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Minimax Regret Estimation of Exponential Distribution Based on Record Values Under Weighted Square Error Loss Function

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Abstract. In this paper, we consider the problem of estimating the scale parameter of exponential distribution after preliminary test when the record values are available. The optimal significance levels based on the minimax regret criterion and the corresponding critical values are obtained. This estimator is illustrated by a numerical example.

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1. Introduction

The exponential distribution is one of the most commonly used models in life testing and reliability studies. The probability density function of exponential distribution is given by

$$f(x) = \frac{1}{\theta}\exp\{-\frac{1}{\theta}(x-\eta)\}, \quad x > \eta,\ \theta > 0, \quad -\infty < \eta < \infty,$$

and cumulative distribution function is

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$$F(x) = 1 - \exp\{-\frac{1}{\theta}(x-\eta)\}, \quad x > \eta.$$

In some applications, the experimenter possesses some knowledge about the parameter θ. This knowledge may be obtained from past experiences, or from the acquaintance with similar situations. Thus, one is in position to give an educated guess or prior estimate θ_0. This prior information may be incorporated in the estimation process using a preliminary test estimator.

In this paper, we present a preliminary test estimator for the scale parameter of exponential distribution based on record values. The optimum level of statistical significance for the usual preliminary test estimator is obtained by using the minimax regret criterion [5]. The preliminary test estimator always depend on the significance level of the preliminary test. The methods to seek the optimal level of significance for the preliminary test have been investigated by [7,8,9]. Baklizi in [2] found the preliminary test estimator for the scale parameter of exponential distribution based on censored data, also in [3] Baklizi studied the preliminary test estimator of this parameter based on record values without any optimizations.

Record values are important in many real life application involving data relating to methodology, sport, economics and life testing. Let $X_1, X_2, ...$ be a sequence of independent and identically distributed random variables having the same distribution as the (population) random variable X. An observation X_j will be called an upper record value if exceeds in value all of the preceding observations, i.e., if $X_j > X_i$, for every $i < j$. The sequence of record times $T_n, n \geqslant 1$ is defined as follows: T_1=1 with probability 1 and, for $n \geqslant 2$, $T_n = \min\{j : X_j > X_{T_{n-1}}\}$. A sequence of upper record values is defined by $X_{U(n)} = X_{T_n}$, $n = 1, 2, \ldots$. For details on record values and other interesting topics related to records see [1]. This paper is organized as follows, in Section 2. we drive the preliminary test estimation and define the minimax regret criterion, the optimal level of α is computed numerically in Section 3. A numerical example is given in this Section.

2. Preliminary Test Estimation

Suppose that we observe n upper record values $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ from the exponential model. The likelihood function is

$$L(\theta|\boldsymbol{x}) = \theta^{-n} \exp(-\frac{x_n - \eta}{\theta}), \qquad \eta < x_1 < x_2 < \cdots < x_n. \tag{1}$$

We observe that $T = (X_{U(1)}, X_{U(n)})$ is a joint sufficient statistic for (η, θ). The maximum likelihood estimations of θ and η are

$$\hat{\theta} = \frac{1}{n}(X_{U(n)} - X_{U(1)}) \quad \text{and} \quad \hat{\eta} = X_{U(1)}.$$

The joint pdf of $(X_{U(1)}, X_{U(n)})$ is

$$f_{X_{U(1)},X_{U(n)}}(x,y) = \frac{\theta^{-n}}{\Gamma(n-1)}(y-x)^{n-2} \exp(-\frac{x-\eta}{\theta}), \qquad \eta < x < y < \infty.$$

Lemma 2.1. *Let $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ be record data from the exponential distribution then $W = \frac{2}{\theta}(X_{U(n)} - X_{U(1)})$ has a chi-square distribution with $2n-2$ degrees of freedom.*

Proof. The proof is straight forward. $\square$

Consider $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ be record data from the exponential distribution. Assume that θ_0 is a prior guess of θ. For testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$, the likelihood ratio test statistic is

$$\lambda(\boldsymbol{x}) = \frac{L(\theta_0)}{L(\hat{\theta})} = \frac{{\theta_0}^{-n} \exp(-\frac{x_{U(n)} - x_{U(1)}}{\theta_0})}{\hat{\theta}^{-n} \exp(-\frac{x_{U(n)} - x_{U(1)}}{\hat{\theta}})}.$$

So the acceptance region at level α is $C_1 < 2(X_{U(n)} - X_{U(1)})/\theta_0 < C_2$, where $C_1 = {\chi^2}_{\alpha/2,2n-2}$ and $C_2 = {\chi^2}_{1-\alpha/2,2n-2}$. A preliminary test estimator $\tilde{\theta}$ of θ may be obtained as follows:

$$\tilde{\theta} = \begin{cases} \theta_0, & C_1 < \frac{2}{\theta_0}(X_{U(n)} - X_{U(1)}) < C_2 \\ \hat{\theta}, & \text{otherwise.} \end{cases}$$

The mean of $\tilde{\theta}$ is given by

$$E(\tilde{\theta}) = \theta_0 \int_{\frac{c_1\theta_0}{\theta}}^{\frac{c_2\theta_0}{\theta}} g(w)\, dw + \theta - \frac{\theta}{2n} \int_{\frac{c_1\theta_0}{\theta}}^{\frac{c_2\theta_0}{\theta}} wg(w)dw,$$

where $g(w)$ is the pdf of chi-square random variable with $2n-2$ degrees of freedom. Similarly, the second moment of $\tilde{\theta}$ is

$$E(\tilde{\theta^2}) = {\theta_0}^2 \int_{\frac{c_1\theta_0}{\theta}}^{\frac{c_2\theta_0}{\theta}} g(w)\, dw + \theta^2(1+\frac{1}{n}) - \frac{\theta^2}{4n^2} \int_{\frac{c_1\theta_0}{\theta}}^{\frac{c_2\theta_0}{\theta}} w^2 g(w)\, dw.$$

Therefore, the mean squared error of $\tilde{\theta}$ is

$$\begin{aligned} MSE(\tilde{\theta}) &= {\theta_0}^2 \int_{\frac{c_1\theta_0}{\theta}}^{\frac{c_2\theta_0}{\theta}} g(w)\, dw + \theta^2(1+\frac{1}{n}) - \frac{\theta^2}{4n^2} \int_{\frac{c_1\theta_0}{\theta}}^{\frac{c_2\theta_0}{\theta}} w^2 g(w)\, dw \\ &\quad -2\theta(\theta_0 \int_{\frac{c_1\theta_0}{\theta}}^{\frac{c_2\theta_0}{\theta}} g(w)\, dw + \theta - \frac{\theta}{2n} \int_{\frac{c_1\theta_0}{\theta}}^{\frac{c_2\theta_0}{\theta}} wg(w)\, dw) + \theta^2. \end{aligned}$$

If we consider the weighted square error loss function $L(d;\theta) = (\frac{d}{\theta}-1)^2 = \frac{(d-\theta)^2}{\theta^2}$, the quantity $\frac{MSE(\tilde{\theta})}{\theta^2}$ can be considered as a risk function. Let $\delta = \frac{\theta_0}{\theta}$. Then

$$\begin{aligned} RISK(\delta,\alpha) &= (\delta^2 - 2\delta) \int_{C_1\delta}^{C_2\delta} g(w)\, dw - \frac{1}{4n^2} \int_{C_1\delta}^{C_2\delta} w^2 g(w)\, dw \\ &\quad +\frac{1}{n} \int_{C_1\delta}^{C_2\delta} wg(w)\, dw + \frac{1}{n}. \end{aligned}$$

If $\delta \to 0$ or ∞ then $RISK(\delta,\alpha)$ converges to $RISK(\delta,1)$ which is the risk of the maximum likelihood estimator $\hat{\theta}$. An optimal value of α is $\alpha = 1$ if $\delta \leqslant \delta_1$ or $\delta \geqslant \delta_2$, and $\alpha = 0$ otherwise, where δ_1 and δ_2 are intersections of $RISK(\delta,0) = (\delta-1)^2$ with $RISK(\delta,1) = \frac{1}{n}$. The intersections are $\delta_1 = 1 - \sqrt{\frac{1}{n}}$ and $\delta_2 = 1 + \sqrt{\frac{1}{n}}$.

The aim is to find the optimum values of α, according to the minimax regret criterion. Since δ is unknown we seek an optimal value $\alpha = \alpha^*$

which gives a reasonable risk for all values of δ. The regret function is defined as

$$REG(\delta, \alpha) = RISK(\delta, \alpha) - \inf_{\alpha} RISK(\delta, \alpha),$$

where

$$\inf_{\alpha} RISK(\delta, \alpha) = \begin{cases} RISK(\delta, 1), & \delta \leqslant \delta_1 \ \ or \ \ \delta \geqslant \delta_2 \\ RISK(\delta, 0), & \text{otherwise.} \end{cases}$$

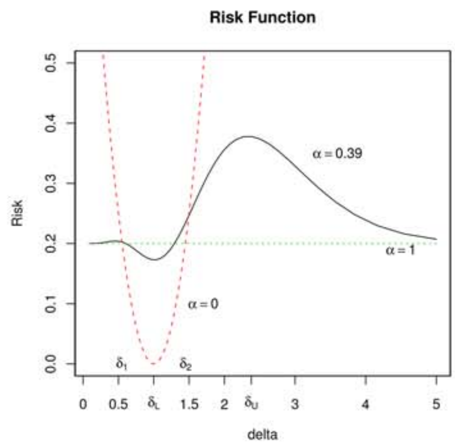

Figure 1. The risk function for some α.

The minimax regret criterion determines α^* such that

$$\sup_{\delta}\{Reg(\delta, \alpha^*)\} \leqslant \sup_{\delta}\{Reg(\delta, \alpha)\},$$

for every significance level $\alpha \neq \alpha^*$. In Figure 1, we plot the risk function for $n = 5$ and $\alpha = 0$, $\alpha = 1$ and $\alpha = 0.39$. We can see that for $\delta \leqslant \delta_2$, $REG(\delta, \alpha)$ takes a maximum value at δ_L, and for $\delta > \delta_2$, it takes a maximum value at δ_U. Thus the minimax regret criterion determines α^* such that $REG(\delta_L, \alpha^*) = REG(\delta_U, \alpha^*)$. An estimator for θ that uses the minimax regret significance levels now can be defined as

$$\tilde{\theta}^* = \begin{cases} \theta_0, & C_1^* < \frac{2}{\theta_0}(X_{U(n)} - X_{U(1)}) < C_2^* \\ \hat{\theta}, & \text{otherwise,} \end{cases}$$

where C_1^* and C_2^* are such that $P_{\theta_0}(W < C_1^*) = P_{\theta_0}(W > C_2^*) = \alpha^*/2$ and $W \sim \chi^2_{2n-2}$.
For the scale parameter exponential distribution ($\eta = 0$), the maximum likelihood estimation of θ is $\hat{\theta} = \frac{1}{n} X_{U(n)}$, and $W = \frac{2}{\theta} X_{U(n)}$ has a chi-square distribution with $2n$ degrees of freedom. Therefore, a preliminary test estimator for θ may be obtained as follows:

$$\tilde{\theta} = \begin{cases} \theta_0, & C_1 < \frac{2}{\theta_0} X_{U(n)} < C_2 \\ \hat{\theta}, & \text{otherwise.} \end{cases}$$

Further steps are the same as general case but with $2n$ degrees of freedom.

3. Numerical Calculations

The aim of this section is to find the optimum values of α, C_1 and C_2 according to the minimax regret criterion (see [2]). We found numerically the optimum significance levels α^* and the corresponding critical values for some degrees of freedoms. The results are given in Table 1.

Table 1: Optimum significance levels and the corresponding critical values.

df	4	6	8	10	12	14	16	18	20
α^*	0.57	0.48	0.43	0.39	0.37	0.35	0.34	0.33	0.32
C_1^*	2.11	3.38	4.74	6.12	7.61	9.09	10.66	12.23	13.81
C_2^*	5.02	7.97	10.77	13.54	16.14	18.74	21.22	23.70	26.18

df	22	24	26	28	30	32	34	36	38
α^*	0.31	0.30	0.30	0.29	0.29	0.28	0.28	0.28	0.27
C_1^*	15.39	16.97	18.67	20.25	21.97	23.56	25.29	27.03	28.61
C_2^*	28.65	31.13	33.43	35.91	38.19	40.67	42.93	45.19	47.68

Example 3.1. The following example is based on a data set discussed by [6] and [4]. A rock crushing machine is kept working as long as the size of the crushed rock is larger than the rocks crushed before. Otherwise it is reset. The data given below represent the sizes of the crushed rocks up to the third reset of the machine

$$9.3, 0.6, 24.4, 18.1, 6.6, 9.0, 14.3, 6.6, 13.0, 2.4, 5.6, 33.8.$$

The upper records are 9.3, 24.4, 33.8. It follows that the MLE's of θ is $\hat{\theta}_{ML} = 8.167$. Based on these record values, the Table 3. gives the values of the preliminary test estimator for various choices of the prior guesses of the scale parameter (θ_0). It can be seen that preliminary test estimation and MLE of θ are equal, for θ_0 less than 10 but the null hypothesis is not rejected for $\theta_0 \geqslant 10$, and so $\tilde{\theta}$ is equal to prior guess and the estimators are different.

Table 2: Preliminary test estimation for the scale parameter.

θ_0	5	6	7	8	9	10	11	12	13	14	15
$\tilde{\theta}^*$	8.167	8.167	8.167	8.167	8.167	10	11	12	13	14	15

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