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Original Research Paper

Endo-Artinian Modules

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Abstract. Let R be a commutative ring. A right R-module M is said to be endo-Artinian if it is Artinian with regard to the left L-module structure, where $L = \mathrm{End}_R(M)$. This study demonstrates that if R is a Dedekind domain, then all injective R-modules with finitely many simple components, all unfaithful modules, and arbitrary direct sums of an endo-Artinian module are endo-Artinian modules. Moreover, the following result is established: if R is a Dedekind domain and M is an indecomposable injective torsion right R-module, then M is an Artinian module. It can therefore be demonstrated, on the basis of the preceding arguments, that if S is a simple R-module, then its injective hull $E(S)$ is an Artinian module. Finally, assuming that the ring R is a Dedekind domain, we will present a necessary and sufficient condition for an R-module to be an endo-Artinian module.

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1 Introduction

Throughout this paper, R is assumed to be a commutative ring with identity. A ring R is said to be right (resp. left) *hereditary* if every right

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(resp. left) ideal of R is projective as a right (resp. left) R-module. In commutative algebra, a *Dedekind domain* is defined to be a commutative hereditary domain. By [5, Theorem 2.17], it follows that Dedekind domains must be Noetherian domains. Dedekind domains may be characterized as commutative Noetherian domains that are integrally closed with *Krull dimension* ≤ 1. Alternatively, Dedekind domains may also be characterized as commutative domains in which every ideal is a finite product of prime ideals. The *injective hull* of an R-module M, denoted by $E(M)$, is defined to be the maximal essential extension of M, which may be equivalently expressed as the minimal injective module over M. It is known that indecomposable injective modules over a commutative Noetherian ring R, are not necessarily right Artinian. For instance, $\mathbb{Z}_{p^\infty}$ can be expressed as an example. But, if this property holds for all indecomposable injective R-modules, it follows that all prime ideals of the ring R must be maximal, equivalently, the Krull dimension of R being equal to one. This is because for each prime ideal P of R, the module $E(R/P)$ is an indecomposable injective R-module. In [5, Corollary 3.86], it is demonstrated that, if R is a commutative Noetherian ring, then the necessary and sufficient condition for the indecomposable injective R-modules to be Noetherian is that the ring R is Artinian. In [7, Theorem 5], the sufficient and necessary condition for the injective hull of a simple R-module S to have finite length is that R_P is an Artinian ring, where $P = \mathrm{ann}_R(S)$.

In the section 2, we begin by establishing the necessary condition for the injective hull of an indecomposable injective module M to be Artinian simplified. In the following section, we present evidence that this phenomenon occurs on certain specific rings. The third section of this study is primarily concerned with examining modules that satisfy the descending chain condition on their fully invariant submodules. These modules, which are known as *endo-Artinian* modules, are of particular interest. Endo-Artinian abelian groups have been investigated in [3]. In order to access the classical definitions and theorems that are not given in this article, the reader is invited to refer to the references [2], [4], and[5].

2 When Are Indecomposable Injective R-Modules Right Artinian?

Let R be an integral domain that is not a field, and let Q be the field of fractions of R. It is clear that Q is an indecomposable injective torsion-free R-module that is not a right Artinian module. In this section, we will attempt to answer the following question: Under what conditions are indecomposable injective torsion modules Artinian? In the case where R is a Noetherian integral domain, Matlis' theorem ([5, Matlis' Theorem 3.62]) implies that indecomposable injective R-modules are isomorphic to $E(R/P)$, where P is a prime ideal of R. If we accept the assumption that indecomposable injective torsion R-modules are Artinian, then we have that non-zero prime ideals of R are maximal ideals. Consequently, it can be demonstrated that, in the event that R is a Noetherian domain with Krull dimension at least two, there exists an indecomposable injective torsion R-module that is not Artinian. Consequently, for the purposes of this discussion, it is necessary to assume that the rings in question are Dedekind domains. The following section will demonstrate that, in the case of R being a Dedekind domain, the injective hull of each simple R-module is an Artinian module.

Proposition 2.1. *Let R be a Dedekind domain and E be an injective R-module. Then there exists the family $\{E_i\}_{i\in I}$ of indecomposable injective submodules of E such that $E = \oplus_{i\in I} E_i$. Moreover, for each $i \in I$, either E_i is isomorphic to the injective hull of a simple R-module S or E_i is isomorphic to the field of fraction of R.*

Proof. By [5, Theorem 3.48], since R is a Noetherian ring, E is a direct sum of indecomposable injective submodules. On the other hand by *Matlis' Theorem* [5, Theorem 3.62] the indecomposable injective R-module T is isomorphic to the injective hull R/P, for some prime ideal P of R. Since $\text{krull.dim}\, R \leq 1$, either $\text{krull.dim}\, R = 0$ or $\text{krull.dim}\, R = 1$. If $\text{krull.dim}\, R = 0$, then any prime ideal is a maximal ideal and hence R/P is a simple R-module. As desired. If $\text{krull.dim}\, R = 1$, then either $P = (0)$ or P is a non-zero prime ideal. In case that $P = (0)$, $T \cong E(R)$ which is isomorphic to the field of fraction R. Otherwise, P is a maximal ideal and T is isomorphic to the injective hull of simple R-module R/P. □

Remark 2.2. Suppose R is a Dedekind domain and E an injective right R-module. In Proposition 2.1, we have proved that E is decomposed to direct sum of indecomposable injective submodules E_i ($i \in I$), such that for each $i \in I$, either $E_i \cong E(S)$, where S is a simple R-module or $E_i \cong Q(R)$, where $Q(R)$ is the field of fraction R. For a simple R-module S, set $\Omega(E,S) = \sum\{E_i; E_i \cong E(S)\}$ as the *S-component* (simple component with respect to S) of E, and $\mathcal{Q}(E) = \sum\{E_j : E_j \cong Q(R)\}$. It is evident that for any simple submodule S of E, there exists an index set I_S such that $\Omega(E,S)$ is isomorphic to $(E(S))^{(I_S)}$. In this case, there exists a family $\mathcal{S}$ of simple submodules of E such that E is isomorphic to the direct sum of the following: $\oplus_{S\in\mathcal{S}}(E(S))^{(I_S)} \oplus \mathcal{Q}(E)$. By Lemma 3.2, for each simple R-module S belonging to E, $\Omega(E,S)$ is a fully invariant submodule of E because both $\mathrm{Hom}_R(\Omega(E,S),\mathcal{Q}(E)) = (0)$ and for any two non-isomorphic simple submodules S and S' of E, $\mathrm{Hom}_R(\Omega(E,S),\Omega(E,S')) = (0)$.

Definition 2.3. Let R be a commutative ring and M be a right R-module. For each non-zero element $a \in R$, define the map $f_a : M \to M$ by $f_a(x) = xa$ for all $x \in M$. It is readily apparent that $f_a \in \mathrm{End}_R(M)$.

Proposition 2.4. *Let R be a Dedekind domain, S a simple right R-module, and $E(S)$ be the injective hull of S. Let P be the annihilator of S as an R-module, and let R_P be the localization of R at P. The following assertions hold.*

1. *For each $a \in R \setminus P$, f_a is an isomorphism.*
2. *$E(S)$ has a structure as a right R_P-module.*
3. *$E(S)$ as an R_P-module is injective.*
4. *Every R_P-submodule of $E(S)$ is an R-submodule.*
5. *If N is an R_P-submodule of $E(S)$, then* $\mathrm{Hom}_{R_P}(E(S)/N, E(S)) = \mathrm{Hom}_R(E(S)/N, E(S))$
6. $\mathrm{End}_R(E(S)) = \mathrm{End}_{R_P}(E(S))$.

Proof. The initial three statements are presented in [5, Proposition 3.77]. In light of the proceeding statements, however, it is necessary to

construct this structure anew.
(1) Let $a \in R \setminus P$. The injectivity of $E(S)$ implies that $\operatorname{Im} f_a = Ma = M$. Assume $x \in E(S)$ is a non-zero element of $\ker f_a$. Since S is an essential submodule of $E(S)$, it follows that there exists $r \in R$ such that $0 \neq xr \in S \subseteq E(S)$. Then $0 = (xa)r = (xr)a$ implies $a \in \operatorname{ann}_R(xr) = \operatorname{ann}_R(xrR) = \operatorname{ann}_R(S) = P$, it is a contradiction.
(2) For each $\frac{r}{s} \in R_P$ and $m \in E(S)$, inasmuch as $f_s : M \to M$ is an isomorphism, there exists a unique element $x_{(mr,s)} \in E(S)$ such that $f_s(x_{(mr,s)}) = x_{(mr,s)}s = mr$. Define the map $\phi : E(S) \times R_P \longrightarrow E(S)$ with,

$$\phi((m, \frac{r}{s})) := m \star \frac{r}{s} = x_{(mr,s)}, \quad \text{for all } m \in E(S) \text{ and } \frac{r}{s} \in R_P.$$

It is enough to show that ϕ is a function. For, assume that $(m, \frac{r}{s})$ and $(m, \frac{b}{t})$ are two elements of $E(S) \times R_P$ which are equal. There exists an element $u \in R \setminus P$ such that $u(rt - bs) = 0$. Since R is a domain, $(rt - bs) = 0$. Thus $mrt = mbs$. Inasmuch as $x_{(mr,s)}s = mr$ and $x_{(mb,t)}t = mb$, then

$$x_{(mr,s)}st = mrt = mbs = x_{(mb,t)}ts.$$

This implies that $x_{(mr,s)} - x_{(mb,t)} \in \ker f_{ts}$. By (1), $x_{(mr,s)} - x_{(mb,t)} = 0$. Thus $x_{(mr,s)} = x_{(mb,t)}$, as desired.
(3) This assertion was stated and proven in the proof of [5, Proposition 3.77].
(4) Assume N is an P_P-submodule of $E(S)$. It is sufficient to show that for each $x \in N$ and $r \in R$, $xr \in N$. It is clear that $xr = x \star \frac{r}{1}$, which is contained in N.
(5) Assume N is an R_P-submodule of $E(S)$ and $f \in \operatorname{Hom}_R(E(S)/N, E(S))$. We show that $f \in \operatorname{Hom}_{R_P}(E(S)/N, E(S))$. For, it is enough to show that for each $\bar{m} \in E(S)/N$ and $\frac{r}{s} \in R_P$, $f(\bar{m} \star \frac{r}{s}) = f(\bar{m}) \star \frac{r}{s}$. For,

$$f(\bar{m} \star \frac{r}{s})s = f((\bar{m} \star \frac{r}{s})s) = f(\bar{x}_{(mr,s)}s) = f(\bar{m}r) = f(\bar{m})r.$$

On the other hand,

$$(f(\bar{m}) \star \frac{r}{s})s = x_{(f(\bar{m})r,s)}s = f(\bar{m})r.$$

By above argument, $s \in \mathrm{ann}_R(f(\bar{m} \star \frac{r}{s}) - f(\bar{m}) \star \frac{r}{s})$. Again by (1), $f(\bar{m} \star \frac{r}{s}) - f(\bar{m}) \star \frac{r}{s} = 0$. Conversely, assume $f \in \mathrm{Hom}_{R_P}(E(S)/N, E(S))$. We show that $f \in \mathrm{Hom}_R(E(S)/N, E(S))$. For, it is enough to show that for each $\bar{m} \in E(S)/N$ and $r \in R$, $f(\bar{m}r) = f(\bar{m})r$. For, we have

$$f(\bar{m}r) = f(x_{(\bar{m}r,1)}) = f(\bar{m} \star \frac{r}{1}) = f(\bar{m}) \star \frac{r}{1} = x_{(f(\bar{m})r,1)} = f(\bar{m})r.$$

As desired.
(6) In accordance with the aforementioned assertion, it becomes evident that the matter addressed in this statement is readily apparent. □

Remark 2.5. For the sake of argument, let us assume that R is a commutative P.I.D. and that S is a simple right R-module. Given that R is a PID, it follows that $P = \mathrm{ann}_R(S) = pR$ for some $p \in R$. For each positive integer n, put $E_n = \mathrm{ann}_E(p^n)$ and $E = E(S)$. It is evident that the following ascending chain exists:

$$0 \subseteq E_1 \subseteq E_2 \subseteq \cdots,$$

and $E = \cup_{n \geq 0} E_n$. Since $p \in \mathrm{ann}_R(E_n/E_{n-1})$, for each $n \geq 1$, E_n/E_{n-1} is a simple R-module. By induction, it can be demonstrated that for each $y \in E_n - E_{n-1}$, $E_n =< y >, E_{n-1} =< yp >, \cdots, E_1 = S =< yp^{n-1} >$. To see this, it is clear that $yp^{n-2} \in E_2 \setminus E_1$ and $yp^{n-1} \in E_1 \setminus \{0\}$. Then $E_1 =< yp^{n-1} >$ and $E_2/E_1 =< \overline{yp^{n-2}} >$. For any $z \in E_2$, $\bar{z} \in E_2/E_1$ and hence for some $r, s \in R$, $z - yp^{n-2}r = yp^{n-1}s$. Thus $z = yp^{n-2}(r - ps) \in < yp^{n-2} >$. By induction, the claim is proved. Now, for a submodule N of $E(S)$, set $\mathcal{O}(N) = \{m \in \mathbb{N} : yp^m = 0, \text{for some } y \in N\}$. If $\mathcal{O}(N)$ is an infinite subset of $\mathbb{N}$, then the preceding arguments imply that $N = E(S)$. Otherwise, $N = E_k$, for some positive integer k. Consequently, $E(S)$ is a uniserial right R-module.

Definition 2.6. Let R be a Dedekind domain, S be a simple R-module, $P = \mathrm{ann}_R(S)$ and $E = E(S)$. For each R-submodule N of $E(S)$, the *closure* of N in $E(S)$ is denoted by $\bar{N}$ and it is defined by

$$\bar{N} = \{x_{(n,t)} : n \in N \text{ and } t \in R \setminus P\}$$

Lemma 2.7. *Let R be a Dedekind domain, S a simple R-module, $P = \mathrm{ann}_R(S)$ and $E = E(S)$. Then,*

1. *For each R-submodule N of $E(S)$, $\bar{N}$ is an R_P-submodule (and hence an R-submodule) of $E(S)$ which contains N.*

2. *If N is a proper submodule of $E(S)$, then $\bar{N}$ is also a proper submodule.*

Proof. (1) For each $x_{(n_1,t_1)}$, $x_{(n_2,t_2)}$ and $x_{(n,t)}$ in $\bar{N}$ and $\frac{r}{s} \in R_P$, we have $x_{(n_1,t_1)} + x_{(n_2,t_2)} = x_{(n_1t_2+n_2t_1,t_1t_2)}$ and $x_{(n,t)} \star \frac{r}{s} = x_{(nr,ts)}$ which are contained in $\bar{N}$. On the other hand, for each $n \in N$, $n = x_{(n,1)} \in \bar{N}$. (2) On the contrary, assume $\bar{N} = E(S)$. Then for each $y \in E(S)$, there exist $n \in N$ and $t \in R \setminus P$ such that $y = x_{(n,t)}$. Consequently, $yt = n \in N$. Given that R is a hereditary ring and that N is a proper submodule of $E(S)$, it follows that $E(S)/N$ is an injective R-module that contains a simple submodule, such as N_1/N, which is isomorphic to S. Let us suppose that $x \in N_1 \setminus N$. By the aforementioned assumptions, there exist an element $n \in N$ and an element $t \in R \setminus P$ such that $xt = n \in N$. This implies that $t \in \mathrm{ann}_R(x + N) = \mathrm{ann}_R(N_1/N) = P$. This is a contradiction. □

Proposition 2.8. *Let (R, P) be a local Dedekind domain and S a simple right R-module. Then for each proper submodule N of $E(S)$, $E(S)/N \cong E(S)$.*

Proof. By Remark 2.5, there exists a positive integer m such that $N = E_m =< y >$, for some $y \in E_m \setminus E_{m-1}$. Therefore $f_{p^m} : E(S) \to E(S)$ is a non–zero homomorphism with $\ker f_{p^m} = \mathrm{ann}_E(p^m) = E_m = N$. Inasmuch as R is a right hereditary ring, then $\mathrm{Im}\, f_{p^m} \cong E(S)/N$ is an injective R-module. Hence $\mathrm{Im}\, f_{p^m}$ is a direct summand of $E(S)$. Since $E(S)$ is an indecomposable R-module, $\mathrm{Im}\, f_{p^m} = E(S)$. As desired. □

Theorem 2.9. *Let R be a Dedekind domain and S a simple right R-module. Then the following assertions hold.*

1. *Every proper R-submodule of $E = E(S)$ is a module of finite length.*

2. *$E(S)$ is a right Artinian R-module.*

3. *Every proper factor of $E(S)$, is an indecomposable injective R-module.*

4. *For every proper submodule N of $E(S)$, $E(S)/N$ is isomorphic to $E(S)$.*

Proof. (1) Let us assume that x is a non-zero element of S and that $P = \mathrm{ann}_R(x)$ is a maximal right ideal of R. Since R is a domain, for each positive integer n, it follows that $P^n \neq (0)$. Let E_n be defined as the annihilator of P^n in E. By [5, Corollary 3.85], for each n, E_n is an R-module of finite length. Clearly, we have the following ascending chain

$$0 \subseteq E_1 \subseteq E_2 \subseteq \cdots,$$

and $E = \cup_{n\geq 0} E_n$. Since R is a Dedekind domain and P is a maximal right ideal of R, R_P is a local Dedekind domain with unique maximal ideal $M_P = S^{-1}P$, and using [2, Proposition 9.2], R_P is a principally integral domain (P.I.D). By Proposition 2.4, $E(S)$ has a structure as an R_P-module. Using materials mentioned in Remark 2.5, $E(S)$ as an R_P-module is uniserial and $(0) \subseteq E_1^* \subseteq E_2^* \subseteq \cdots$ is the unique (ascending) chain of its R_P-submodules, where for each n, $E_n^* = \mathrm{ann_E}((\mathrm{M_P})^\mathrm{n})$. Using [2, Proposition 9.2], we know there exist $p_0 \in P$ and $t_0 \in R \setminus P$ such that for each positive integer n, $(M_P)^n =< (\frac{p_0}{t_0})^n >$.

Now, we show that for each positive integer n, $E_n^* \subseteq E_n$. For, suppose $y \in E_n^*$ and $\{p_1, p_2, \cdots, p_n\}$ is a subset of P. Clearly, $\frac{\prod_{i=1}^n p_i}{1} \in (M_P)^n =< (\frac{p_0}{t_0})^n >$. Thus, there exists $\frac{r}{s} \in R_P$ such that $\frac{\prod_{i=1}^n p_i}{1} = \frac{p_0^n}{t_0^n}\frac{r}{s}$. Therefore,

$$y(\prod_{i=1}^n p_i) = x_{(y(\prod_{i=1}^n p_i),1)} = y \star \frac{\prod_{i=1}^n p_i}{1} = y \star (\frac{p_0^n}{t_0^n}\frac{r}{s}) = (y \star \frac{p_0^n}{t_0^n})\frac{r}{s} = 0$$

This implies that $P^n \subseteq \mathrm{ann}_R(y)$ or equivalently $y \in \mathrm{ann}_E(P^n) = E_n$. As desired.

Now suppose that N be a proper R-submodule of $E(S)$. By Lemma 2.7, $\bar{N}$ is a proper R_P submodule of $E(S)$. Then for some positive integer k, $\bar{N} = E_k^*$. Therefore $N \subseteq \bar{N} = E_k^* \subseteq E_n$. Inasmuch as E_n is an R-module of finite length, then N is also a module of finite length.

(2) Any descending chain of R-submodules of $E(S)$, such as $N_1 \supseteq N_2 \supseteq \cdots$, terminates because N_1 is an Artinian R-module.

(3) Assume A and B are submodules of $E(S)$ such that $E(S) = A + B$.

Using materials mentioned in proof of part (1), there exists positive integers n and m, such that $A \subseteq E_n$ and $B \subseteq E_m$. This implies that for some positive integer k, $E(S) \subseteq E_k$. It is a contradiction. Moreover, since R is a right hereditary ring, every proper factor of an injective R-module is injective.
(4) By before argument, $E(S)/N$ is an injective R-module. By Proposition 2.1 and Remark 2.2, there exists a family $\{S_i\}_{i \in I}$ of simple R-modules such that $E(S)/N \cong [\oplus_{i \in I} \Omega(E, S_i)] \oplus \mathcal{Q}(E)$. Inasmuch as for each $y \in E(S)$, there exists a positive integer n such that $y \in E_n$, thus $yP^n = 0$. This implies that $E(S)/N \cong \Omega(E, S)$. Since by part (3), $E(S)/N$ is indecomposable, $E(S)/N \cong E(S)$. $\square$

3 Endo-Artinian Modules

Let R be a commutative ring, M a right R-module, and $L = \mathrm{End}_R(M)$. It is evident that M possesses a structure as a left L-module. The primary objective of this section is to ascertain when M is an Artinian L-module. In light of the fact that the L-submodules of M are precisely the fully invariant R-submodules, the aforementioned question can be posed in the following manner: When does M satisfy the descending chain condition with respect to the fully invariant submodules? Such a module is referred to as an "endo-Artinian" R-module. In this section, we will provide a necessary and sufficient condition for an R-module to be endo-Artinian, assuming that the ring R is a Dedekind domain. It is evident that Artinian modules are endo-Artinian. It is advisable to provide examples of endo-Artinian modules that are not necessarily Artinian before plotting and verifying the results.

Example 3.1.

1. Every right Artinian module is endo-Artinian. The converse is not true generally. For, see the following.

2. Assume F is a field and V an infinite dimensional F-vector space with $L = \mathrm{End}_F(V)$. Clearly, V as a left L-module is simple but V as a right F-module is not Artinian.

3. The abelian group $\mathbb{Q}$ as a $\mathbb{Z}$-module is not Artinian but it is clearly an endo-Artinian module.

4. For each prime number p, the abelian group $(\mathbb{Z}_{p^\infty})^{(\mathbb{N})}$ is an endo-Artinian $\mathbb{Z}$-module (see Proposition 3.6 and Corollary 3.7).

5. For each Artinian R-module M and index set I, $M^{(I)}$ is an endo-Artinian R-module.

For a better study of endo-Artinian modules, it is necessary to state and prove some known lemmas (citing the source) and other basic results that play important roles in the analysis of such modules.

Lemma 3.2. *[6, Lemma 1.9] Let a module $M = M_1 \oplus M_2$ be a direct sum of submodules M_1 and M_2. Then M_1 is a fully invariant submodule of M if and only if $\mathrm{Hom}_R(M_1, M_2) = 0$.*

Lemma 3.3. *[6, Lemma 2.1] Let a module $M = \oplus_I M_i$ be a direct sum of submodules M_i $(i \in I)$ and N a fully invariant submodule of M. Then $N = \oplus_I (N \cap M_i)$.*

Lemma 3.4. *1. Let a module $M = M_1 \oplus M_2$ be a direct sum of submodules M_1 and M_2. If N is a fully invariant submodule of M, then $N \cap M_1$ is a fully invariant submodule of M_1.*

2. Let $N \subseteq K$ be submodules of right R-module M. If N is a fully invariant submodule of K and K is a fully invariant submodule of M, then N is a fully invariant submodule of M.

Proof. (1). Assume $f \in \mathrm{End}_R(M_1)$. Define $\bar{f} \in \mathrm{End}_R(M)$, by $\bar{f}(x + y) = f(x)$ for each $x \in M_1$ and $y \in M_2$. Since N is a fully invariant submodule, $\bar{f}(N) \subset N$. This implies that for each $x \in N \cap M_1$, $\bar{f}(x) = f(x) \in N \cap M_1$. As desired.
(2). The verification is immediate. $\square$

Proposition 3.5. *Let R be a ring, M a right R-module and $\{M_i\}_{i=1}^n$ a family of endo-Artinian submodules of M such that $M = \oplus_{i=1}^n M_i$. Then M is an endo-Artinian module. Moreover, if for each $1 \leq t \leq n$, M_t is a fully invariant submodule of M, then the converse is true.*

Proof. Assume $N_1 \supseteq N_2 \supseteq \cdots$ is a descending chain of fully invariant submodules of M. By Lemma 3.3, for each $t \geq 1$, $N_t = \oplus_{i\in 1}^n (N_t \cap M_i)$ and by Lemma 3.4 part (1), for each $t \geq 1$ and $1 \leq i \leq n$, set $N_{(t,i)} =$

$N_t \cap M_i$, which is a fully invariant submodule of M_i. For each $1 \leq i \leq n$, there exists positive integer m_i such that $N_{(m_i,i)} = N_{(m_i+1,i)} = \cdots$. If we set $m = \max\{m_1, m_2, \cdots, m_n\}$, then $N_m = N_{m+1} = \cdots$. The second part is elementary because for each $1 \leq t \leq n$, fully invariant submodules of M_t are also fully invariant submodules of M. □
The following statement plays a pivotal role in the identification and construction of endo-Artinian modules.

Proposition 3.6. *Let R be a ring and M an R-module. Then N is a fully invariant submodule of $M^{(I)}$ if and only if $N = A^{(I)}$, where A is a fully invariant submodule of M.*

Proof. Let us assume that N is a fully invariant submodule of $M^{(I)}$. It is evident that, by defining $M_i = \iota_i(M)$ for each $i \in I$, we have $M = \oplus_{i\in I} M_i$. For each $i \in I$, define $A_i = \{m \in M : \iota_i(m) \in N\}$. We will now demonstrate that for each distinct elements $i, j \in I$, $A_i = A_j$. To do so, suppose $x \in A_i$. Since N is a fully invariant submodule of $M^{(I)}$, $\iota_i(x) \in N$ and $\iota_j \circ \pi_i \in \mathrm{End}_R(M^{(I)})$, it follows that $\iota_j \circ \pi_i(\iota_i(x)) = \iota_j(x) \in N$, which implies that $x \in A_j$. This implies that $A_i \subseteq A_j$. With the same arguments, one can show that $A_j \subseteq A_i$. Set $A = A_i$, for some $i \in I$. For each $x \in N$ and $i, j \in I$, we have $\iota_i \circ \pi_j(x) \in N$. This implies that $\pi_j(x) \in A$ for each $j \in I$. We now proceed to demonstrate that A is a fully invariant submodule of M. Let $x \in A$ and $f \in \mathrm{End}_R(M)$. Fix an element $i \in I$ and define $g = \iota_i \circ f \circ \pi_i \in \mathrm{End}_R(M^{(I)})$. Since $x \in A$, it follows that $\iota_i(x) \in N$. On the other hand, N is a fully invariant submodule of $M^{(I)}$. Therefore, $g(\iota_i(x)) = \iota_i \circ f \circ \pi_i(\iota_i(x)) = \iota_i(f(x)) \in N$, which implies that $f(x) \in A$. Conversely, let us suppose that A is a fully invariant submodule of M, $x \in A^{(I)}$ and $f \in \mathrm{End}_R(M^{(I)})$. There exist two finite subsets, I_0 and I_1, of I, a finite subset $\{x_i\}_{i\in I_0}$ of A and a finite $\{y_t\}_{t\in I_1}$, of M such that $x = \sum_{i\in I_0} \iota_i(x_i)$ and $f(x) = \sum_{t\in I_1} \iota_t(y_t)$. For each $t \in I_1$,

$$y_t = \pi_t(f(x)) = \pi_t(f(\sum_{i\in I_0} \iota_i(x_i))) = \pi_t(\sum_{i\in I_0} f\circ\iota_i(x_i)) = \sum_{i\in I_0} \pi_t\circ f\circ\iota_i(x_i) \in A.$$

This implies that $f(x) \in A^{(I)}$. As desired. □

Corollary 3.7. *Let R be ring and M a right endo-Artinian R-module. For each index set I, $M^{(I)}$ is an endo-Artinian module (It is not necessarily a right Artinian module).*

Proof. According to the Proposition 3.6, the proof is obvious. □

In the subsequent phase of the discussion, our objective is to examine the structure of endo-Artinian modules on Dedekind domains. To achieve this objective, we will initially introduce two crucial classes of such modules. The theorem 3.14, explicitly demonstrates that these two classes of modules can describe the structure of any endo-Artinian module.

Definition 3.8. Let R be a commutative ring and M an R module. M is said to be *totally bounded* provided that there exists a non-zero element $a \in R$ such that $Ma = (0)$.

Definition 3.9. Let R be a commutative ring, P a prime ideal of R and M an R-module. M is said to be a P-module if for each $m \in M$, there exists a positive integer n, such that $\operatorname{ann}_R(m) = P^n$. A P-module M is called bounded provided that

$$\{n \in \mathbb{Z}^+ : \text{for some } x \in M, \operatorname{ann}_R(x) = P^n\},$$

is finite.

The following proposition demonstrates that, over Dedekind domains, totally bounded modules can be decomposed into the finite direct sum of their fully invariant bounded P-submodules.

Proposition 3.10. *Let R be a Dedekind domain and M a totally bounded R-module. Then there exist a finite subset $\{P_1, P_2, \cdots, P_k\}$ of distinct prime ideals of R and fully invariant bounded P_i-submodules M_i, for each $1 \leq i \leq k$, of M, such that $M = \oplus_{i=1}^k M_i$.*

Proof. Since R is a Dedekind domain, every non-zero ideal of R, has a unique factorization as a product of prime ideals. Then there exist a finite subset $\{P_1, P_2, \cdots, P_k\}$ of distinct prime ideals of R and positive integers $n_1, n_2, \cdots, n_k$, such that $aR = P_1^{n_1} P_2^{n_2} \dots P_k^{n_k}$. By the same arguments, for each non-zero element $x \in M$, $\operatorname{ann}_R(x)$ has a unique factorization as a product of prime ideals such as $\prod_{i=1}^d (P_i')^{m_i}$. By assumption, $\prod_{i=1}^k P_i^{n_i} \subseteq \prod_{i=j}^d (P_j')^{m_j}$. Inasmuch as any non-zero prime ideal of R is a maximal ideal, then for every $1 \leq j \leq d$, there exists $1 \leq i \leq n$ such that $P'_j = P_i$. In other words, we have shown that for each non-zero element $x \in M$, there exist non-negative integers $m_1, m_2, \cdots m_k$, such that $\mathrm{ann}_R(x) = \prod_{i=1}^{k} P_i^{m_i}$. Now, we show that for each $1 \leq i \leq k$, $m_i \leq n_i$. On the contrary, suppose for some $1 \leq i \leq k$, $m_i > n_i$. Since P_i's are maximal ideals, $P_i^{(m_i - n_i)}$ and $\prod_{i \neq j=1}^{k} P_j$ are co-prime (co-maximal). So $P_i^{(m_i - n_i)} + \prod_{i \neq j=1}^{k} P_j^{n_j} = R$. This implies that

$$P_i^{n_i} \subseteq P_i^{m_i} + \prod_{j=1}^{k} P_j^{n_j} \subseteq P_i^{m_i} + \prod_{j=1}^{k} P_j^{m_j} \subseteq P_i^{m_i} \subseteq P_i^{n_i}.$$

This implies that $P_i^{n_i} = P_i^{m_i} = P_i^{n_i}.P_i^{(m_i - n_i)}$. Since $P_i^{n_i}$ is a finitely generated right ideal, there exists $x \in R$ such that $1 - x \in P_i^{(m_i - n_i)}$ and $P_i^{n_i}.x = 0$. It is a contradiction because R is a domain.

Now, for each $1 \leq i \leq k$, put $M_i = \{x \in M : P_i^t \subseteq \mathrm{ann}_R(x), \text{ for some } 1 \leq t \leq n_i\}$. It is clear that M_i's are submodules of M. We show that, $M = \oplus_{i=1}^{n} M_i$. Assume $x \in M$. Since $P_1^{n_1}$ and $\prod_{j=2}^{k} P_j^{n_j}$ are co-prime, there exist $b \in P_1^{n_1}$ and $b' \in \prod_{j=2}^{k} P_j^{n_j}$ such that $b + b' = 1$. Then $xb + xb' = x$. By view of $xb'P_1^{n_1} = 0$, then $xb' \in M_1$. Again, since $P_2^{n_2}$ and $\prod_{j=3}^{k} P_j^{n_j}$ are co-prime, there exist $c \in P_2^{n_2}$ and $c' \in \prod_{j=3}^{k} P_j^{n_j}$ such that $c + c' = 1$. Then $xbc + xbc' = xb$. Inasmuch as $xbc'P_2^{n_2} = 0$, then $xbc' \in M_2$. Thus $x = xb + xb' = xbc + xbc' + xb'$, where $xb' \in M_1$ and $xbc' \in M_2$. By induction, we can show that $M = \sum_{i=1}^{k} M_i$. Since for each $1 \leq i \leq k$, $\mathrm{Hom}_R(M_i, \oplus_{(i \neq) j=1}^{k} M_j) = (0)$, by using Lemma 3.2, M_i's are fully invariant submodules of M. □

Lemma 3.11. *Let R be a Dedekind domain and P a prime ideal of R. Then bounded P-modules are endo-Artinian.*

Proof. Supposing M is a right bounded P-module. Then M has a structure as an R_P-module. Inasmuch as R_P is a P.I.D., there exist positive integers $n_1, n_2, \cdots, n_k$ and index sets $I_1, I_2, \cdots, I_k$ such that

$$M \cong (\frac{R}{P^{n_1}})^{(I_1)} \times (\frac{R}{P^{n_2}})^{(I_2)} \times \cdots \times (\frac{R}{P^{n_k}})^{(I_k)}.$$

Considering that, for each $1 \leq t \leq k$, $\frac{R}{P^{n_t}}$ has finitely many R-submodule, it is an Artinian R-module. By Corollary 3.7, $(\frac{R}{P^{n_t}})^{(I_t)}$ is an endo-

Artinian module. Hence by Proposition 3.5, M is an endo-Artinian R-module. □

Corollary 3.12. *Let R be a Dedekind domain. Then totally bounded R-modules are endo-Artinian.*

Proof. The verification is immediate by applying Proposition 3.5, Proposition 3.10 and Lemma 3.11. □

Proposition 3.13. *Let R be a Dedekind domain and E be an injective R module. Then E is an endo-Artinian module if and only if the number of simple components of E is finite.*

Proof. In light of the aforementioned Remark 2.2, it follows that there exists a finite family of simple submodules of the R-module E, namely $\mathcal{S}$, such that $E \cong \oplus_{S\in\mathcal{S}}(E(S))^{(I_S)} \oplus \mathcal{Q}(E)$. This is consistent with the desired outcome. Conversely, by Theorem 2.9 and Corollary 3.7, for each simple submodule S of E, the simple component due to S, $\Omega(S,E)$, is an endo-Artinian module. Consequently, the Remark 2.2 and the proposition 3.5 will yield the desired result. □

Theorem 3.14. *Let R be a Dedekind domain and M a right R-module. The following statements are equivalent.*

1. *M is an endo-Artinian module.*
2. *There exist a fully invariant injective with finitely many simple component $D(M)$ and a totally bounded submodule $T(M)$ of M such that $M = D(M) \oplus T(M)$.*

Proof. (1) ⇒ (2) If there exists a non-zero element $a \in R$ such that $Ma = (0)$, then the proof is complete upon defining $T(M) = M$. Now, let us assume that for each non-zero element $c \in R$, $Mc \neq (0)$. For any non-zero element $a \in R$, define the mapping $f_a : M \to M$ by $f_a(m) = ma$ for all $m \in R$. It is evident that the image of f_a is Ma and the kernel of f_a is $\mathrm{ann}_M(a)$. These are fully invariant submodules of M. Since M_R is an endo-Artinian module, the set $\mathcal{S} = \{Ma : a \in R \setminus \{0\}\}$ has a minimal element, which may be taken to be Ma_0, for some non-zero element $a_0 \in R$. For each non-zero element $b \in R$, it follows that $Ma_0b \subseteq Ma_0$, and thus $(Ma_0)b = Ma_0$. This implies that Ma_0 is

a divisible right R-module. Since R is a Dedekind domain, it follows from [5, Corollary 3.24] that divisible right R-modules are injective. Let $D(M) = Ma_0$ denote the divisible part of M. Then there exists a submodule $T(M)$ of M such that $M = D(M) \oplus T(M)$. Since $T(M)a_0 \subseteq Ma_0 \cap T(M)$, it follows that $T(M)a_0 = (0)$, which implies that $T(M)$ has no injective submodule. We now demonstrate that $\mathrm{Hom}_R(D(M), T(M)) = (0)$. Otherwise, suppose that f is a non-zero element of $\mathrm{Hom}_R(D(M), T(M)) = (0)$. Since R is a right hereditary ring, the quotient of any right injective R-module is injective. Consequently, $\mathrm{Im}\, f$ is an injective submodule of $T(M)$ because $\mathrm{Im}\, f \cong D(M)/\ker f$. This is a contradiction. By Lemma 3.2, $D(M)$ is a fully invariant submodule of M. It follows that all fully invariant submodules of $D(M)$ are fully invariant submodules of M, and thus $D(M)$ is an endo-Artinian module. Proposition 3.13 indicates that the number of simple components of $D(M)$ must be finite, as desired.
(2) $\Rightarrow$ (1) In light of Assumption 2, it is sufficient to demonstrate that $D(M)$ and $T(M)$ are endo-Artinian modules. These consequences are also the results of Proposition 3.5, Corollary 3.12, and Proposition 3.13. □

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