ)-Amenable-Like Properties of Banach Algebras

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Abstract. In this paper, we present examples contrasting the two main results of the theory of (φ, ψ)-amenability which were introduced in [3]. In particular, we show that being (φ, ψ)-biflatness does not imply (φ, ψ)-approximate biprojectivity, and thus prove that [3, Theorem 3.4] is false. Our main example shows that for an infinite discrete amenable semigroup G, the Banach algebra $\ell^1(G)$ is (id, ψ)-biflat for any $\psi \in \Delta(\ell^1(G))$, but not (id, ψ)-approximately biprojective. The reason for this is that we can prove that (id, ψ)-approximately biprojective implies the left ψ-contractivity of $\ell^1(G)$, which in turn implies that G is finite—a contradiction. Using an essentially identical argument, we show that [3, Theorem 3.7], which connects (φ, ψ)-pseudo-amenability with (φ, ψ)-approximate biprojectivity, is also false. Our examples thus point to the need for more structural hypotheses in the theory of generalized amenability.

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1 Introduction and Preliminaries

The theory of amenability for Banach algebras was introduced by B. E. Johnson, which establishes a profound relationship between the cohomological form of an algebra, and the analytic nature of the space from which the algebra originates. A Banach algebra A is called *amenable* if there is a bounded net (m_α) in the projective tensor product $A \otimes_p A$ such that

$$a \cdot m_\alpha - m_\alpha \cdot a \longrightarrow 0 \quad \text{and} \quad \pi_A(m_\alpha) \cdot a \longrightarrow a \quad \text{for all } a \in A,$$

where $\pi_A : A \otimes_p A \longrightarrow A$ denotes the product map given by $\pi_A(a \otimes b) = ab$. This fundamental result has important implications for abstract harmonic analysis; for example, Johnson's famous result that provides a characterisation of the amenability of the group algebra $L^1(G)$ with respect to the amenability of the locally compact group G. Dales et al. then showed that the measure algebra $M(G)$ is amenable, if and only if, G is discrete and amenable [8].

The use of homotopic techniques is a powerful method to study the structural properties of Banach algebras. In this sense, Helemskii introduced the notion of biflatness [5]. A Banach algebra A is biflat if there exists a bounded A-bimodule morphism $\rho : A \to (A \otimes_p A)^{**}$ such that $\pi_A^{**} \circ \rho = \mathrm{Id}_A$. The concept of biflatness has strong ties to amenability, since a Banach algebra is amenable if and only if it is biflat and has a bounded approximate identity. Since amenability does not capture certain finer structural properties, several generalizations have been studied of amenability that have been studied. Zhang introduced the notion of approximate biprojectivity, where the approximate diagonal is replaced by a net of A-bimodule morphisms $\rho_\alpha : A \longrightarrow A \otimes_p A$ such that $\pi_A \circ \rho_\alpha(a) \longrightarrow a$ [9]. Ghahramani et al. in [4] gave a notion of pseudo-amenable Banach algebras. A Banach algebra A is said to be pseudo-amenable if there is a (not necessarily bounded) net (m_α) in $A \otimes_p A$ such that

$$a.m_\alpha - m_\alpha.a \longrightarrow 0, \qquad \text{and} \qquad \pi_A(m_\alpha)a \longrightarrow a,$$

for any $a \in A$. There exist some examples among matrix algebras and sequence algebras which are pseudo-amenable but are not amenable [1, 2].

Extending these generalizations, Ghorbani and Baradaran proposed an additional refinement in [3], by assuming (φ, ψ)-biflatness and (φ, ψ)-pseudo-amenability, so called because one has a bounded homomorphism $\varphi \in \mathrm{Hom}(A)$ and a multiplicative linear functional $\psi \in \Delta(A)$. One of the main claims they made is that (φ, ψ)-biflatness (φ, ψ)-approximates biprojectivity [3, Theorem 3.4]. However, a major gap in the literature is that these results have not been independently rigorously investigated and the proof contains assumptions that may not hold universally, which could pose an acceptable risk to the protectability of the generalized amenability.

The main purpose of this paper is to deal with this problem by critically evaluating the results in [3]. We show that the above implication is generally false. Our primary contribution is to provide a concrete counterexample: We show that the Banach algebra $\ell^1(G)$ for any infinite, amenable, discrete semigroup G, is (id_A, ψ)-biflat for any character $\psi \in \Delta(\ell^1(G))$, but is not (id_A, ψ)-approximatively biprojective. The proof relies on the fact that the latter must be the case to imply left $\psi-$contractibility of $\ell^1(G)$ is a necessary condition for G to be finite (which is a contradiction). A similar proof shows that we can also contradict [3, Theorem 3.7]. The counterexamples given here not only correct the original literature, but also point to the need for stronger conditions in the study of $(\varphi, \psi)-$amenability-type properties and show the importance of paying attention to the strict nature of abstract elements in harmonic analysis.

For the reader's convenience, we summarize this section with the relevant definitions. For a Banach algebra A, let $\Delta(A)$ denote the character space—that is, the collection of all non-zero multiplicative linear functionals on A, and let $\mathrm{Hom}(A)$ denote the set of all bounded linear homomorphisms from A to itself. For Banach A-bimodules X and Y, a map $\rho : X \longrightarrow Y$ is an A-bimodule morphism if it is bounded, linear, and satisfying

$$\rho(a \cdot x) = a \cdot \rho(x) \quad \text{and} \quad \rho(x \cdot a) = \rho(x) \cdot a$$

for all $a \in A$ and $x \in X$.

2 Counterexamples

In this section, we give concrete counterexamples disproving two important claims in the theory of (φ, ψ)-amenability for Banach algebras from [3]. Before we begin, we review the relevant definitions and establish a preliminary lemma which acts as a key device for our construction.

Let us first recall the formal definition of (ϕ, ψ)-biflatness as introduced in [3]. This definition generalizes the classical biflatness condition, which states that the bimodule morphism must satisfy a compatibility condition with respect to the homomorphism ϕ and the character ψ.

Definition 2.1 ([3])**.** *Suppose that* $\phi \in \mathrm{Hom}(A)$ *and* $\psi \in \Delta(A)$*. Then* A *is called* (ϕ, ψ)*-pseudo-amenable if* A *possesses a net* (m_α) *in* $A \otimes_p A$ *such that*

$$m_\alpha \cdot \phi(a) - \phi(a) \cdot m_\alpha \longrightarrow 0 \quad \text{and} \quad \psi \circ \pi_A(m_\alpha) \longrightarrow 1,$$

for all $a \in A$.

Definition 2.2 ([3], Definition 2.9)**.** *Let* A *be a Banach algebra,* $\phi \in \mathrm{Hom}(A)$ *and* $\psi \in \Delta(A)$*. Then* A *is called* (ϕ, ψ)*-biflat if there exists a bounded* A*-bimodule map* $\rho : A \longrightarrow (A \otimes_p A)^{**}$ *such that*

$$\tilde{\psi} \circ \pi_A^{**} \circ \rho \circ \phi = \psi \circ \phi.$$

Then, we define the notion of (ϕ, ψ)-approximate biprojectivity. This idea changes the structure of an approximate diagonal by putting a net of bimodule morphisms in its place, and the relation between (ϕ, ψ)-biflatness is the main subject of the false theorem we want to refute.

Definition 2.3 ([3], Definition 3.2)**.** *Let* A *be a Banach algebra,* $\phi \in \mathrm{Hom}(A)$ *and* $\psi \in \Delta(A)$*. Then* A *is called* (ϕ, ψ)*-approximately biprojective if there exists a net* $\rho_\alpha : A \longrightarrow A \otimes_p A$ *of continuous* A*-bimodule morphisms such that*

$$\psi \circ \pi_A \circ \rho_\alpha \circ \phi(a) \longrightarrow \psi \circ \phi(a),$$

for any $a \in A$.

To ease the proof of our main lemma, let us recall the definition of left ψ-contractibility. This is a strong condition that guarantees the existence of a certain type of identity element corresponding to the character ψ, and we prove it is implied by a stronger version of approximate biprojectivity.

Definition 2.4 ([7]). *For a Banach algebra A and $\psi \in \Delta(A)$, A is called left ψ-contractible if there exists an element $m \in A$ such that:*

$$a \cdot m = \psi(a) \cdot m \quad \textit{and} \quad \psi(m) = 1,$$

for any $a \in A$.

The main claim we want to refute is the notion that (ϕ, ψ)-biflatness implies (ϕ, ψ)-approximative biprojectivity. We show this theorem here as it appears in [3] before developing the proof of its falsity.

Theorem 2.5 ([3], Theorem 3.4). *Suppose that A is a Banach algebra, $\phi \in \mathrm{Hom}(A)$ and $\psi \in \Delta(A)$. If A is (ϕ, ψ)-biflat, then A is (ϕ, ψ)-approximately biprojective.*

Our method of constructing a counterexample depends on an important intermediate result. The next lemma shows that the existence of a bounded approximate identity means that (id_A, ψ)-biflatness implies a left ψ-amenability. The existence of this relation means that we can use existing results about ψ-amenability to generate a contradiction.

We remind that for a Banach algebra A with $\psi \in \Delta(A)$, A is called left ψ-amenable if there exists an element $m \in A^{**}$ such that

$$am = \psi(a)m \quad \text{and} \quad \tilde{\psi}(m) = 1,$$

for any $a \in A$. Here $\tilde{\psi}$ is the unique extension of ψ to A^{**}, which is defined by

$$\tilde{\psi}(F) = F(\psi),$$

for any $F \in A^{**}$ [6].

Lemma 2.6. *Let A be a Banach algebra, $\phi = \mathrm{id}_A$ and $\psi \in \Delta(A)$. Suppose that A has a bounded approximate identity. If A is (id_A, ψ)-biflat, then A is left ψ-amenable.*

Proof. Let A be (id_A, ψ)-biflat. Then there is a bounded A-bimodule morphism $\rho : A \longrightarrow (A \otimes_p A)^{**}$ such that

$$\tilde{\psi} \circ \pi_A^{**} \circ \rho(a) = \psi(a),$$

for any $a \in A$. Suppose that (e_α) is a bounded approximate identity. Define

$$m_\alpha := \rho(e_\alpha) \in (A \otimes_p A)^{**}.$$

One can see that (m_α) is a bounded net in $(A \otimes_p A)^{**}$ such that

$$a.m_\alpha - m_\alpha.a = a.\rho(e_\alpha) - \rho(e_\alpha).a = \rho(ae_\alpha) - \rho(e_\alpha a) = \rho(ae_\alpha - e_\alpha a) \longrightarrow 0.$$

Also

$$\tilde{\psi} \circ \pi_A^{**}(m_\alpha) = \tilde{\psi} \circ \pi_A^{**} \circ \rho(e_\alpha) = \psi(e_\alpha) \longrightarrow 1.$$

Using Goldstine's theorem, put

$$M := w^*\text{-}\lim m_\alpha \in (A \otimes_p A)^{**}.$$

We claim that $a.M = M.a$ and $\tilde{\psi} \circ \pi_A^{**}(M) = 1$ for any $a \in A$. To see these, let $N \in (A \otimes_p A)^{**}$ be an arbitrary element. It is known that

$$(N.a)(f) = N(a.f) \quad \text{and} \quad a.N(f) = N(f.a),$$

for any $a \in A$ and $f \in (A \otimes_p A)^*$. Hence for any $f \in (A \otimes_p A)^*$, we have

$$|(a.M - M.a)(f)| = |\lim(a.m_\alpha - m_\alpha.a)(f)| \leq \lim \|a.m_\alpha - m_\alpha.a\| \|f\| \longrightarrow 0,$$

which implies $a.M = M.a$. On the other hand, $\tilde{\psi}$ and π_A^{**} are w^*-continuous maps, so

$$\tilde{\psi} \circ \pi_A^{**}(M) = \lim \tilde{\psi} \circ \pi_A^{**}(m_\alpha) = 1.$$

Define $T : A \otimes_p A \longrightarrow A$ with $T(a \otimes b) = \psi(b)\phi(a)$. It is easy to see that

$$a.T^{**}(x) = T^{**}(a.x), \quad T^{**}(x.a) = \psi(a)T^{**}(x), \quad \text{and} \quad \tilde{\psi} \circ T^{**}(x) = \tilde{\psi} \circ \pi_A^{**}(x),$$

for any $a \in A$ and $x \in (A \otimes_p A)^{**}$. Put $\tilde{M} = T^{**}(M) \in A^{**}$. Then we have

$$a\tilde{M} = aT^{**}(M) = T^{**}(a.M) = T^{**}(M.a) = \psi(a)T^{**}(M) = \psi(a)\tilde{M},$$

for any $a \in A$. Also

$$\tilde{\psi}(\tilde{M}) = \tilde{\psi}(T^{**}(M)) = \tilde{\psi} \circ \pi_A^{**}(M) = 1.$$

It implies that A is left ψ-amenable. $\square$

We now present our first counterexample, which is in direct contradiction to the statement of Theorem [3, Theorem 3.4]. We consider the Banach algebra $\ell^1(G)$, where G is an infinite amenable discrete semigroup. We first show that this algebra is (id, ψ)-biflat for any character ψ. Then we show that if it was (id, ψ)-approximately biprojective, it would be left ψ-contractible, which would require G to be finite, which is a contradiction.

Example 2.7. Let G be any infinite, amenable and discrete semigroup (for instance the group of integer numbers). Then by Johnson's theorem, $\ell^1(G)$ is an amenable Banach algebra. Hence there exists $m \in (\ell^1(G) \otimes_p \ell^1(G))^{**}$ such that

$$a \cdot m = m \cdot a, \quad \text{and} \quad \pi_{\ell^1(G)}^{**}(m)a = a,$$

for any $a \in \ell^1(G)$. Applying $\tilde{\psi}$ on the equation $\pi_{\ell^1(G)}^{**}(m)a = a$ yields

$$\tilde{\psi}(\pi_{\ell^1(G)}^{**}(m)) = 1.$$

Define $\rho : \ell^1(G) \to (\ell^1(G) \otimes_p \ell^1(G))^{**}$ by $\rho(a) = a \cdot m$ for all $a \in \ell^1(G)$. It is easy to see that ρ is a bounded $\ell^1(G)$-bimodule morphism and

$$\tilde{\psi} \circ \pi_{\ell^1(G)}^{**} \circ \rho \circ \mathrm{id}_{\ell^1(G)}(a) = \psi(a) = \psi \circ \mathrm{id}_{\ell^1(G)}(a),$$

for any $a \in \ell^1(G)$. So $\ell^1(G)$ is $(\mathrm{id}_{\ell^1(G)}, \psi)$-biflat.

On the other hand, since G is discrete, $\ell^1(G)$ is a Banach algebra with unit e. We claim that $\ell^1(G)$ is not $(\mathrm{id}_{\ell^1(G)}, \psi)$-approximately biprojective. Assume for contradiction that it is. Then there exists a net $\rho_\alpha : \ell^1(G) \to \ell^1(G) \otimes_p \ell^1(G)$ of $\ell^1(G)$-bimodule morphisms such that

$$\psi \circ \pi_{\ell^1(G)} \circ \rho_\alpha(a) \longrightarrow \psi(a),$$

for any $a \in \ell^1(G)$. Define $m_\alpha := \rho_\alpha(e)$. Clearly

$$a.m_\alpha = a.\rho_\alpha(e) = \rho_\alpha(ae) = \rho_\alpha(ea) = \rho_\alpha(e)a = m_\alpha a,$$

and

$$\psi \circ \pi_{\ell^1(G)}(m_\alpha) \longrightarrow \psi(e) = 1.$$

For sufficiently large α, we may assume $\psi \circ \pi_{\ell^1(G)}(m_\alpha) \neq 0$, and replace m_α with $\dfrac{m_\alpha}{\psi \circ \pi_{\ell^1(G)}(m_\alpha)}$ to ensure $\psi \circ \pi_{\ell^1(G)}(m_\alpha) = 1$.

Define $T : \ell^1(G) \otimes_p \ell^1(G) \to \ell^1(G)$ by $T(a \otimes b) = \psi(b)a$. Then $T(m_\alpha) \in \ell^1(G)$ satisfies

$$aT(m_\alpha) = T(am_\alpha) = T(m_\alpha a) = \psi(a)T(m_\alpha),$$

for any $a \in \ell^1(G)$, and

$$\psi(T(m_\alpha)) = \psi \circ \pi_{\ell^1(G)}(m_\alpha) = 1.$$

Thus $\ell^1(G)$ is left ψ-contractible. By [7], this implies G is compact. Since G is discrete, compactness implies G is finite-a contradiction.

Remark 2.8. [3, Theorem 3.7] is also not valid. Following the same arguments as in Example 2.7, we can see that $\ell^1(\mathbb{Z})$ is (id_A, ψ)-pseudo-amenable but it is not (id_A, ψ)-approximately biprojective.

Remark 2.9. In [3, Definition 3.2], "a net $\Theta_\alpha : A \to A \otimes_p A$ of A-bimodule morphisms" should be replaced by "a net $\Theta_\alpha : A \to A \otimes_p A$ of approximately A-bimodule morphisms that is

1. $a.\Theta_\alpha(b) - \Theta_\alpha(ab) \longrightarrow 0$,
2. $\Theta_\alpha(ab) - \Theta_\alpha(a).b \longrightarrow 0$,

for any $a, b \in A$." After changing this hypothesis, the results could be valid.

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