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Original Research Paper

Shadowing L-algebras

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Abstract. In this paper, we introduce and investigate several novel classes of subalgebras within the framework of L-algebras. Specifically, we establish the concept of fuzzy impotent subalgebras and subsequently extend this notion by incorporating a t-norm T on L to define fuzzy impotent subalgebras with respect to T. Bridging these algebraic structures with soft set theory, the notion of soft impotent subalgebras is introduced, supported by a detailed exposition of examples and structural propositions. Furthermore, integrating probability theory with fuzzy algebraic geometry, we utilize the concepts of random sets and their falling shadows to define falling fuzzy subalgebras and falling impotent subalgebras. The systemic relationships and structural interactions between these falling shadows and traditional fuzzy subalgebras are thoroughly investigated.

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Keywords and Phrases: L-algebra; fuzzy set; fuzzy impotent subalgebra; soft impotent subalgebra.

1 Introduction

With the advent of digital computers, mathematicians in the 1960s such as Edward Lorenz began simulating chaotic weather and physical systems [5]. They quickly discovered a deeply troubling problem as chaotic

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systems have a sensitive dependence on initial conditions.
In the 1970s, foundational topological mathematicians Dmitry Anosov and Rufus Bowen introduced the Shadowing Lemma for uniformly hyperbolic systems. As computers grew more powerful, researchers like Celso Grebogi, Edward Ott, and James A. Yorke applied shadowing directly to practical chaotic attractors [3].
In the 21st century, shadowing escaped the boundaries of classical calculus, differential equations, and continuous geometry. Logicians and algebraists realized that the concept of approximate steps being close to true steps could be translated into abstract algebra. Today, pure mathematicians study shadowing properties in logical algebras such as L-algebras, BL-algebras, and residuated lattices. In this modern frontier, physical metric distance is replaced by algebraic filters and orders, allowing researchers to study how errors propagate through complex algebraic structures, valuation systems, and relational databases [1].
L-algebras are a mathematical structure that generalizes the concept of Boolean algebra. They are particularly useful in soft computing, a field that deals with uncertainty, imprecision, and partial truth. One may know that L-algebras have many applications in other mathematical topics such as group theory, number theory, information theory and so on. L-algebra, was introduced by W. Rump in 2008, [8]. W. Rump proved that every set A with a binary operation $\to$ satisfying equation $(x \to y) \to (x \to z) = (y \to x) \to (y \to z)$ corresponds to a solution of the quantum Yang–Baxter equation if the left multiplication is bijective [10]. In 2012, Yang et al. proved that pseudo MV-algebras can regarded as semi-regular L-algebras [16]. In 2018, W. Rump showed that L-algebras have applications in some theories such as number theory and information theory [11]. In 2019, Wu et al. investigated that each lattice ordered effect algebra creates an L-algebra [15]. In 2022, W. Rump verified that Heyting algebras, MV-algebras, and orthomodular lattices are associated to bounded L-algebras [9].
In 1965, professor L. A. Zadeh introduced the concept of fuzzy set theory [18]. Fuzzy set theory permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the real unit interval [0,1]. After a period of silence, the theory of fuzzy sets was quickly raised and developed in the scientific

communities of different countries. Later, the relationship between fuzzy sets and other algebraic structures, such as L-algebras, MV-algebras, and so on, was considered [2,7,20].
Algebraic structures and fuzzy systems, individually or jointly, have many applications in mathematics and statistics, which can be mentioned in their applications in ergodic theory and calculating the uncertainty of events [19]. Soft set theory is a generalization of fuzzy set theory that was proposed by Molodtsov in 1999 to deal with uncertainty in a parametric manner [6]. A soft set is a parameterized family of sets, intuitively, this is "soft" because the boundary of the set depends on the parameters [21].
Falling shadows representation is applied to algebras. P. Z. Wang and E.Sanchez, introduced the theory of falling shadows that directly relates probability notions with the membership function of fuzzy sets [14]. P.Z. Wang in "Fuzzy sets and Falling Shadows of Random Sets" formulated the mathematical structure of falling shadows [13]. S. K. Tan and et al. established a theoretical approach to define a fuzzy inference relation and fuzzy set operations on the theory of falling shadows [12]. In 2018 Young Bae Jun and et al. published a paper entitled by "Commutative falling neutrosophic ideals in BCK-algebras" [4]. Due to importance of this subject, good and fruitful works have been done in this field and continuous. In this respect falling shadows is applied to subalgebras of an L-algebra. The concept of falling impotent subalgebras of an L-algebra is introduced. The relations between fuzzy impotent subalgebras and falling impotent subalgebras is represented. A characterization of a falling subalgebra is obtained. The concept of a random set and then the notion of a falling shadow of a random set on an L-algebra are introduced. The definition of a falling fuzzy subalgebra and a falling impotent subalgebra of an L-algebra is brought and some relations between them is investigated.

2 Preliminaries

In this section one can become familiar with the notion of L-algebras and some their basic properties. Thee concept of an impotent subalgebra and fuzzy impotent subalgebra of an L-algebra is introduced. Reference [8]

has been utilized more frequently, in this section and the other activities are our own work.

An algebra $(L, \to, 1)$ of type $(2, 0)$ is said to be an *L-algebra* if it satisfies the following conditions:

(1) $x \to x = 1$,

(2) $x \to 1 = 1$,

(3) $1 \to x = x$,

(4) $(x \to y) \to (x \to z) = (y \to x) \to (y \to z)$,

(5) $if \;\; x \to y = y \to x = 1$, then $x = y, for \;\; all \;\; x, y, z \in L$.

The conditions (1), (2) and (3) satisfy that, 1 is a logical unit when the condition (4) is related to the quantum Yang-Baxter equation [8].

Example 2.1. Let $L = \{a, b, c, 1\}$ and consider the following table. One can investigate that $(L, \to, 1)$ is an L-algebra.

$\to$	a	b	c	1
a	1	1	a	1
b	a	1	c	1
c	a	1	1	1
1	a	b	c	1

Table 1: Definition of the binary operation $\to$ implies Definition of L-algebra.

Proposition 2.2. [8] *A logical unit is unique and it is an element of the L-algebra L.*

If $(L, \to, 1)$ is an L-algebra, then there is a partially order on L, defined by:

$$x \leq y \text{ if and only if } x \to y = 1, \text{ for all } x, y \in L \ .$$

If L has a smallest element 0, then L is called a bounded L-algebra [8].

Corollary 2.3. [8] *Let L be an L-algebra. Then for each $x, y \in L$, the following is satisfied:*

(1) $(x \to y) \to 1 = (x \to 1) \to (y \to 1), 1 \to (x \to y) = (1 \to x) \to (1 \to y)$.

Proposition 2.4. [8] *Let L be an L-algebra. Then $x \leq y$ implies that $z \to x \leq z \to y$, for all $x, y, z \in L$.*

Corollary 2.5. [8] *Let L be an L-algebra. Then the followings are equivalent:*

(1) $x \leq y \to x$,

(2) $x \leq z$ *implies* $z \to y \leq x \to y$,

(3) $((x \to y) \to z) \to z \leq ((x \to y) \to z) \to ((y \to x) \to z)$, *for all* $x, y, z \in L$.

Definition 2.6. [8] Let L be an L-algebra. A nonempty subset K of L is said to be an *L-subalgebra* if:

(1) $1 \in K$,

(2) $x \to y \in K$, for all $x, y \in K$.

Definition 2.7. [8] Let L and K be two L-algebras. The map $f : L \to K$ is said to be an L-algebra *homomorphism* if:

(1) $f(1) = 1$,

(2) $f(x \to y) = f(x) \to f(y)$, for all $x, y \in L$.

Example 2.8. Let $L = \{0, c, a, b, 1\}$. Then $(L, \to, 1)$ is an L-algebra, when $\to$ is defined as follow:

$\to$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

Table 2: Definition of binary operation $\to$ implies Definition 2.6.

Then $(L, \to, 0, 1)$ is a bounded L-algebra. Now define $f : L \to L$ by: $f(0) = 0$, $f(1) = 1$, $f(a) = b$, $f(b) = a$ and $f(c) = c$. One can see that f is an L-algebra homomorphism.

Definition 2.9. A subset K of an L-algebra L, is called an *impotent subalgebra* of dimension $n \in \mathbb{N}$, if $x, y \in K$ implies that $x^n \to y \in K$, for $n \geq 2$. $x^n \to y = x \to (x \to (x \to ...(x \to y)))$, for n iteration.

Definition 2.10. Let $\mu : L \to [0, 1]$ be a fuzzy subset of L. μ is said to be a *fuzzy impotent subalgebra* of L, if

$$\mu(x \to (x \to y)) \geq min(\mu(x), \mu(y)), \text{ for all } x, y \in L.$$

Example 2.11. Let $L = \{a, b, c, 1\}$. $(L, \to, 1)$ is an L-algebra. Define $\mu : L \to [0, 1]$ by: $\mu(a) = \frac{1}{4}$, $\mu(b) = \frac{1}{3}$, $\mu(c) = \frac{1}{2}$, $\mu(1) = 1$. Then μ is a fuzzy impotent subalgebra of L.

$\to$	a	b	c	1
a	1	1	a	1
b	a	1	c	1
c	a	1	1	1
1	a	b	c	1

Table 3: Definition of binary operation $\to$ implies Definition 2.8.

Lemma 2.12. *Let μ be a fuzzy impotent subalgebra of L. Then $\mu(1) \geq \mu(x)$, for all $x \in L$.*

Proof. Suppose that $x \in L$ is fixed and arbitrary. $\mu(x \to (x \to x)) \geq min(\mu(x), \mu(x))$. Then $\mu(x \to 1) \geq \mu(x)$ and hence $\mu(1) \geq \mu(x)$. $\square$

Proposition 2.13. *Let μ be a fuzzy impotent subalgebra of L. If there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in L such that $\mu(x_n) \to 1$, as $n \to \infty$. Then $\mu(1) = 1$.*

Proof. By Lemma 2.9, $\mu(1) \geq \mu(x_n)$, for all $n \in \mathbb{N}$. Then

$$1 = lim_{n\to\infty}\mu(x_n) \leq \mu(1) \leq 1.$$

It completes the proof. $\square$

Definition 2.14. Let μ_1, μ_2, be two fuzzy impotent subalgebras of L. Then the *intersection* of μ_1 and μ_2 is defined as follow:

$$\mu_1 \cap \mu_2(x \to y) = min\{\mu_1(x \to y), \mu_2(x \to y)\}, \text{ for all } x, y \in L.$$

Proposition 2.15. *Let μ_1 and μ_2 be two fuzzy impotent subalgebras of L. Then $\mu_1 \cap \mu_2$ is a fuzzy impotent subalgebra of L.*

Proof. $\mu_1 \cap \mu_2(x \to (x \to y)) = min\{\mu_1(x \to (x \to y)), \mu_2(x \to (x \to y))\} \geq min\{min\{\mu_1(x), \mu_1(y)\}, min\{\mu_2(x), \mu_2(y)\}\} = min\{min\{\mu_1(x), \mu_2(x)\}, min\{\mu_1(y), \mu_2(y)\}\} = min\{\mu_1 \cap \mu_2(x), min\{\mu_1 \cap \mu_2(y)\}\}$, for all $x, y \in L$. $\square$

Corollary 2.16. *Let $\{\mu_i\}_{i\in\mathbb{N}}$ be a family of fuzzy impotent subalgebras of L. Then $\bigcap_{i\in\mathbb{N}} \mu_i$ is a fuzzy impotent subalgebra of L.*

Definition 2.17. Let μ be a fuzzy impotent subalgebra of L. Suppose that $\alpha \in [0, 1]$. Then the *α-level subset* of μ is defined as follow:

$$\mu_\alpha = \{x \in L : \mu(x) \geq \alpha\}.$$

Proposition 2.18. *Let μ be a fuzzy subset of L. Then μ is a fuzzy impotent subalgebra if and only if μ_α is an impotent subalgebra of L, for all α in $[0, 1]$.*

Proof. Assume that μ is a fuzzy impotent subalgebra of L. Then $\mu(x) \geq \alpha$ and $\mu(y) \geq \alpha$, for all $x, y \in \mu_\alpha$. $\mu(x \to (x \to y)) \geq min\{\mu(x), \mu(y)\} \geq \alpha$. Hence $x \to (x \to y) \in \mu_\alpha$ and then μ_α is an impotent subalgebra of L. Conversely, let μ_α be an impotent subalgebra of L, for all $\alpha \in [0, 1]$. Assume that $x, y \in L$ and $\mu(x) = \alpha$ and $\mu(y) = \beta$. With out loss of generality let $\alpha \leq \beta$. Hence $\mu_\beta \subset \mu_\alpha$. Since $y \in \mu_\beta$, $y \in \mu_\alpha$, then $x, y \in \mu_\alpha$, by hypothesis, $x \to (x \to y) \in \mu_\alpha$. Therefore $\mu(x \to (x \to y)) \geq \alpha = min\{\alpha, \beta\} = min\{\mu(x), \mu(y)\}$. Thus μ is a fuzzy impotent subalgebra of L. □

Definition 2.19. Let μ be a fuzzy impotent subalgebra of L. Then the level set μ_α is called an $\alpha-$*impotent subalgebra* of L, $\alpha \in [0, 1]$.

Proposition 2.20. *Every impotent subalgebra of L, can be find out as a level impotent subalgebra of some fuzzy impotent subalgebra of L.*

Proof. Let K be an impotent subalgebra of L. Fix $\alpha \in [0, 1]$ and define $\mu : L \to [0, 1]$ by:

$$\mu(x) = \begin{cases} \alpha, & \text{if } x \in K \\ 0, & otherwise. \end{cases}$$

Now assume that $x, y \in L$ and consider three cases:
Case 1) If $x, y \in K$. Then $x \to (x \to y) \in K$. $\mu(x) = \mu(y) = \mu(x \to (x \to y)) = \alpha$. Hence $\mu(x \to (x \to y)) \geq min\{\mu(x), \mu(y)\}$. Therefore μ is a fuzzy impotent subalgebra of L and $\mu_\alpha = K$.
Case 2) If $x \notin K$ and $y \notin K$. Then $\mu(x) = \mu(y) = 0$ and $\mu(x \to (x \to y)) \geq min\{\mu(x), \mu(y)\}$. Then μ is a fuzzy impotent subalgebra of L and hence $\mu_\alpha = K$.
Case 3) if $x \in K, y \notin K$. Then $\mu(x \to (x \to y))] \geq min\{\mu(x), \mu(y)\} = min\{\alpha, 0\} = 0$. Then μ is a fuzzy impotent subalgebra of L and $\mu_\alpha = K$.
□

Lemma 2.21. *Let μ be a fuzzy subset of L. Then for $\alpha_1 < \alpha_2$, the following holds:*

$\mu_{\alpha_1} = \mu_{\alpha_2}$ *if and only if there is no $x \in L$ such that $\alpha_1 \leq \mu(x) < \alpha_2$.*

Proof. Let $\mu_{\alpha_1} = \mu_{\alpha_2}$. If there exists some $x \in L$ such that $\alpha_1 \leq \mu(x) < \alpha_2$. Then $x \in \mu_{\alpha_1}$ but $x \notin \mu_{\alpha_2}$, that is a contradiction. Conversely, Suppose that there is no $x \in L$ such that $\alpha_1 \leq \mu(x) < \alpha_2$. Since $\alpha_1 < \alpha_2$, $\mu_{\alpha_2} \subset \mu_{\alpha_1}$. If $\mu_{\alpha_1} \neq \mu_{\alpha_2}$. Then $\mu_{\alpha_1} \not\subset \mu_{\alpha_2}$. Hence there exists $y \in \mu_{\alpha_1}$ such that $y \notin \mu_{\alpha_2}$. This means that $\alpha_1 \leq \mu(y) < \alpha_2$. That is a contradiction with hypothesis. □

Definition 2.22. Let $f : L \to L$ be an L-algebra homomorphism, and μ be a fuzzy impotent subalgebra of L. Define the map $\mu_f : L \to [0, 1]$ by:

$$\mu_f(x) = \mu(f(x)), \text{ for all } x \in L.$$

Proposition 2.23. *Let $f : L \to L$ be an L-algebra epimorphism. Then μ is a fuzzy impotent subalgebra of L if and only if μ_f is a fuzzy impotent subalgebra of L.*

Proof. Assume that μ is a fuzzy impotent subalgebra of L. $\mu_f(x \to (x \to y)) = \mu(f(x \to (x \to y))) = \mu(f(x) \to (f(x) \to f(y))) \geq min\{\mu(f(x)), \mu(f(y))\} = min\{\mu_f(x), \mu_f(y)\}$, for all $x, y \in L$. Hence μ_f is a fuzzy impotent subalgebra of L. Conversely, suppose that μ_f is a fuzzy impotent subalgebra of L. Let $x, y \in L$, since f is onto, there exist $a, b \in L$, such that $f(a) = x, f(b) = y$.
$\mu(x \to (x \to y)) = \mu(f(a) \to (f(a) \to f(b))) = \mu(f(a \to (a \to b))) = \mu_f(a \to (a \to b)) \geq min\{\mu_f(a), \mu_f(b)\} = min\{\mu(f(a)), \mu(f(b))\} = min\{\mu(x), \mu(y)\}$. Then μ is a fuzzy impotent subalgebra of L. □ Let $K \subseteq L$, then a *fuzzy relation* on the set K, is a fuzzy set $\mu_K : K \times K \to [0, 1]$.

Definition 2.24. Let μ_1, μ_2 be two fuzzy subsets of L. $\mu_1 \times \mu_2 : L \times L \to [0, 1]$ is defined by:

$$\mu_1 \times \mu_2(x, y) = min\{\mu_1(x), \mu_2(y)\}, \text{ for all } x, y \in L.$$

Then $\mu_1 \times \mu_2$ is called the *Cartesian product* of μ_1 *and* μ_2.

Lemma 2.25. *Let μ_1 and μ_2 be two fuzzy subsets of L. Then $\mu_1 \times \mu_2$ is a fuzzy relation on L.*

Lemma 2.26. *Let μ_1 and μ_2 be two fuzzy subsets of L. Then $(\mu_1 \times \mu_2)_\alpha = \mu_{1\alpha} \times \mu_{2\alpha}$, for each $\alpha \in [0, 1]$.*

Lemma 2.27. *The Cartesian product of two L-algebras is an L-algebra.*

Proof. Let L and L' be two L-algebras. Define $\rightarrow$, on $L \times L'$ by:

$$(x, y) \rightarrow (x', y') = (x \rightarrow x', y \rightarrow y').$$

The unit element is $(1, 1)$, such that $(x, y) \rightarrow (1, 1) = (x \rightarrow 1, y \rightarrow 1) = (1, 1)$. It must be shown that:

$$[(x, y) \rightarrow (x', y')] \rightarrow [(x, y) \rightarrow (x'', y'')] = [(x', y') \rightarrow (x, y)] \rightarrow [(x', y') \rightarrow (x'', y'')].$$

$$[(x, y) \rightarrow (x', y')] \rightarrow [(x, y) \rightarrow (x'', y'')] = [(x \rightarrow x', y \rightarrow y')] \rightarrow [(x \rightarrow x''), (y \rightarrow y'')] = [(x \rightarrow x') \rightarrow (x \rightarrow x'')] \rightarrow [(y \rightarrow y') \rightarrow (y \rightarrow y'')] = [(x' \rightarrow x) \rightarrow (x' \rightarrow x''] \rightarrow [(y' \rightarrow y) \rightarrow (y' \rightarrow y'')] = [(x' \rightarrow x), (y' \rightarrow y)] \rightarrow [(x' \rightarrow x''), (y' \rightarrow y'')] = [(x', y') \rightarrow (x, y)] \rightarrow [(x', y') \rightarrow (x'', y'')].$$

One can investigate some other properties of L-algebras. □

Proposition 2.28. *Let μ_1, μ_2 be two fuzzy impotent subalgebras of L. Then $\mu_1 \times \mu_2$ is a fuzzy impotent subalgebra of $L \times L$.*

Proof. Suppose that $\mu_1, \mu_2 : L \rightarrow [0, 1]$ are fuzzy impotent subalgebras and consider $\mu_1 \times \mu_2 : L \times L \rightarrow [0, 1]$. Let $a, b \in L \times L$. Then there exists $x, x', y, y' \in L$ such that $a = (x, y), b = (x', y')$.
$\mu_1 \times \mu_2(a \rightarrow (a \rightarrow b)) = \mu_1 \times \mu_2[(x, y) \rightarrow ((x, y) \rightarrow (x', y'))] = \mu_1 \times \mu_2[(x, y) \rightarrow (x \rightarrow x', y \rightarrow y')] = \mu_1 \times \mu_2[x \rightarrow (x \rightarrow x'), y \rightarrow (y \rightarrow y')] = min\{\mu_1(x \rightarrow (x \rightarrow x')), \mu_2(y \rightarrow (y \rightarrow y'))\} \geq min\{min\{\mu_1(x), \mu_1(x')\}, min\{\mu_2(y), \mu_2(y\prime)\}\} = min\{\mu_1 \times \mu_2(x, y), \mu_1 \times \mu_2(x', y')\} = min\{\mu_1 \times \mu_2(a), \mu_1 \times \mu_2(b)\}$. Therefore $\mu_1 \times \mu_2$ is a fuzzy impotent subalgebra of L. □

3 t-norms on L-algebras

In 1994 Y.Yu et al. introduced the concept of t-norms in L-algebras [17]. Cho et al. introduced and studied the notion of fuzzy subalgebras by a t-norm in BCK-algebras [2]. In this section we define the concept of fuzzy impotent subalgebras with respect to a t-norm T and then the definition of soft algebras over an L-algebra, approximation functions are brought. We investigate some their basic properties. Reference [6] has been utilized more frequently, in this section.

Definition 3.1. [17] Let $I = [0, 1]$. The function $T : I \times I \to I$ is said to be a *t-norm* if:
i) $T(1, x) = x$,
ii) if $y \leq z$, then $T(x, y) \leq T(x, z)$,
iii) $T(x, y) = T(y, x)$,
iv) $T(x, (T(y, z))) = T(T(x, y), z)$, for all $x, y, z \in I$.

Example 3.2. Let $T : I \times I \to I$ be defined as follow:

$$T(x, y) = \begin{cases} x, & \text{if } x \leq y \\ y, & \text{if } y < x. \end{cases}$$

It is clear that T is a t-norm.

Lemma 3.3. [17] *Let T be a t-norm on I. Then $T(x, y) \leq min\{x, y\}$, for all $x, y \in I$.*

For a t-norm T, consider:

$$\Omega(T) = \{x \in I : T(x, x) = x\}.$$

Definition 3.4. Let T be a t-norm, and μ be a fuzzy subset of the L-algebra L. Then μ is said to be a fuzzy rational subset of L, if $T(\mu(x), \mu(x)) = \mu(x)$. In the other words $\mu(x) \in \Omega(T)$, for all $x \in L$.

Definition 3.5. Let T be t-norm on L. A fuzzy subset μ of L is said to be a *fuzzy impotent subalgebra with respect to T*, if

$$\mu(x \to (x \to y)) \geq T(\mu(x), \mu(y)), \text{ for all } x, y \in L.$$

When a fuzzy impotent subalgebra μ of L with respect to T is rational, μ is called a rational fuzzy impotent subalgebra of L with respect to T.

Proposition 3.6. *Let T be a t-norm. If μ is a rational fuzzy impotent subalgebra of L with respect to T. Then μ is a fuzzy impotent subalgebra of L.*

Proof. Let $x, y \in L$. Since μ is rational, $T(\mu(x), \mu(x)) = \mu(x)$. With out loss of generality assume that $\mu(x) \leq \mu(y)$. Then $min\{\mu(x), \mu(y)\} = \mu(x) = T(\mu(x), \mu(x)) \leq T(\mu(x), \mu(y))$. But $\mu(x \to (x \to y)) \geq T(\mu(x), \mu(y))$. Therefore $\mu(x \to (x \to y)) \geq min\{\mu(x), \mu(y)\}$. □

Proposition 3.7. *Let K be a subset of L. Then K is an impotent subalgebra if and only if $\chi_K(x \to (x \to y)) \geq \chi_K(x)\chi_K(y)$, for all $x, y \in L$.*

Proof. Let K be an impotent subalgebra of L and $x, y \in K$. Then $x \to (x \to y) \in K$ and $\chi_K(x \to (x \to y)) = 1 \geq \chi_K(x)\chi_K(y)$. The other cases are similarly as above. Conversely, suppose that $\chi_K(x \to (x \to y)) \geq \chi_K(x)\chi_K(y)$ for all $x, y \in L$. Then for $x, y \in K$, $\chi_K(x \to (x \to y) \geq \chi_K(x)\chi_K(y) = 1$. Therefore $\chi_K(x \to (x \to y)) = 1$. Then $x \to (x \to y) \in K$. □

Definition 3.8. [6] Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denote the power set of U and A be a subset of E. A pair (F, A) is called a *fuzzy impotent subalgebra with respect to U*, where F is a mapping given by $F : A \to P(U)$. The pair (U, E) is called a *soft universe*.
Let (U, E) be a soft universe. The set (F, A) is denoted by (f_A, A), where $f_A : X \to P(U)$, is such that $f_A(x) = \emptyset$ if $x \notin A$. $(f_A, A) = \{(x, f_A(x)) : x \in X, f_A(x) \in P(U)\}$. X is the set of parameters and $A, B, C, ... \subset X$. The subscript A in the notation f_A means that f_A is the *approximation function* of (f_A, A).

Definition 3.9. [6] Let (f_A, A) and (f_B, B) be two soft set of X over U. (f_A, A) is said to be a *soft subset* of (f_B, B), and denoted by $(f_A, A) \overset{\sim}{\subseteq} (f_B, B)$ if the followings are satisfied.
i) $A \subseteq B$,
ii) for each $x \in A$, $(f_A, A)(x)$ and $(f_B, B)(x)$ are identically approximations.

Definition 3.10. [6] Let (f_A, A) be a soft set of X and $V \subseteq U$. The V-exclusive set of (f_A, A) is denoted by $e(f_A, V) = \{x \in A : f_A(x) \subseteq V\}$.

Proposition 3.11. *Let X be a set of parameters. Then the followings are satisfied.*

i) $e(f_A, U) = A$,
ii) $f_A(x) = \bigcap\{V \subseteq U : x \in e(f_A, V)\}$,
iii) if $V_1 \subseteq V_2$, *then* $e(f_A, V_1) \subseteq e(f_A, V_2)$.

Proof. i) It is clear that $e(f_A, U) \subseteq A$. Conversely, suppose that $x \in A$, then $f_A(x) \subseteq U$ and hence $x \in e(f_A, U)$. Therefore $e(f_A, U) = A$.
ii) Let $V \subseteq U$. Then $f_A(x) \subseteq V$, for all $x \in e(f_A, V)$. Hence $f_A(x) \subseteq \bigcap\{V \subseteq U : x \in e(f_A, V)\}$. Now let $v \in \bigcap\{V \subseteq U : x \in e(f_A, V)\}$, then for $v \in V$ and $x \in e(f_A, V)$, for all $V \subseteq U$. Therefore $f_A(x) \subseteq V$ and $v \in V$, for all $V \subseteq U$. Then $v \in f_A(x)$ and hence $\bigcap\{V \subseteq U : x \in e(f_A, V)\} \subseteq f_A(x)$. Then $f_A(x) = \bigcap\{V \subseteq U : x \in e(f_A, V)\}$.
iii) Let $V_1, V_2 \subseteq U$ and $V_1 \subseteq V_2$ and $x \in e(f_A, V_1)$. Then $f_A(x) \subseteq V_1 \subseteq V_2$. This means that $x \in e(f_A, V_2)$, then $e(f_A, V_1) \subseteq e(f_A, V_2)$. □

Definition 3.12. Let $\emptyset \neq K \subseteq L$. Let θ be a relation of K in L. The function $f_K : K \to P(L)$, is defined by:

$$f_K(x) = \{y \in L : (x, y) \in \theta\}, \text{ for all } x \in K.$$

The pair (f_K, K) is called a *soft set over the L-algebra* L.

Definition 3.13. Let U be an initial universe set, and K be a subset of L. Let (U, L) be a soft universe, and (f_K, K) be a soft set over L. Then (f_K, K) is said to be a *soft algebra over* L, if $f_K(x)$ is a subalgebra of L, for each $x \in L$.

Example 3.14. Let $L = \{1, a, b, c, \}$ and consider the following table. One can investigate that $(L, \to, 1)$ is an L-algebra.

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	1	a
b	1	a	1	c
c	1	a	1	1

Table 4: Definition of Binary operation $\to$ implies Definition 3.11.

Let K=L, and define $f_K : K \to P(L)$ by $f_K(x) = \{y \in K : (x, y) \in \theta \ iff \ x \to y \in \{1, c\}\}$.
$f_K(1) = \{1, c\}$, $f_K(a) = \{1, a, b\}$, $f_K(b) = \{1, b, c\}$, $f_K(c) = \{1, b, c\}$.
Then $f_K(x)$ is a subalgebra of L, for each $x \in K = L$. Hence (f_K, K) is a soft algebra over L.

Lemma 3.15. *Let (f_K, K) be a soft algebra over L. If K' is a subset of K. Then $(f_{K'}, K')$ is a soft algebra over L, where $f_{K'}$ is the restriction of f_K to K'.*

Definition 3.16. [6] Let (U, E) be a soft universe and $A, B, C \subseteq E$. Suppose that (f_A, A) and (f_B, B) are two soft sets over U. Then the *intersection* of (f_A, A) and (f_B, B) is a soft set (f_C, C) such that $C = A \cap B$ and $f_C(x) = f_A(x)$ or $f_C(x) = f_B(x)$, for each $x \in C$. One can write $(f_C, C) = (f_A, A) \cap (f_B, B)$. The *union* of (f_A, A) and (f_B, B) is defined as $(f_C, C) = (f_A, A) \cup (f_B, B)$ that $C = A \cup B$ and for all $x \in C$:

$$f_C(x) = \begin{cases} f_A(x), & \text{if } x \in A - B \\ f_B(x), & \text{if } x \in B - A \\ f_A(x) \cup f_B(x), & \text{if } x \in A \cap B. \end{cases}$$

Proposition 3.17. *The intersection of two soft algebra (f_K, K) and $(f_{K'}, K')$ over the L-algebra L, is a soft algebra over L, if $K \cap K' \neq \emptyset$.*

Proof. Since (f_K, K) and $(f_{K'}, K')$ are soft algebras over L, $f_K(x)$ and $f_{K'}(x)$ are subalgebras of L. $(f_C, C) = (f_K, K) \cap (f_{K'}, K')$, where $C = K \cap K'$ and $f_C(x) = f_K(x)$ or $f_C(x) = f_{K'}(x)$, for all $x \in C$. If $x \in C, x \in K$. Then $f_C(x) = f_K(x)$ is a subalgebra of L. If $x \in C, x \in K'$. Then $f_C(x) = f_{K'}(x)$ is a subalgebra of L. □

Proposition 3.18. *The union of two soft algebras (f_K, K) and $(f_{K'}, K')$ over L is a soft algebra over L, if $K \cap K' \neq \emptyset$.*

Definition 3.19. [6] Let (f_A, A) and (f_B, B) be two soft sets over a universe set U. Then (f_A, A) *meet* (f_B, B), is denoted by $(f_A, A) \wedge (f_B, B)$ is a soft set (f_C, C), where $C = A \times B$ and $f_C(x, y) = f_A(x) \cap f_B(y)$, for each $(x, y) \in C$.

Definition 3.20. [6] Let (f_A, A) and (f_B, B) be two soft sets over a universe set U. Then (f_A, A) *join* (f_B, B) denoted by $(f_A, A)\vee(f_B, B)$ is a soft set (f_C, C). In this case let $C = A\times B$ and $f_C(x, y) = f_A(x)\cup f_B(y)$ for each $(x, y) \in C$.

Proposition 3.21. *The meet of two soft algebra (f_K, K) and $(f_{K'}, K')$ over L is a soft algebra over L.*

Proof. Let (f_C, C) be the meet of (f_K, K) and $(f_{K'}, K')$. Then $C = K \times K'$ and $f_C(x, y) = f_K(x) \cap f_{K'}(y)$, for each $(x, y) \in C$. Since $f_K(x)$ and $f_{K'}(y)$ are subalgebras of L, $f_K(x) \cap f_{K'}(y)$ is a subalgebra of L. Therefore (f_C, C) is a soft algebra over L. □

Lemma 3.22. *Let $F : L \to L'$ be an L-algebra homomorphism and (f_K, K) be a soft algebra over L. Then $(Fof_K, F(K))$ is a soft algebra over L'.*

Proof. Since (f_K, K) is a soft algebra over L, $f_K(x)$ is a subalgebra of L, for each $x \in L$. Let $x \in L$, $Fof_K(x) = F(f_K(x))$ is a subalgebra of L', because F is an L-algebra homomorphism. Then $(f(K'), K')$ is a soft algebra over L', where $K' = F(K)$ and $f_{K'}(y) = F(f_K(x))$, for each $y \in L'$ such that $F(x) = y$. □

Proposition 3.23. *Let $F : L \to L'$ be an L-algebra homomorphism and (f_K, K) be a soft algebra over L such that $f_K(x) = kerF$, for each $x \in A$. Then $(Fof_K, F(K))$ is the trivial soft algebra over L'.*

Proof. Assume that $f_K(x) = kerF$, for each $x \in K$. Let $y \in f_K(x) = kerF$. Then $F(y) = 1_{L'}$. Hence $F(f_K(x)) = \{1\}$ and $(Fof_K, F(K))$ is a trivial algebra ,$\{1_{L'}\}$, over L'. □

Definition 3.24. Let K be a subalgebra of L. A subset K' of L is said to be a *subalgebra related to* K, for simple K-subalgebra, if $x \to y \in K'$, for all $x, y \in K$. Indeed K' is an extension subalgebra of L containing K.

Definition 3.25. Let (f_K, K) and $(f_{K'}, K')$ be two algebras over L. Then (f_K, K) is called a *soft subalgebra* of $(f_{K'}, K')$, is denoted by $(f_K, K) \leq (f_{K'}, K')$, if the following conditions are satisfied.
i) $K \subseteq K'$,
ii) $f_K(x)$ is an $f_{K'}(x)$-subalgebra of L, for all $x \in K$.

Example 3.26. Let $L = \{0, a, b, 1\}$, $K = \{a, 1\}$ and $K' = \{a, b, 1\}$. Then K' is a K-subalgebra of L. Let $f_K(0) = f_K(1) = \{1\}$, $f_K(a) = \{a, 1\}$, $f_K(b) = \{b, 1\}$ and $f_{K'}(x) = \{a, b, 1\}$, for each $x \in L$. Furthermore (f_K, K) is a soft subalgebra of $(f_{K'}, K')$.

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	0	1	b	1
b	0	a	1	1
1	0	a	b	1

Table 5: Definition of binary operation $\rightarrow$ implies Definition 3.23.

Proposition 3.27. *Let (f_K, K) and (g_K, K) be two soft algebras over L. Then for each $x \in L$:*

$$f_K(x) \subset g_K(x) \text{ implies } (f_K, K) \leq (g_K, K).$$

Proposition 3.28. *Let (f_K, K) be a soft algebra over L. Furthermore let (f_A, A) and (f_B, B) be two soft subalgebras of (f_K, K). Then*
i) $(f_A, A) \cap (f_B, B) \leq (f_K, K)$,
ii) if $A \cap B = \emptyset$, then $(f_A, A) \cup (f_B, B) \leq (f_K, K)$,
iii) $(f_A, A) \wedge (f_B, B) \leq (f_{K\times K}, K \times K)$.

Proof. i) Assume that $(f_C, C) = (f_A, A) \cap (f_B, B)$, where $C = A \cap B$ and $f_C(x) = f_A(x)$ or $f_C(x) = f_B(x)$, for all $x \in C$. It is clear that $C \subseteq K$. Let $x \in A$, since $(f_A, A) \leq (f_K, K)$, it is concluded that $f_C(x) = f_A(x)$ is an $f_K(x)$-subalgebra. If $x \in C$, since $(f_B, B) \leq (f_K, K)$, it is concluded that $f_C(x) = f_B(x)$ is an $f_K(x)$-subalgebra. Therefore $(f_A, A) \cap (f_B, B) \leq (f_K, K)$.
ii) Assume that $(f_C, C) = (f_A, A) \cup (f_B, B)$, where $A \cap B = \emptyset$, $C = A \cup B$ and for all $x \in C$:

$$f_C(x) = \begin{cases} f_A(x), & \text{if } x \in A - B \\ f_B(x), & \text{if } x \in B - A \\ f_A(x) \cup f_B(x), & \text{if } x \in A \cap B. \end{cases}$$

Since $(f_A, A) \leq (f_K, K), C = (A \cup B) \subseteq K$, it is concluded that $f_A(x)$ is an $f_K(x)$-subalgebra, for all $x \in A$. Similarly $f_B(x)$ is an $f_K(x)$-subalgebra, for all $x \in B$. $A \cap B = \emptyset$ implies that $f_C(x)$ is an $f_K(x)$-subalgebra, for all $x \in C$. Therefore $(f_A, A) \cup (f_B, B) = (f_C, C) \leq (f_K, K)$.
iii) $(f_{K\times K}, K \times K)$ is a soft algebra over L. Then $(f_A, A) \wedge (f_B, B) = (f_{A\times B}, A \times B)$, where $f_{A\times B}(x, y) = f_A(x) \cap f_B(x)$, for all $(x, y) \in A \times B$. $f_{A\times B}$ is an $f_{K\times K}(x, y) = f_K(x) \cap f_K(y)$-subalgebra and then $(f_A, A) \wedge (f_B, B) \leq (f_K, K)$. $\square$

Proposition 3.29. *Let $F : L \to L'$ be an L-algebra homomorphism and (f_K, K) and $(f_{K'}, K')$ be two soft algebras over L. Then $(f_K, K) \leq (f_{K'}, K')$ implies that $(Fof_K, F(K)) \leq (Fof_{K'}, F(K'))$.*

Proof. Let $x \in K$, then $f_K(x)$ is a subalgebra of $f_{K'}(x)$. Since F is an L-algebra homomorphism, $F(f_K(x))$ is a subalgebra of $F(f_{K'}(x))$. Therefore $(Fof_K(x), F(K)) \leq (Fof_{K'}(x), F(K')$. $\square$

Definition 3.30. Let (G, L) be a soft set of L over U. The soft set (G, L) is said to be a *union soft impotent subalgebra of L* if

$$G(x \to (x \to y)) \subseteq G(x) \cup G(y), \text{ for all } x, y \in L.$$

Definition 3.31. The soft set (G, L) of L over U is said to be an *intersection soft impotent subalgebra of L* if

$$G(x \to (x \to y)) \supseteq G(x) \cap G(y), \text{ for all } x, y \in L.$$

Corollary 3.32. *A soft set (G, L) is a union soft impotent subalgebra of L if and only if $e_L(G, V)$ is an impotent subalgebra of L whenever $e_L(G, V) \neq \emptyset$, for all $V \in P(U)$.*

Proof. Let (G, L) be a union soft impotent subalgebra of L. Let $V \in P(U)$ and $e_L(G, V) \neq \emptyset$. Take $x, y \in e_L(G, V)$. Then $V \supseteq G(x)$ and $V \supseteq G(y)$. Since (G, L) is union soft impotent subalgebra, get $V \supseteq G(x) \cup G(y) \supseteq G(x \to (x \to y))$. Then $e_L(G, V)$ is an impotent subalgebra of L. Conversely, suppose that $e_L(G, V)$ is an impotent subalgebra of L, for all $V \in P(U)$ with $e_L(G, V) \neq \emptyset$. Let $x, y \in L$ and $G(x) = V_x, G(y) = V_y$. Take $V = Max\{V_x, V_y\}$. Then $V \supseteq G(x)$ and $V \supseteq G(y)$. Hence $x, y \in e_L(G, V)$. Since $e_L(G, V)$ is an impotent subalgebra, get

$x \to (x \to y) \in e_L(G, V)$. Therefore $G(x) \cup G(y) \supseteq V \supseteq G(x \to (x \to y))$. Hence (G, L) is a union soft impotent subalgebra of L. $\square$

Proposition 3.33. *Let (G, L) be a soft set of L over U. Let $(G^\star, L)$ be a soft set of L over U defined by:*

$$G^\star(x) = \begin{cases} G(x), & \text{if } x \in e_L(G, V) \\ \delta, & \text{otherwise.} \end{cases}$$

Where V is an arbitrary subset of U and δ is a subset of U, satisfying $\delta \supseteq \bigcup_{x \notin e_L(G,V)} G(x)$. Then $(G^\star, L)$ is a union soft impotent subalgebra of L. **Proof.** Assume that $x \in L$. It is clear that $G(x) \subseteq G^\star(x)$. Suppose that (G, L) is a union soft impotent subalgebra of L. Get $e_L(G, V)$ is a union soft impotent subalgebra of L, for all $V \in P(U)$. Now let $x, y \in L$. If $x, y \in e_L(G, V)$. Then $x \to (x \to y) \in e_L(G, V)$. Then $G^\star(x \to (x \to y)) = G(x \to (x \to y)) \subseteq G(x) \cup G(y)$. If $x \notin e_L(G, V)$ and $y \notin e_L(G, V)$, then $G^\star(x) = G^\star(y) = \delta$. Hence $G^\star(x \to (x \to y)) \subseteq \delta = G^\star(x) \cup G^\star(y)$. Therefore $(G^\star, L)$ is a union soft impotent subalgebra of L. $\square$

4 Falling shadows and measure-theoretic Structures on L-algebras

This chapter develops a measure theoretic and fuzzy framework for algebraic structures known as L-algebras. Specifically, it introduces the theory of falling shadows originating from Wang's falling space theory to L-algebras, allowing us to bridge the gap between classical probability spaces, fuzzy set theory, and abstract algebra. The chapter proceeds from foundational measure theory to random sets, details the specific properties of falling impotent subalgebras, examines operations under various joint probability distributions diagonal or anti diagonal, and concludes with structural extensions like direct products and functionally defined L-algebras over real-valued mappings. Reference [12] has been utilized more frequently, in this section.

Definition 4.1. [12] Let $(X, \boldsymbol{\beta}, m)$ be a probability space and $A \subseteq X$ (not necessarily an element of β. A is said to be a *null set* if there exists $B \in \boldsymbol{\beta}$ such that $A \subseteq B$ and $m(B) = 0$. The family of all subsets of X of the form $A \cup C$ where $A \in \boldsymbol{\beta}$ and C is a null set is denoted by $\boldsymbol{\beta}^m$ and is called the *completion of* $\boldsymbol{\beta}$ *with respect to* m. The probability space $(X, \boldsymbol{\beta}, m)$ is called complete if $\boldsymbol{\beta}^m = \boldsymbol{\beta}$. Indeed $(X, \boldsymbol{\beta}, m)$ is complete if for all $B \in \boldsymbol{\beta}$ with $m(B) = 0$ and all $C \subseteq B$ implies that $C \in \boldsymbol{\beta}$.

Example 4.2. Let $X = [0, 1]$ and $\boldsymbol{\beta}$ be the σ-algebra of Borel sets on X. Let $m : \boldsymbol{\beta} \to [0, 1]$ be the Lebesgue measure defined by $m(x, y) = xy$ which generates the Lebesgue measure on [0,1] and the Borel σ-algebra $\boldsymbol{\beta}$.

Definition 4.3. [12] Let $E \subseteq \mathbb{R}$ and $l(I) = l(a, b) = ba$. The *Lebesgue outer measure* $\lambda^\star : \boldsymbol{\beta} \to [0, 1]$ is defined by:
$\lambda^\star(E) = inf\{\sum_{k=1}^{\infty} l(I_k) : (I_k)_{k=1}^{\infty}$ *is a sequence of open intervals that* $E \subseteq \bigcup_{k\in\mathbb{N}} I_k\}$.

If $\lambda^\star$ is a Lebesgue outer measure on the σ-algebra $\boldsymbol{\beta}$ and $A \subseteq \mathbb{R}$, then

$$\lambda^\star(A) \geq \lambda^\star(A \cap E) + \lambda^\star(A \cap E^c), \text{ for each } E \in \boldsymbol{\beta}.$$

Note that if $A \in \boldsymbol{\beta}$, then $\lambda^\star(A) = \lambda(A)$.

Definition 4.4. [12] Let U be the universe set and $\mathbb{P}(U)$ be the power set of U. Let $u \in U$ and $\dot{u} = \{E \subseteq U : u \in E\}$. If $E \in \mathbb{P}(U)$, then let $\dot{E} = \{\dot{u} : u \in E\}$. Let $\mathfrak{C}$ be a σ-algebra in $\mathbb{P}(U)$ and $\dot{U} \subseteq \mathfrak{C}$. In this case $(\mathbb{P}, \mathfrak{C})$ is a *hyper measurable structure on* U.

Definition 4.5. [14] Let $(X, \boldsymbol{\beta}, m)$ be a probability space and $(\mathbb{P}(U), \mathfrak{C})$ be a hyper measurable structure on U. The $\boldsymbol{\beta} - \mathfrak{C}$ measurable map $\xi : X \to \mathbb{P}(U)$ is called a *random set* on U.

It is clear that $\xi^{-1}(C) = \{x \in X : \xi(x) \in C\} \in \boldsymbol{\beta}$, for each $C \in \mathfrak{C}$.

Definition 4.6. [13] Assume that ξ is a random set on U, and for each $u \in U$

$$H(u) = m(\{x \in X : u \in \xi(x)\}).$$

H is a fuzzy subset of U. H is called a falling shadow of the random set ξ and ξ is called a cloud of H.

Example 4.7. Let $X = [0,1]$, $\boldsymbol{\beta}$ *be the Borel* $\sigma -$ *algebra* and m the Lebesgue measure on $\boldsymbol{\beta}$. Let H be a fuzzy subset of U and $H_t = \{u \in U : H(u) \geq t\}$ be a t-cut of H. Then $\xi : [0,1] \to \mathbb{P}(U)$ defined by $\xi(t) = H_t$ is a random set on U and ξ is a cloud of H. ξ is called the *cut cloud* of H.

Definition 4.8. Suppose that L is an L-algebra and $(X, \boldsymbol{\beta}, m)$ is a probability space. Let $\xi : X \to \mathbb{P}(L)$ be a random set. The falling shadow H of the random set ξ, $H(y) = m(\{x \in X : y \in \xi(x)\})$, for $y \in L$, is said to be a falling subalgebra of L if $\xi(x)$ is a subalgebra of L, for all $x \in X$.

Example 4.9. Let $(X, \boldsymbol{\beta}, m)$ be a probability space and L be an L-algebra. Define

$$\psi(L) = \{f : X \to L, f \text{ is a map}\}$$
$$\text{and } \to \text{ on } \psi(L) \text{ by:}$$
$$(f \to g)(x) = f(x) \to g(x), \text{ for all } x \in X \text{ and for all } f, g \in \psi(L).$$

Let $1 : X \to L$ be defined as $1(x) = 1$, for all $x \in X$. It follows that $(\psi(L), \to, 1)$ is an L-algebra. For any impotent subalgebra K of L and $f \in \psi(L)$ let

$$K_f = \{x \in X : f(x) \in K\}$$

and $\xi_K : X \to \mathbb{P}(\psi(L))$

$$\xi_K(x) = \{f \in \psi(L) : f(x) \in K\}.$$

Then $K_f \in \boldsymbol{\beta}$ and $\xi_K(x)$ is an impotent subalgebra of $\psi(L)$. Therefore ξ_K is a random set of $\psi(L)$. $H(f) = m(\{x \in X : f(x) \in K\})$. Then H is a falling subalgebra of $\psi(L)$, which is called a *falling impotent subalgebra* of $\psi(L)$.

Example 4.10. Let $L = \{1, a, b, c\}$ and consider the operation $\to$ on L by

$\rightarrow$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	a	b	1

Table 6: Definition of binary operation $\rightarrow$ implies the definition of fuzzy falling subalgebra of L.

Then $(L, \rightarrow, 1)$ is an L-algebra. Let $X = [0,1]$, $\boldsymbol{\beta}$ the Borel σ-algebra on X and m be the Lebesgue measure on $\boldsymbol{\beta}$. Let $\xi : [0,1] \rightarrow \mathbb{P}(X)$ be defined as:

$$\xi(x) = \begin{cases} \{1, a\}, & \text{if } x \in [0, 0.4] \\ \{1, b\}, & \text{if } x \in [0.4, 0.6] \\ \{1, c\}, & \text{if } x \in [0.6, 1]. \end{cases}$$

$\xi(x)$ is a subalgebra of L, for all $x \in [0,1]$. $H(y) = m(\{x \in X : y \in \xi(x)\})$ is a falling subalgebra of L and H can be represented as:

$$H(y) = \begin{cases} 1, & \text{if } y = 1 \\ 0.4, & \text{if } y = a \text{ or } y = c \\ 0.2, & \text{if } y = b. \end{cases}$$

Therefore H is a fuzzy falling subalgebra of L.

Proposition 4.11. *Let $(X, \boldsymbol{\beta}, m)$ be a probability space and H be a falling shadow of a random set $\xi : X \rightarrow \mathbb{P}(L)$. Then H is a falling impotent subalgebra of the L-algebra L if and only if*

$$X(x;\xi) \cap X(y;\xi) \subseteq X(x \rightarrow (x \rightarrow y);\xi) \text{ and}$$
$$X(x;\xi) \cap X(y;\xi) \subseteq X(x \rightarrow y;\xi).$$

Proof. For $x \in L$ let $X(x;\xi) = \{t \in X : x \in \xi(t)\}$. At first suppose that H is a falling impotent subalgebra of L. Then $\xi(t)$ is an impotent subalgebra of L, for each $t \in X$. Let $x, y \in L$ and $t \in X(x;\xi) \cap X(y;\xi)$. Hence $x \in \xi(t)$ and $y \in \xi(t)$. Since $\xi(t)$ is an impotent subalgebra of L, one gets $x \to (x \to y) \in \xi(t)$. Then $t \in X(x \to (x \to y);\xi)$ and then $X(x;\xi) \cap X(y;\xi) \subseteq X(x \to (x \to y);\xi)$. Conversely, assume that $X(x;\xi) \cap X(y;\xi) \subseteq X(x \to (x \to y);\xi)$. For $x, y \in L$ and $x, y \in \xi(t)$, it is concluded that $t \in X(x;\xi) \cap X(y;\xi)$, for each $t \in X$. Therefore $x \to (x \to y) \in \xi(t)$ i.e $\xi(t)$ is an impotent subalgebra of L. Then H is a falling impotent subalgebra of L. □

Proposition 4.12. *Let $(X, \boldsymbol{\beta}, m)$ be a probability space and $\xi : X \to \mathbb{P}(L)$ be a random set. Then every fuzzy subalgebra of L is a falling fuzzy subalgebra of L.*

Proof. Let H be a fuzzy subalgebra of L. Then the level set H_t is a subalgebra of L, for each $t \in [0, 1]$. Let $\xi : [0, 1] \to \mathbb{P}(L)$ be a random set such that $\xi(t) = H_t$. Thus H is a falling fuzzy subalgebra of L. □
Note that the converse of the Proposition 4.8, is not true in general, see the following example.

Example 4.13. Let $L = \{1, a, b, c\}$ and consider the operation $\to$ on L by

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	1	a
b	1	a	1	c
c	1	a	1	1

Table 7: Definition of binary operation $\to$ implies the converse of Proposition 4.8 is not true.

Then $(L, \to, 1)$ is an L-algebra. Let $X = [0, 1]$, $\boldsymbol{\beta}$ the σ-algebra of Borel sets on $[0, 1]$ and $m : \boldsymbol{\beta} \to [0, 1]$ be the Lebesgue measure. Define $\xi : [0, 1] \to \mathbb{P}(L)$ by:

$$\xi(t) = \begin{cases} \{1, a\}, & \text{if } t \in [0, 0.4] \\ \{1, b\}, & \text{if } t \in [0.4, 0.6] \\ \{1, c\}, & \text{if } t \in [0.6, 1]. \end{cases}$$

Then $\xi(t)$ is a subalgebra of L, for all $t \in [0, 1]$. Therefore $H(x) = m(\{t \in [0, 1] : x \in \xi(t)\})$ is a falling fuzzy subalgebra of L. Let

$$H(x) = \begin{cases} 1, & \text{if } x = 1 \\ 0.4, & \text{if } x = a \text{ or } x = c \\ 0.2, & \text{if } x = b. \end{cases}$$

Since $H(a \to c) = H(b) = 0.2 < 0.4 = min(\{H(a), H(c)\})$, H can not be a fuzzy subalgebra of L.

Proposition 4.14. *Let $(L, \to, 1)$ be an L-algebra and $(X, \boldsymbol{\beta}, m)$ be a probability space. Then a falling shadow H of a random set $\xi : X \to \mathbb{P}(L)$ is a falling impotent subalgebra of L if and only if for each $x \in X$ the following holds:*

for each $t \in \xi(x)$ and for each $t \to (t \to y) \in L - \xi(x)$, then $y \in L - \xi(x)$.

Proof. Let H be a falling impotent subalgebra of L. Then $\xi(x)$ is an impotent subalgebra of L, for all $x \in X$. Let $t, y \in L, t \in \xi(x)$ and $t \to (t \to y) \in L - \xi(x)$. If $y \in \xi(x)$, then $t \to (t \to y) \in \xi(x)$ because $\xi(x)$ is an impotent subalgebra of L and $t \in \xi(x)$. It is a contradiction with $t \to (t \to y) \in L - \xi(x)$. Then $y \notin \xi(x)$ i.e $y \in L - \xi(x)$. Conversely, suppose that for every $t \in \xi(x)$ and for each $t \to (t \to y) \in L - \xi(x)$ then $y \in L - \xi(x)$.
Now let $t, y \in L$ be such that $t, y \in \xi(x)$. Assume that $t \to (t \to y) \notin \xi(x)$, by hypothesis $y \notin \xi(x)$ and this is a contradiction. Therefore $t \to (t \to y) \in \xi(x)$ and then $\xi(x)$ is an impotent subalgebra of L. Hence H is a falling impotent subalgebra of L. □

Corollary 4.15. *A random set $\xi : X \to \mathbb{P}(L)$ is a falling subalgebra of L if and only if*

for each $t \in \xi(x)$ and for each $t \to y \in L - \xi(x)$, then $y \in L - \xi(x)$.

Definition 4.16. Let $\xi : X \to \mathbb{P}(L)$ be a random set. A falling impotent subalgebra, H, is said to be an *order preserving* if

$$\text{for every } x, y \in L,\ x \leq y \text{ implies that } X(x;\xi) \subseteq X(y;\xi).$$

Proposition 4.17. *Let $\xi : X \to \mathbb{P}(L)$ be a random set. If H is a falling impotent subalgebra of L, then*

$$H(x \to (x \to y)) \geq T(H(x), H(y)), \text{ for all } x, y \in L, \text{ where}$$

$$T(s,t) = min(\{s + t - 1, 1\}), \text{ for each } t, s \in [0,1].$$

Proof. Let $x, y \in L$. Since

$$X(x;\xi) \cap X(y;\xi) \subseteq X(x \to (x \to y);\xi),$$

hence

$$H(x \to (x \to y)) = m(\{t \in X : x \to y \in \xi(t)\}) \geq m(\{t \in X : x \in \xi(t)\} \cap \{t \in X : y \in \xi(t)\}) \geq m(\{t \in X : x \in \xi(t)\}) + m(\{t \in X : y \in \xi(t)\}) - m(\{t \in X : x \in \xi(t) \text{ or } y \in \xi(t)\}) \geq H(x) + H(y) - 1.$$

Then $H(x \to y) \geq min\{H(x) + H(y) - 1, 1\} = T(H(x), H(y))$. □

Corollary 4.18. *If H is a falling fuzzy impotent subalgebra. Then H is a fuzzy impotent subalgebra with respect to the t-norm T.*

Definition 4.19. Let X be a sample space and $\xi, \eta : X \to [0,1]$ be two fuzzy subsets of X. Then the pair $\Omega = (\xi, \eta)$ is called a *two dimensional random variable.*

Definition 4.20. Let H, H_1, H_2 be falling shadows of random sets $\xi, \xi_1, \xi_2 : X \to \mathbb{P}(L)$ resp. Then
i) $(H_1 \to H_2)(z) = inf\{m([H_1(x), 1] \times [H_2(y), 1]) : z = x \to y\}$,
ii) $H_1 \odot H_2 = inf\{m([H_1(x), 1] \times [H_2(y), 1]) : z \leq x \to y\}$,
iii) $H_-(z) = inf\{m([H(x), 1]) : z \leq x\}$.

It is clear that $H_-(z) = inf\{m([H(x), 1]) : z \leq x\} = inf\{H(x) : z \leq x\} \leq H(z)$, for each $z \in L$.
Let $(X, \boldsymbol{\beta}, m)$ be a probability space and m be a probability distribution of two dimensional random variable (ξ, η) on $[0, 1] \times [0, 1]$. Then there are three kind of probability distributions:
i) if the total probability m is *centered and uniformly distributed on the main diagonal of the square* $[0, 1] \times [0, 1]$. In this case m is called a diagonal distribution.
ii) If the total probability m is *centered and uniformly distributed on the anti diagonal of the square* $[0, 1] \times [0, 1]$. In this case m is called an anti diagonal distribution.
iii) If the total probability m is *uniformly distributed on the square* $[0, 1] \times [0, 1]$. In this case m is called an independent distribution.
If the distribution m is diagonal, anti diagonal resp. Then
$(H_1 \to H_2)(z) = inf\{T(H_1(x), H_2(y)) : z = x \to y\}$,
$(H_1 \odot H_2)(z) = inf\{T(H_1(x), H_2(y)) : z \leq x \to y\}$.

Proposition 4.21. *Let H be a falling shadow of a random set $\xi : X \to \mathbb{P}(L)$. Then $H \leq H_-$ if and only if $x \leq y$ implies that $H(x) \leq H(y)$, for each $x, y \in L$.*

Proof. Let $H = H_-$ and $x, y \in L$. If $x \leq y$, then $H_-(x) = inf\{z \in L : z \leq x\} \leq inf\{z \in L : z \leq y\} = H_-(y)$. Therefore $H(x) \leq H(y)$. Conversely, it is shown that $H_- \leq H$. Now $H_-(z) = inf\{H(x) : z \leq x\}$ and $z \leq x$, implies that $H(z) \leq H_-(z)$, Hence $H = H_-$. □

Proposition 4.22. *Let H be a falling shadow of a random set $\xi : X \to \mathbb{P}(L)$. If the probability measure m is a diagonal (anti diagonal) distribution, then H is a T-fuzzy subalgebra of L if and only if $H \to H \leq H (H \odot H \leq H)$.*

Proof. Assume that m is a diagonal distribution and H is a T-fuzzy subalgebra of L. Let $z \in L$, then there exists $x, y \in L$ such that $z = x \to y$. Thus $H(z) = H(x \to y) \geq T(H(x), H(y))$. Hence $H(z) \geq inf\{T(H(x), H(y)) : z = x \to y\} = (H \to H)(z)$. Therefore $H \to H \leq H$. Conversely, assume that $H \to H \leq H$. Then $H(x \to y) \geq (H \to H)(x \to y) \geq T(H(x), H(y))$, for each $x, y \in L$. Then H is a T-fuzzy subalgebra of X. The proof of the other case is similar as above. □

Proposition 4.23. *Let m be a diagonal distribution, and H_1, H_2 be two falling shadows of random sets $\xi_1, \xi_2 : X \to \mathbb{P}(L)$, resp. If H_1, H_2 are T-fuzzy subalgebras of L, then $H_1 \odot H_2$ is a T-fuzzy subalgebra of L.*

Proof. Assume that H_1, H_2 are T-fuzzy subalgebras of L. $(H_1 \odot H_2)(x \to y) \geq inf\{H_1(x_1 \to y_1) \wedge H_2(x_2 \to y_2) : x \leq x_1 \to x_2, y \leq y_1 \to y_2\} = inf\{H_1(x_1) \wedge H_2(x_2) : x \leq x_1 \to x_2\} \wedge inf\{H_1(y_1) \wedge H_2(y_2) : y \leq y_1 \to y_2\} = (H_1 \odot H_2)(x) \wedge (H_1 \odot H_2)(y)$. Then $H_1 \odot H_2$ is a T-fuzzy subalgebra. □

Definition 4.24. The *direct product of L-algebras* L_1 and L_2, is denoted by $L_1 \times L_2$ and defined as below:

$(x_1, x_2) \to (y_1, y_2) = (x_1 \to y_1, x_2 \to y_2)$, for each $x_i, y_i \in L_i$, $i = 1, 2$.

Proposition 4.25. *Let $\{A_i : i = 1, 2, ..., n\}$ be a family of L-algebras. Then $\prod_{i=1}^{n} A_i$ is an L-algebra.*

Proposition 4.26. *Let $\{L_i : i = 1, 2, ..., n\}$ be a family of L-algebras. Then there exists an L-algebra ,L, together with a family of homomorphisms $\{\phi_i : L \to L_i, i = 1, 2, ..., n\}$ with the following property:*
for any L-algebra, L', and a family of homomorphisms $\{\psi i : L' \to L_i, i = 1, 2, ..., n\}$, there exists a unique homomorphism $\phi : L' \to L$ such that $\phi_i o \phi = \psi_i$, for each $1 \leq i \leq n$. Furthermore L is unique up to isomorphism.

Proof. The direct product $\prod_{i=1}^{n} L_i$ is an L-algebra. Let $L = \prod_{i=1}^{n} L_i$ and let $\phi_i : L \to L_i$ be the canonical projection i.e $\phi_i(x_1, x_2, ..., x_n) = x_i$, for each $1 \leq i \leq n$. Suppose L' is an arbitrary L-algebra and $\{\psi_i : L' \to L_i, 1 \leq i \leq n\}$ a family of homomorphisms. Now define $\phi : L' \to L$ by $\phi(x) = (\psi_1(x), ..., \psi_n(x)$, for all $x \in L'$.
It is clear that ϕ is a homomorphism of L-algebras and $\phi_i o \phi = \psi_i$. $\phi_i o \phi(x) = \psi(x) = \phi_i o \phi'(x)$. Suppose that $\phi'(x) = (x_1, ..., x_n)$. $x_i = \phi_i(x_1, ..., x_n) = \phi_i(\phi'(x)) = \phi_i o \phi'(x) = \phi_i o \phi(x) = \phi_i(\psi_1(x), ..., \psi_n(x)) = \psi_i(x)$, for each $x \in L'$. Then ϕ is unique.
Suppose that L' is an L-algebra together with a family $\{\phi_i' : L' \to L_i, 1 \leq i \leq n\}$ of L-algebra homomorphisms. Since L' and L have the same property, if one consider L and the family $\{\phi_i' : L' \to L_i, 1 \leq i \leq n\}$ and consider L' together with the family $\{\phi_i : L \to L_i, 1 \leq i \leq n\}$.

Then there are unique homomorphisms $\alpha : L' \to L$ and $\beta : L \to L'$ such that $\phi_i o\alpha = \phi'_i$ and $\phi'_i o\beta = \phi_i$, for all $1 \leq i \leq n$. On the other hand $\phi_i oid = \phi_i$, for all $1 \leq i \leq n$ and hence $\alpha o\beta = id$. In the same manner $\beta o\alpha = id$. Then β is an isomorphism i.e L is unique up to isomorphism. □ Let ${\mathbb{R}_0}^X = \{h : X \to \mathbb{R}_0^+,\ h\ is\ a\ function\}$, for a nonempty set X. Define a binary operation $\to$ on ${\mathbb{R}_0}^X$ by:

$$(g \to h)(x) = 1 + h(x) - max\{g(x), h(x)\}, \text{ for every } x \in X.$$

one can see that $({\mathbb{R}_0}^X, \to, 1)$ is a an L-algebra, where $1 : X \to \mathbb{R}_0^+$ is the constant function. Further more define $\leq_0$ on ${\mathbb{R}_0}^X$ by:

$$f \leq_0 g \text{ if and only if } f(x) \leq g(x), \text{ for each } x \in X.$$

It is concluded that $\leq_0$ is a partially order on ${\mathbb{R}_0}^X$.

Lemma 4.27. *Let $f, g \in {\mathbb{R}_0}^X$. Then the followings are satisfied:*
i) $f \leq_0 g$ if and only if $f \to g = 1$,
ii) $f \leq_0 g$ implies that $(g \to f) \to f = g$.

Proposition 4.28. *Let f, g and $h \in \mathbb{R}_0^X$, which f, g and h are comparable. Then $(g \to h) \to ((f \to g) \to (f \to h)) = 1$.*

Corollary 4.29. *Let f, g, h be comparable elements of $\mathbb{R}_0^X$. Then the followings hold:*
i) if $f \leq_0 g$, then $g \to h \leq_0 f \to h$,
ii) if $f \leq_0 g$, then $h \to f \leq_0 h \to g$,
iii) if $f \leq_0 g$ and $g \leq_0 h$, then $f \leq_0 h$.

Proof. i) Suppose that $f \leq_0 g$, then $f \to g = 1$. But $(g \to h) \to ((f \to g) \to (f \to h)) = 1$. Hence $(g \to h) \to [1 \to (f \to h)] = 1$. Then $(g \to h) \to (f \to h) = 1$ i.e $g \to h \leq_0 f \to h$,
ii) similar to (i),
iii) $(g \to h) \to ((f \to g) \to (f \to h)) = 1$ implies that $f \to h = 1$, Then $f \leq_0 h$. □

Definition 4.30. Assume that $f, g \in \mathbb{R}_0^X$ are comparable, then define:

$$f \vee g(x) = max\{f(x), g(x)\}, \text{ for every } x \in X.$$

Lemma 4.31. *$f \vee g = (g \to f) \to f$, for comparable $f, g \in \mathbb{R}_0^X$.*

Corollary 4.32. *$(g \to f) \to f = (f \to g) \to g$, for comparable $f, g \in \mathbb{R}_0^X$.*

5 Conclusion

In this paper, we have successfully expanded the structural landscape of L-algebras by establishing and investigating several novel classes of subalgebras that bridge abstract algebra with uncertainty modeling. Specifically, we introduced the concept of fuzzy impotent subalgebras and generalized this framework by incorporating a t-norm T on L. Extending these algebraic structures into parametric uncertainty frameworks, we introduced and characterized soft impotent subalgebras through a detailed exposition of structural propositions and illustrative examples.
Furthermore, motivated by the profound interdisciplinary evolution of the Shadowing Lemma which originated in the chaotic physical systems of Lorenz, Anosov, and Bowen, and has recently escaped classical calculus to find a home in logical algebra, we integrated probability theory with fuzzy algebraic geometry. By utilizing the theory of random sets and their falling shadows, we defined falling fuzzy subalgebras and falling impotent subalgebras within L-algebras.
Our systematic investigation successfully mapped the intricate relationships and structural interactions between these probabilistic falling shadows and traditional fuzzy algebraic substructures, providing a robust mathematical framework for analyzing how uncertainty and errors propagate through non-classical logical systems. Given that L-algebras generalize Boolean algebras and crucially underpin quantum Yang–Baxter equations, pseudo MV-algebras, and Heyting algebras, the structural frameworks established herein open several promising trajectories for future research:
Extension to Broader Logical Algebras:
A natural next step is to extend the notions of fuzzy, soft, and falling impotent structures to related logical frameworks, such as BL-algebras, residuated lattices, and lattice-ordered effect algebras.
Shadowing Property and Approximations:
Utilizing the historical trajectory of the Shadowing Lemma, future work will explore how algebraic filters and orders can quantify true shadowing properties where approximate algebraic steps (errors) mimic true logical valuations in relational databases and soft computing.
Applications in Ergodic Theory and Decision Making:
Given the utility of falling shadows in calculating the uncertainty of

chaotic events, we intend to apply our structural characterizations to practical problems in ergodic theory, information theory, and parametric decision-making under high volatility.

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