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Original Research Paper

Numerical Solution for 2D-VOFOCPs by Ritz Method and Gegenbauer Operational Matrix

F. Soufivand
Islamic Azad University

Abstract. This research introduces a Ritz method to address two-dimensional variable-order fractional optimal control problems (2D-VO FOCPs) with nonlinear dynamical system. The system incorporates variable-order Caputo-type fractional derivatives, which are widely recognized in modeling memory-dependent and nonlocal behaviors. To construct the solution, shifted Gegenbauer polynomials are employed as orthogonal basis functions to approximate the state and control variables. These approximations are substituted into the performance index and the governing equations, resulting in a system of algebraic equations. Solving this system yields the numerical solution to the (2D-VOFOCPs). We conduct a rigorous convergence analysis is conducted to verify the stability and reliability of the proposed method. Furthermore, two numerical examples are provided to demonstrate the accuracy and computational efficiency of the technique in comparison with existing integer-order approaches.

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1 Introduction

Fractional calculus is recognized as a powerful mathematical framework for modeling complex dynamical systems with memory and hereditary properties. Compared to integer-order models, fractional-order formulations offer more accurate representations of phenomena such as heat conduction in materials and fluid flow in porous media ([30], [12], [27]). Optimal control theory plays a vital role across various scientific and engineering disciplines, including physics, quantum dynamics and porous media flow ([39],[6].[23]).

Optimal control theory has been extensively studied and applied across various scientific and engineering disciplines, including physics and quantum systems ([28], [9], [5]). Within the framework of fractional calculus, fractional optimal control problems (FOCPs) constitute a prominent class of problems in which both the dynamic constraints and the cost functionals may involve derivatives and integrals of integer and fractional orders ([1], [13]).

Fractional optimal control problems (FOCPs) can be formulated using various definitions of fractional derivatives, among which the Caputo and Riemann–Liouville types are the most widely used. In this paper, the Caputo definition is adopted due to its suitability for initial value problems and physical interpretations. Extensive research has been conducted in this area([37], [34], [8], [17]).

In another study [18], a general method for solving time-varying linear optimal control problems based on Chebyshev wavelet is proposed. In [33], an operational matrix method based on Gegenbauer polynomials was developed to solve a specific class of fractional optimal control problems. Two-dimensional fractional optimal control problems (2D-FOCPs) arise in various applications, such as porous media flow, steam heating, and air drying processes [21]. Numerical methods have been extensively employed to solve these problems, and recent studies have introduced novel techniques for addressing 2D-FOCPs.

In [24], the use of eigenfunctions was pioneered for the fractional optimal control of two-dimensional distributed systems. Subsequent contributions by [35] and [4] extended this approach by utilizing continuous-time two-dimensional and multi-dimensional systems, along with operational matrices, to transform 2D-FOCPs into systems of algebraic equations. In [20], a numerical technique based on the Legendre spectral method was introduced to solve a class of two-dimensional variational problems. Furthermore, [22] proposed a numerical method that integrates the Ritz approach with fractional operational matrices to address two-dimensional fractional optimal control problems.

Variable fractional operators offer a powerful tool for modeling complex phenomena across various scientific and engineering disciplines, including mechanical [29], telegraph [14] and diffusion-wave [7].
Two-dimensional variable-order fractional optimal control problems (2D-VOFOCPs) represent a novel frontier in optimal control theory. To date, this field remains relatively unexplored. While some researchers have delved into variable-order fractional optimal control problems, their focus has primarily been on one-dimensional systems. For instance, [38] utilized non-standard finite difference scheme for solving 2D-VOFOCPs. Also [15] a new numerical method utilizing Chebyshev cardinal functions is presented for the solution of variable-order fractional optimal control problems. The practical importance of two-dimensional variable-order fractional optimal control problems 2D-VOFOCPs has led to increased research efforts. [16] solved the 2D-VOFOCPs by representing the state and control variables using Legendre cardinal functions in matrix form. In [32], a numerical method based on the Gegenbauer operational matrix is proposed for solving a class of two-dimensional variable-order fractional optimal control problems. This paper investigates 2D-VOFOCPs characterized by nonlinear fractional dynamical systems and subject to a quadratic performance index We employ Gegenbauer polynomials and their operational matrix to solve these problems.

References [22] and [15] provide rigorous investigations into the existence and uniqueness of solutions for problems structurally analogous to the one considered in this study.

The focus of this work is on the following 2D-VOFOCPs is considered:

$$Minimize\ J(u) = \frac{1}{2}\int_0^1 \int_0^L r^t(a_1 z^2(r,s) + a_2 u^2(r,s))drds, \qquad (1)$$

subject to the dynamic constraint defined by a nonlinear variable-order fractional derivative

$$\frac{\partial^{\eta(r,s)} z(r,s)}{\partial s^{\eta(r,s)}} = \alpha(\frac{\partial^2 z(r,s)}{\partial r^2} + \frac{t}{r}\frac{\partial z(r,s)}{\partial r}) + u(r,s), \tag{2}$$

with conditions

$$z(L,s) = 0, \ s > 0 \ , \qquad z(r,0) = z_0(r), \ 0 < r < L. \tag{3}$$

Let, r and s denote the spatial and temporal variables, respectively. The functions a_1 and a_2 are arbitrary, while $0 < \eta(r,s) \leq 1$ is a positive function. The control and state functions are represented by $u(r,s)$ and $z(r,s)$, respectively, with L being a positive constant. Both $z(r,s)$ and $u(r,s)$ are assumed to be smooth and α is a positive number. For simplicity, we consider values of s less than or equal to 1, although any positive value is permissible. In numerical examples, we typically use $t = 1$ or $t = 2$. The term $\frac{\partial^{\eta(r,s)} u(r,s)}{\partial s^{\eta(r,s)}}$ the variable-order fractional derivative is described by the Caputo definition, with being a positive constant. To approximate the control and state functions, we employ shifted Gegenbauer polynomials with unknown coefficients. This method employs the Ritz method and Gegenbauer polynomials to approximate the control and state functions. To enhance computational efficiency, a novel fractional operational matrix is introduced. Finally, the solution of the nonlinear algebraic equation system is obtained, which is solved numerically using Newton's iterative method. For the existence and uniqueness of the solutions, see references [10] and [2]. The remaining part of this paper is organized as follows: section 2 reviews fundamental concepts of variable-order fractional calculus and the properties of Gegenbauer polynomials. In section 3 presents a new function approximation for fractional optimal control problems. Section 4 introduces the shifted Gegenbauer operational matrix of Caputo derivatives. In section 5 details the numerical implementation of the proposed method. Section 6 analyzes the convergence of the method. At last, our presents numerical results for two examples, comparing our method with other techniques in section 7.

2 Preliminaries and Notations

This section revisits the concept of the partial fractional Caputo derivative and delves into the fundamental properties of Gegenbauer polynomials, which serve as the foundational basis functions for the Ritz method.

2.1 Two-dimensional variable-order fractional derivative

Definition 2.1. Let $n-1<\eta(r,s)\leq n$, where $n\in N$. The Caputo fractional derivative of order $\beta(r,s)$ of the function $u(r,s)$ with respect to the variable s is defined as follows [16]:

$$\frac{\partial^{\eta(r,s)}u(r,s)}{\partial s^{\eta(r,s)}}=\begin{cases}\frac{1}{\Gamma(n-\eta(r,s))}\int_0^s(s-\beta)^{n-\eta(r,s)-1}\frac{\partial^n u(r,\beta)}{\partial\beta^n}d\beta, \\ \qquad\qquad n-1<\eta(r,s)\leq n,\\ \frac{\partial^n u(r,s)}{\partial s^n},\quad \eta(r,s)=n.\end{cases}\tag{4}$$

The following properties of the Caputo-type variable-order fractional derivative are presented in [16]:

$$\frac{\partial^{\eta(r,s)}s^k}{\partial s^{\eta(r,s)}}=\begin{cases}\frac{\Gamma(k+1)}{\Gamma(k-\eta(r,s)+1)}s^{k-\eta(r,s)}, & k\in N, k\geq n,\\ 0, & k\in N, k<n.\end{cases}\tag{5}$$

$$\frac{\partial^{\eta(r,s)}(w_1u_1+w_2u_2)(r,s)}{\partial s^{\eta(r,s)}}=w_1\frac{\partial^{\eta(r,s)}u_1(r,s)}{\partial s^{\eta(r,s)}}+w_2\frac{\partial^{\eta(r,s)}u_2(r,s)}{\partial s^{\eta(r,s)}},$$

where w_1 and w_2 are constants.

2.2 Shifted gegenbauer polynomials and properties

In this section, we begin by introducing the Gegenbauer polynomials. Subsequently, we derive the shifted Gegenbauer polynomials through an appropriate variable transformation. The Gegenbauer polynomials, denoted by $C_n^\eta(t)$ a family of orthogonal polynomials on the interval [-1, 1] with degree $n\in Z^+$ and parameter $\eta>-\frac{1}{2}$. These polynomials are defined by [36] and [11]

$$C_n^\eta(s)=\sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^k\Gamma(n-k+\eta)}{k!\Gamma(\eta)(n-2k)!}(2s)^{n-2k}.$$

Another way to generate the Gegenbauer polynomials is to use the following recurrence formula

$$(n+2\eta)C_{n+1}^{(\eta)}(s) = 2(n+\eta)sC_n^{(\eta)}(s) - nC_{n-1}^{(\eta)}(s),$$

starting with

$$C_0^{(\eta)}(s) = 1, \quad C_1^{(\eta)}(s) = s.$$

The Gegenbauer polynomials are orthogonal with respect to the L^2-space on the interval $[-1,1]$ and their orthogonality relation is given by [31].

We define the shifted Gegenbauer polynomials on the interval $[0,1]$ by applying the linear transformation $2s-1$, $s \in [0,1]$ to the standard Gegenbauer polynomials [8]. Let $C_n^{(\eta)}(2s-1)$ denote the shifted Gegenbauer polynomial of degree n, which is defined as: $C_{S,n}^{(\eta)}(s)$, so

$$C_{S,0}^{(\eta)}(s) = 1, \quad C_{S,1}^{(\eta)}(s) = 2s-1.$$

The explicit form of the shifted Gegenbauer polynomial of degree n can be produced using the following methods, as referenced in [8] and [36]:

$$C_{S,n}^{(\eta)}(s) = \frac{\Gamma(\eta+\frac{1}{2})}{\Gamma(2\eta)} \sum_{k=0}^{n} \frac{(-1)^{(n-k)}\Gamma(n+k+2\eta)}{(n-k)!k!\Gamma(k+\eta+\frac{1}{2})} s^k.$$

The shifted Gegenbauer polynomials are defined on the interval $[0,1]$. Their orthogonality condition with respect to the L^2-space on the interval $[0,1]$ can be obtained from [8].

$$\int_0^1 C_{S,m}^{(\eta)}(s) C_{S,n}^{(\eta)}(s) \vartheta^{(\eta)}(s) dt = \lambda_n^{(\eta)} \delta_{m,n},$$

where

$$\vartheta^{(\eta)}(s) = (s-s^2)^{(\eta-\frac{1}{2})},$$

$$\lambda_n^{(\eta)} = (\frac{1}{2})^{2\eta} \eta_n^{(\eta)}.$$

It should be noted that these polynomials can be adapted for use on the interval $[0,L]$ by replacing the variable s with $\frac{2s}{L}-1$, $0 \le s \le L$, as follows:

$$C_{S,j}^{(\eta)}(s) = C_j^{(\eta)}(\frac{2s}{L}-1), \quad C_{S,0}^{(\eta)}(s) = 1, \quad C_{S,1}^{(\eta)}(s) = \frac{2s}{L}-1.$$

The explicit form of the shifted Gegenbauer polynomial can indeed be found in reference [3]. This form is crucial for various applications, such as:

$$C_{S,j}^{(\eta)}(s) = \sum_{k=0}^{j} (-1)^{(j-k)} \frac{\Gamma(\eta + \frac{1}{2})\Gamma(j + k + 2\eta)}{\Gamma(2\eta)\Gamma(j + \frac{1}{2} + \eta)(j - k)!k!L^k} s^k,$$

$$C_{S,j}^{(\eta)}(0) = (-1)^{(j)} \frac{\Gamma(j + 2\eta)}{\Gamma(2\eta)j!}.$$

To numerically evaluate the double integral of a sufficiently smooth function, we employ the 2D Legendre-Gauss quadrature rule [22] as follows:

$$\int_a^b \int_{a'}^{b'} g(r,s)drds \simeq \frac{(b'-a')(b-a)}{4} Q^T G Q' \tag{6}$$

where G is an $l \times l^{'}$ matrix and entries are defined by

$$G_{ij} = g(a' + (m_i + 1)\frac{(b'-a')}{2}, a + (n_j + 1)\frac{(b-a)}{2}),$$

where $1 \leq i \leq l, 1 \leq j \leq l'$ denote the nodes of LG on the intervals $[a', b']$ and $[a, b]$ respectively. The column vectors Q and Q' contain the corresponding Christoffel numbers of size l and l', respectively.

3 Function Approximation Based on Gegenbauer Polynomials

Let $H = C^2([0,1] \times [0,1])$, we define $K = \{C_{S,i}^{(\beta)}(r)C_{S,j}^{(\beta)}(s)\}_{i,j=0}^{m,n} \subseteq C^2([0,1] \times [0,1]), m, n \in N \cup \{0\}$ be the set of the Gegenbauer polynomial products.
With the help of the following lemma, a two-dimensional function is approximated in terms of Gegenbauer polynomials.

Lemma 3.1. *Let* $\Gamma_{mn} = span < C_{S,0}^{(\beta)}(r)C_{S,0}^{(\beta)}(s), ..., C_{S,m}^{(\beta)}(r)C_{S,n}^{(\beta)}(s) >$ *and* $\Theta(r,s) \in C^2([0,1]\times[0,1])$*, if* $\Theta_{mn}(r,s)$ *is the best approximation of* $\Theta(r,s)$ *out of* Γ_{mn} *then*

$$\Phi_m(r) = \begin{bmatrix} \bar{C}_0^{(\beta)}(r) \\ \vdots \\ \vdots \\ \bar{C}_m^{(\beta)}(r) \end{bmatrix}, \Phi_n(s) = \begin{bmatrix} \bar{C}_0^{(\beta)}(s) \\ \vdots \\ \vdots \\ \bar{C}_n^{(\beta)}(s) \end{bmatrix}$$

where $\bar{C}_i^{(\beta)}(r) = C_{S,i}^{(\beta)}(r)$ *and* $\bar{C}_j^{(\beta)}(s) = C_{S,j}^{(\beta)}(s)$, $i = 0, ..., m, j = 0, ..., n$

$$\Theta(r,s) \simeq \Theta_{mn}(r,s) = \sum_{i=1}^{m}\sum_{j=1}^{n} e_{ij}\bar{C}_i^{(\beta)}(r)\bar{C}_j^{(\beta)}(s) = \Phi_m^T(r)E\Phi_n(s).$$

Proof. [32] □

4 Operational Matrices

In this section, we will present a novel approach for solving the 2D-VOFOCP defined by Eqs.(1)-(3) based on shifted Gegenbauer polynomials. To effectively handle the variable-order fractional derivative $\frac{\partial^{\eta(r,s)}z(r,s)}{\partial s^{\eta(r,s)}}$ we will construct a dedicated operational matrix. As will be demonstrated later, this operational matrix plays a crucial role in efficiently solving the 2D-VOFOCP. According to the previous discussion and the Ritz method, the state function can be represented in terms of shifted Gegenbauer polynomials in the following form:

$$z(r,s) \simeq v(r,s)\Phi_m(r)^T E\Phi_n(s) + w(r,s). \tag{7}$$

Where $v(r,s)$ and $w(x,t)$ are trial functions and shall be chosen in a way that the estimated function satisfies boundary conditions (3). Hence, the function $v(r,s)$ is selected to satisfy homogeneous initial-boundary condition ($v(r,0) = v(0,s) = 0$) and $w(r,s)$ to satisfy inhomogeneous initial-boundary conditions. Therefore, these auxiliary functions may be taken as $v(r,s) = rs, \quad w(r,s) = \frac{rz(r,0)+sz(L,s)}{r+s}$. If the point $r_0 := r_a$ is selected, we put $v(r,s) = (r - r_a)s$.

$E = [e_{ij}]$ for $i = 0, 1, ..., m$ and $j = 0, 1, ..., n$ is an unknown matrix. Also, the vectors $\Phi_n(s)$ and $\Phi_m(r)$ are represented as:

$$\Phi_m(r) = [\bar{C}_0^{(\eta)}(r), \bar{C}_1^{(\eta)}(r), ..., \bar{C}_m^{(\eta)}(r)]^T = AM_m(r), \tag{8}$$

$$\Phi_n(s) = [\bar{C}_0^{(\eta)}(s), \bar{C}_1^{(\eta)}(s), ..., \bar{C}_n^{(\eta)}(s)]^T = BM_n(s), \tag{9}$$

where

$$M_m(r) = [1, r, ..., r^m]^T, M_n(s) = [1, s, ..., s^n]^T = [\nu_0(s), \nu_1(s), ..., \nu_n(s)]^T,$$

$$\nu_j(s) = s^j, \quad j = 0, 1, 2, ..., n$$

and

$$A = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0m} \\ a_{10} & a_{11} & \cdots & a_{1m} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m0} & a_{m1} & \cdots & a_{mm} \end{bmatrix}, \ B = \begin{bmatrix} a_{00} & a_{01} & \dots & a_{0n} \\ a_{10} & a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n0} & a_{n1} & \dots & a_{nn} \end{bmatrix},$$

$$a_{ij} = \begin{cases} \frac{(-1)^{(i-j)}\Gamma(\eta+\frac{1}{2})\Gamma(i+j+2\eta)}{(i-j)!j!\Gamma(2\eta)\Gamma(j+\eta+\frac{1}{2})}, & i \geq j, \\ 0, & i < j. \end{cases}$$

Fractional derivative of $\nu_j(s)$ of variable-order $0 < \eta(r, s) \leq 1$ in Caputo fractional derivative sense is shown as below:

$$\frac{\partial^{\eta(r,s)}\nu_j(s)}{\partial s^{\eta(r,s)}} = \begin{cases} \frac{\Gamma(j+1)}{\Gamma(j+1-\eta(r,s))} s^{j-\eta(r,s)}, & j = 1, 2, ..., n, \\ 0, & j = 0. \end{cases}$$

Lemma 4.1. *Let $\Phi_n(s)$ be the vector function as defined in Eq. (9), and let $0 < \eta(r, s) \leq 1$ be a positive continuous function over $[0, L] \times [0, 1]$. The Caputo fractional derivative of order $\eta(r, s)$ of the function $\Phi_n(s)$ is expressed as:*

$$\frac{\partial^{\eta(r,s)} z(r, s)}{\partial s^{\eta(r,s)}} \simeq \Phi_m^T(r) E \frac{\partial^{\eta(r,s)} \Phi_n(s)}{\partial s^{\eta(r,s)}} = \Phi_m^T(r) EBO_s^{\eta(r,s)} M_n(s),$$

where $O_s^{\eta(r,s)}$ *is the* $(n+1)\times(n+1)$ *variable-order fractional derivatives operational matrix of order* $\eta(r,s)$ *given by*

$$O_s^{\eta(r,s)} = s^{-\eta(r,s)} \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & \frac{1}{\Gamma(2-\eta(r,s))} & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{n!}{\Gamma(n+1-\eta(r,s))} \end{bmatrix}.$$

Proof. We substitute Eq. (5) into Eq. (9)

$$\frac{\partial^{\eta(r,s)}\Phi_n(s)}{\partial s^{\eta(r,s)}} = B\frac{\partial^{\eta(r,s)}M_n(s)}{\partial s^{\eta(r,s)}}$$

$$= B[0, \tfrac{1}{\Gamma(2-\eta(r,s))}s^{1-\eta(r,s)}, \dots, \tfrac{n!}{\Gamma(n+1-\eta(r,s))}s^{n-\eta(r,s)}]^T$$

$$= Bs^{-\eta(r,s)}[0, \tfrac{1}{\Gamma(2-\eta(r,s))}s, \dots, \tfrac{n!}{\Gamma(n+1-\eta(r,s))}s^n]^T,$$

and the remainder of the proof follows directly. □

The first and second order derivatives of the state function $z(r,s)$ as defined in Eq. (7), can be calculated as follows:

$$\frac{\partial\Phi_m(r)}{\partial r} = A\frac{\partial M_m(r)}{\partial r} = AO^1M_m(r),$$

$$\begin{aligned} \frac{\partial z(r,s)}{\partial r} \simeq \frac{\partial \Phi_m^T(r)E\Phi_n(s)}{\partial r} &= \frac{\partial M_m^T(r)}{\partial r}A^TE\Phi_n(s) \\ &= M_m^T(r)(D^1)^TA^TE\Phi_n(s), \end{aligned} \tag{10}$$

similarly, in (10), one can find

$$\frac{\partial^2 z(r,s)}{\partial r^2} \simeq M_m^T(r)(O^2)^TA^TE\Phi_n(s),$$

so that the $(m+1)\times(m+1)$ matrices O^1, O^2 are referred to as the operational matrices of the first and second order derivatives, respectively, and are defined by:

$$O^1 = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{r} & 0 & \dots & 0 \\ 0 & 0 & \frac{2}{r} & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \frac{m}{r} \end{bmatrix}, \tag{11}$$

$$O^2 = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \frac{(2)(1)}{r^2} & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \frac{(m)(m-1)}{r^2} \end{bmatrix}. \tag{12}$$

5 Numerical Treatment

In this part, our proposed method for solving the 2D-VOFOCP (1)-(3) is explained.

$$Min\ J(u) = \frac{1}{2}\int_0^1\int_0^L r^t(a_1 z^2(r,s) + a_2 u^2(r,s))drds,$$

To do this, we approximate the functions $z(r,s)$ and $u(r,s)$ by Gegenbauer polynomials as:

$$z(r,s) \simeq v(r,s)\Phi_m(r)^T E\Phi_n(s) + w(r,s).$$

$$u(r,s) \simeq -\frac{\partial^{\eta(r,s)} z(r,s)}{\partial s^{\eta(r,s)}} + \alpha(\frac{\partial^2 z(r,s)}{\partial r^2} + \frac{t}{r}\frac{\partial z(r,s)}{\partial r}). \tag{13}$$

By obtaining $u(r,s)$ and $z(r,s)$ from Eqs. (7)-(13) By inserting it into the cost functional J, we can approximate it as:

$$J(u) = \textstyle\int_0^1\int_0^L F(r,s,e_{nm})drds, \tag{14}$$

$$F(r,s,e_{nm}) = G(r,s,v(r,s)\Phi_m(r)^T E\Phi_n(s) + w(r,s),$$

$$-\frac{\partial^{\eta(r,s)} z(r,s)}{\partial s^{\eta(r,s)}} + \alpha(z_{rr}(r,s) + \frac{t}{r} z_r(r,s)))$$

The approximate derivatives of the state function $z(r,s)$ can be expressed as:

$$z_r(r,s) \simeq M_m^T(r)(O^1)^T A^T E\Phi_n(s),$$

$$z_{rr}(r,s) \simeq M_m^T(r)(O^2)^T A^T E\Phi_n(s),$$

$$\frac{\partial^{\eta(r,s)} z(r,s)}{\partial s^{\eta(r,s)}} \simeq \Phi_m^T(r) EBO_s^{\eta(r,s)} M_n(s).$$

The double integral in equation (14) is evaluated using the Legendre-Gauss quadrature rule as described in equation (6). Furthermore, the following necessary conditions for the extremum must hold:

$$\frac{\partial J^*}{\partial e_{ij}} = 0, \qquad j = 0,1,...,m, \quad i = 0,1,...,n. \tag{15}$$

In this manner, the 2D-VOFOCP defined by Eqs. (1)-(3) is reduced to a system of nonlinear algebraic equations, namely Eq. (15). This system can then be efficiently solved using Newton's iterative method. A brief description of the proposed pseudocode algorithm is given by the following steps:
Inputs: Consider $\eta(r,s)$, and define a constant$J$, which must be a very small positive real number. As a result, the functions $z(r,s)$ and $u(r,s)$ are approximated, Additionally, define a constant ϵ_J, which must also be a very small positive real number.
Step 1: Construct the basis vectors from the shifted Gegenbauer polynomials $\Phi_n(s)$ and $\Phi_m(r)$ utilizing Eqs. (8), (9) for appropriate values of $z(r,s)$ and $u(r,s)$, define the unknown matrix E, and compute the operational matrice O^1, O^2 and $O^{\eta(r,s)}$ utilizing Eqs. (11), (12) and (10).
Step 2: Solve the system of equations in the necessary conditions (15) using the dynamic method System (2) and approximate state and control functions (7), (13), boundary and initial conditions (3).
Step 3: Obtain the optimal functions $z(r,s)$ and $u(r,s)$ from equations

(7), (13) and find the optimal function J subject to constraints (1).
Step 4: Increase the values of n and m, and repeat Steps 1 through 3.
Step 5: If $|J_{new} - J_{old}| < \epsilon_J$, the optimal solutions by this method are obtained; otherwise, go back to Step 4.

The outputs of this pseudocode algorithm are the approximate optimal solutions J, $z(r,s)$ and $u(r,s)$.

6 The Convergence Analysis

This section investigates the convergence behavior of the proposed method. We demonstrate that as the degrees of approximation parameters m and n increase in Eqs. (7)-(13), the estimated value of the cost functional J converges to its optimal value. This result is formally established in Theorem 6.3.

Let $C^2([0,1] \times [0,1])$ denote the Banach space of all twice continuously differentiable functions defined on the domain $[0,1] \times [0,1]$, equipped with the uniform norm. This space is equipped with the uniform norm defined by:

$$||\Omega(r,s)|| = ||\Omega(r,s)||_\infty + ||\tfrac{\partial \Omega(r,s)}{\partial r}||_\infty$$
$$+||\tfrac{\partial \Omega(r,s)}{\partial s}||_\infty + ||\tfrac{\partial^2 \Omega(r,s)}{\partial r^2}||_\infty.$$

The following lemma demonstrates the continuity of the cost functional J, on the defined Banach space.

Lemma 6.1. *Suppose that $J : C^2([0,1] \times [0,1]) \longrightarrow R$ is uniformly continuous .*

Proof. Assuming $0 < \eta \leqslant 1$, we apply the definition of the Caputo derivative to Eq. (4), which yields:

$$\begin{aligned} ||\tfrac{\partial^\eta h}{\partial s^\eta}|| &\leq \tfrac{1}{\Gamma(1-\eta)} \max_{r\in[0,a]} |\int_0^s \tfrac{1}{(1-\beta)^\eta} ||\tfrac{\partial h}{\partial \beta}||_\infty d\beta| \\ &\leq \tfrac{a^{(1-\eta)}}{\Gamma(2-\eta)} ||\tfrac{\partial h}{\partial s}||_\infty. \end{aligned} \tag{16}$$

Suppose any $\epsilon > 0$ is given and let $\delta > 0$ and $h \in C^2([0,1] \times [0,1])$. Now suppose that $f \in C^2([0,1] \times [0,1])$, such that

$$||h - f|| = ||h - f||_\infty + ||\tfrac{\partial}{\partial r}(h - f)||_\infty$$

$$+||\tfrac{\partial}{\partial s}(h - f)||_\infty + ||\tfrac{\partial}{\partial r^2}(h - f)||_\infty < \delta.$$

By using Eq. (16), we have

$$||\frac{\partial^\eta h}{\partial s^\eta} - \frac{\partial^\eta f}{\partial s^\eta}|| = ||\frac{\partial^\eta (h - f)}{\partial s^\eta}||_\infty \leq \frac{a^{(1-\eta)}}{\Gamma(2-\eta)}||\frac{\partial h}{\partial s} - \frac{\partial f}{\partial s}||_\infty.$$

Since the function h and its first and second derivatives are continuous, it follows that $\Omega(r, s, h, \frac{\partial h}{\partial r}, \frac{\partial^\eta h}{\partial s^\eta}, \frac{\partial^2 h}{\partial r^2})$ is a continuous function [26]. So there exists $\delta > 0$ with $||f - h_1|| < \delta$ such that:

$$||\Omega_1(r, s, h, \frac{\partial h}{\partial r}, \frac{\partial^\eta h}{\partial s^\eta}, \frac{\partial^2 h}{\partial r^2}) - \Omega_0(r, s, h, \frac{\partial h}{\partial r}, \frac{\partial^\eta h}{\partial s^\eta}\frac{\partial^2 h}{\partial r^2})|| < \epsilon,$$

and we have

$$|J(h) - J(f)| < \epsilon.$$

□

The function $f(x, y)$ can be approximated by a series of two-dimensional Gegenbauer polynomials by the following lemma:

Lemma 6.2. *Let $p_1(r, s)$ denote the set of all continuous functions on the unit square and $\epsilon > 0$ is given, then there exists a set of polynomials $\{p_{m,n}(r, s)\}_{m,n \in N}$ satisfies the following conditions:*

$$\forall m, n \geq N, \qquad |p_1(r, s) - p_{mn}(r, s)| < \epsilon.$$

Proof. [26]. □

The following theorem establishes the convergence of the approximating method for Gegenbauer polynomials.

Theorem 6.3. *Let θ_{mn} denote the minimum of the cost functional J on the subspace $C^2([0,1] \times [0,1]) \cap \Gamma_{mn}$ and let θ denote the global minimum of J on the space $C^2([0,1] \times [0,1])$. Then, the following relationship holds:*

$$\theta_{mn} \longrightarrow \theta.$$

Proof.

Let $\epsilon > 0$ be given. Since J is a continuous functional, and there exists $g^* \in C^2([0,1]\times[0,1])$ such that $J[g^*] < \theta+\epsilon$, we can apply Lemma 6.1. This implies that for any $f \in (C^2([0,1]\times[0,1]\cap\theta_{mn})$ provided that $||f-g^*|| < \delta$ with $||f-g^*|| < \delta$, we have $J[g^*] < \theta+\epsilon$. By Lemma 6.2, for sufficiently large m and n, there exists p_{mn} such that $||p_{mn}-g^*|| < \delta$. Denoting $J[p_{mn}]$ by θ_{mn}, we can then show that

$$\theta \leq \theta_{mn} = |J[p_{mn}] - J[g^*] + J[g^*]| \leq |J[p_{mn}]| - |J[g^*]| + |J[g^*]| \\ < \theta + 2\epsilon,$$

since $\epsilon > 0$ is an arbitrary, we have

$$\lim_{(m,n)\to(\infty,\infty)} \theta_{mn} = \lim_{(m,n)\to(\infty,\infty)} J[p_{mn}] = \theta.$$

□

Figure 1: Approximate solution $z(r,s)$ is presented for $\eta(r,s) = 1$, $m = 3$, $n = 10$ in Example 1

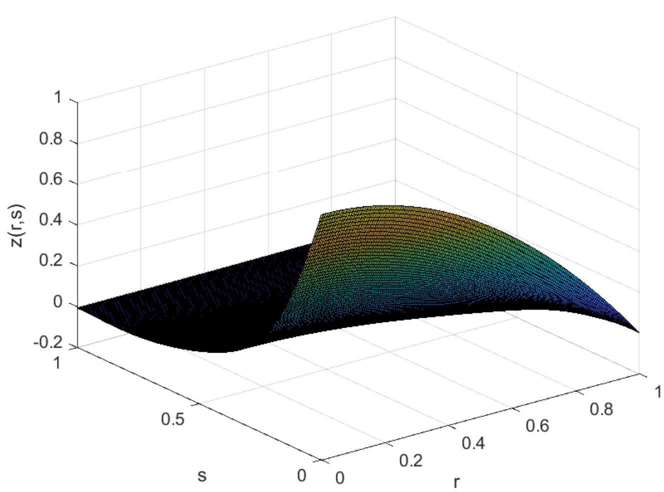


Table 1: : J values are approximated for different selections of m, n and $\eta(r,s)$ in Example 1.

$m\times n$	2×2	2×4	2×6	2×7	3×7	3×9	3×10
$\eta(r,s)=1-0.7cos(rs)$	0.072	0.035	0.02023	0.01674	0.01665	0.000394	0.000310
$\beta(r,s)=1-0.02e^{(-(rs))}$	0.05	0.020	0.0104	0.0087	0.00812	0.00132	0.00118

Table 2: : J values are approximated for the method in [19], [25] and proposed scheme for $\eta(r,s)=1$ in Example 1.

$m\times n$	1×4	2×4	2×6	2×7	3×7	3×9	3×10
Method in [19]	0.081044	0.028790	0.018283	0.016484	0.013027	0.010405	0.007569
Method in [25]	0.015864	0.014089	0.013968	0.014015	0.0047307	0.0047063	0.0046844

7 Illustrative Examples

This section presents two illustrative examples demonstrating the accuracy of the results is compared with those produced by established methods. All numerical computations were performed using MATLAB 2018.

Example 7.1. Consider the following problem:

$$Min \quad J(u)=\frac{1}{2}\int_0^1\int_0^1 r(z^2(r,s)+u^2(r,s))drds,$$

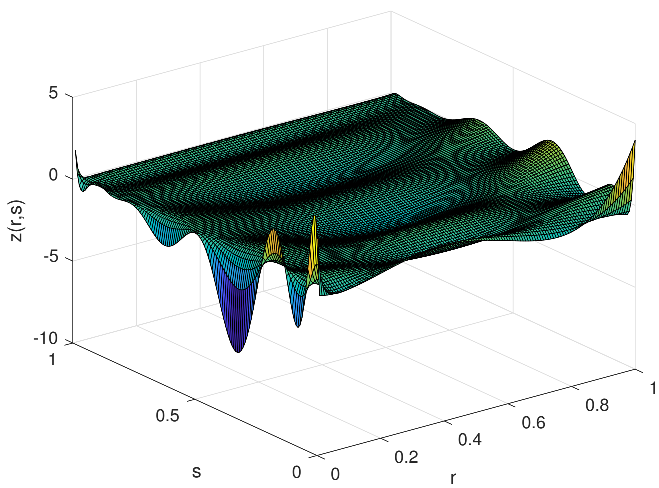
The constraint involves a nonlinear dynamical system with a variable-order fractional derivative

$$\frac{\partial^{\eta(r,s)}z(r,s)}{\partial s^{\eta(r,s)}}=(\frac{\partial^2 z(r,s)}{\partial r^2}+\frac{1}{r}\frac{\partial z(r,s)}{\partial r})+u(r,s),$$

subject to the conditions

$$z(1,s)=0,\ \ s>0,\qquad z_0(r)=1-r^2,\quad 0<r<1.$$

Figure 2: Approximate solution $u(r,s)$ is presented for $\eta(r,s)=1$, $m=3$, $n=10$ in Example 1



According to the proposed method, an approximation of the state function is obtained as

$$z(r,s) \simeq (r-1)s\Phi_m(r)^T E\Phi_n(s) + 1 - r^2.$$

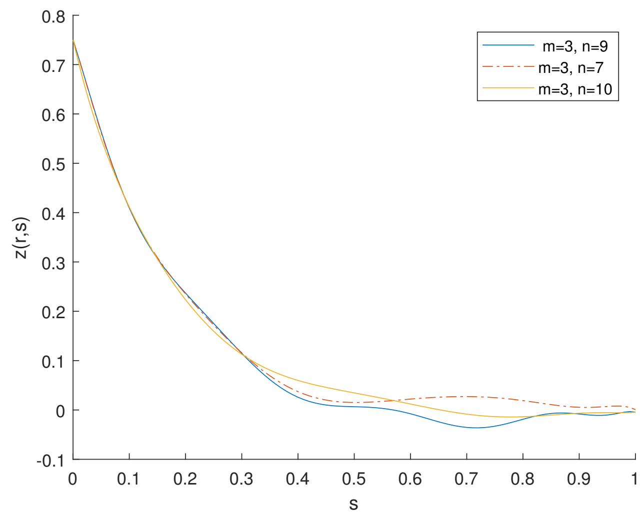
To address the numerical solution, the Ritz spectral method is employed. Following approximation of the state function, the control input is determined from the system dynamics. Subsequently, by substituting the approximated functions into the cost functional, an unconstrained minimization problem arises. The objective is to determine the unknown coefficient matrix e_{mn}.

$$Min \quad J(u) = \frac{1}{2}\int_0^1\int_0^1 r(((r-1)s\Phi_m(r)^T E\Phi_n(s) + 1 - r^2)^2(r,s)$$

$$+(\frac{\partial^{\eta(r,s)} z(r,s)}{\partial s^{\eta(r,s)}} - (\frac{\partial^2 z(r,s)}{\partial r^2} + \frac{1}{r}\frac{\partial z(r,s)}{\partial r}))^2(r,s))drds.$$

After employing the two-dimensional Legendre–Gauss quadrature rule

Figure 3: Approximate solution $z(r, s)$ is presented for $\beta(r, s) = 1$, $r = 0.5$, in Example 1.



to evaluate the double integral, Newton's iterative method is then applied to solve the resulting system. The numerical example is solved for the specific parameters m=1, n=4 and $\eta(r, s) = 1$.

$$\mathrm{E}=\begin{bmatrix} 4.19092119088 & -4.5723815941 & 2.97851005569 & -1.5163527164 & 0.430054751883 \\ 1.54371694607 & -1.8887960334 & 1.4960895875 & -1.0149774235 & 0.38465020123 \end{bmatrix}$$

$$\mathrm{D}_s^{\eta(r,s)} = s^{-\eta(r,s)} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\Gamma(2-\eta(r,s))} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{\Gamma(3-\eta(r,s))} & 0 & 0 \\ 0 & 0 & 0 & \frac{6}{\Gamma(4-\eta(r,s))} & 0 \\ 0 & 0 & 0 & 0 & \frac{24}{\Gamma(5-\eta(r,s))} \end{bmatrix}.$$

The statement indicates that the algorithm has been executed, and the approximate value of the cost functional J is 0.03861. Table 1 presents the achieved values of the cost functional J for various choices of the variable-order parameter $\eta(r, s)$ and different values of$n$ and m.

Figure 4: Approximate solution $u(r, s)$ is presented for $\beta(r, s) = 1$, $r = 0.5$, in Example 1.

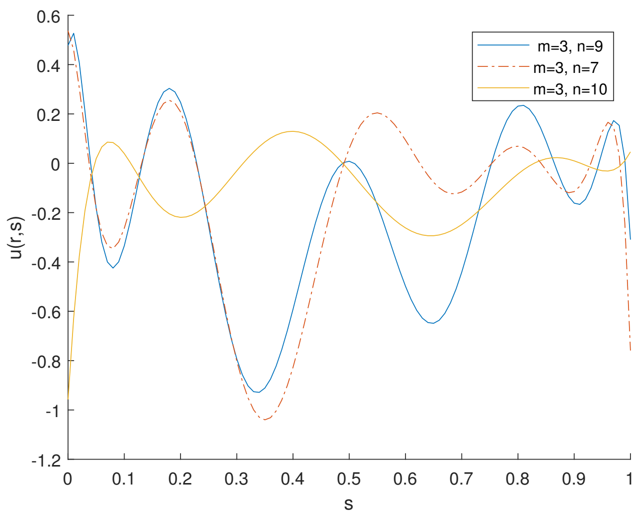
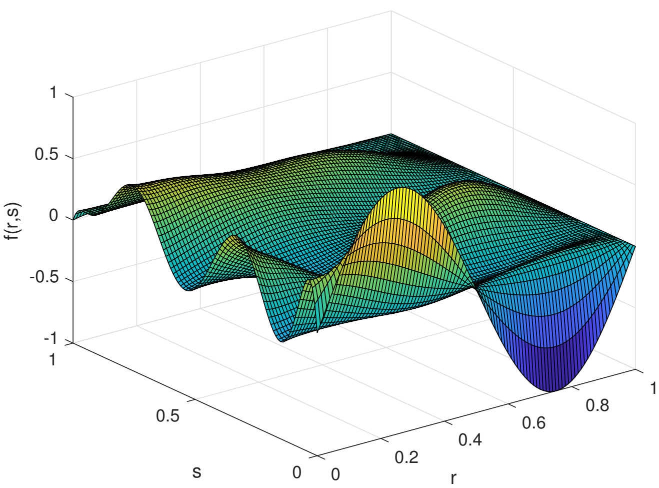


Table 2 compares the values of J for various combinations of n and m with $\eta(r, s) = 1$ obtained via the approaches in [20] and [25], as well as the proposed method. A comparison of the results indicates that the proposed numerical scheme achieves higher accuracy than the methods presented in [20] and [25]. Figures 1 and 2 show the surface plots of the state function and the control function for $\eta(r, s) = 1$, $m = 3$ and $n = 10$. For $\eta(r, s) = 1$ and $r = 0.5$, figure 3 shows the plot of the approximate state function for different values of n and m. Figure 4 shows the plot of the approximate control function for different values of n and m, $\eta(r, s) = 1$ and $r = 0.5$. Figures 3 and 4 illustrate that increasing the number of shifted Gegenbauer basis functions leads to a convergence of the approximate state and control functions towards the zero solution. The cost functional J values obtained using the proposed method, for both constant ($\eta(r, s) = 1$) and variable $\eta(r, s)$ and various combinations of n and m, consistently yield consistently lower values compared to those obtained using the methods presented in [20] and [25]. These results indicate that the proposed method provides more accurate and

Figure 5: Approximate solution $z(r,s)$ is presented for $\eta(r,s) = 1$, $m = 7$, $n = 8$, in Example 2



reliable results and more accurate solutions.

Example 7.2. Consider the following 2D-VOFOCP:

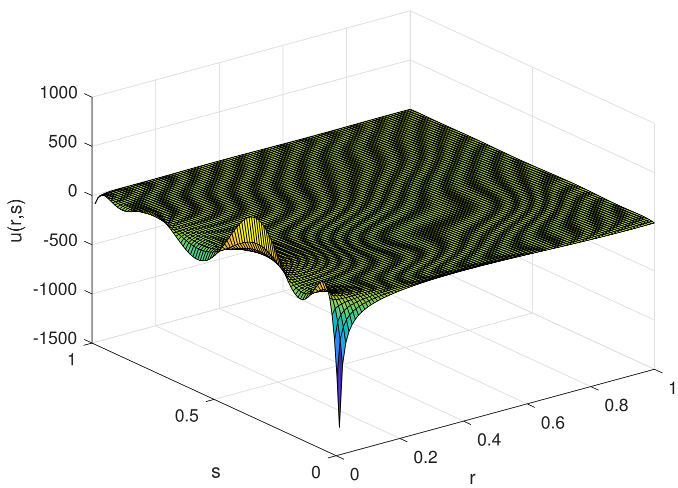
$$Min \quad J(u) = \frac{1}{2}\int_0^1 \int_0^1 r^2(z^2(r,s) + u^2(r,s))drds,$$

subject to the dynamic constraint with nonlinear variable-order frac-

Table 3: : J values for different selections of m, n and $\eta(r,s)$ in Example 2.

$m{\times}n$	4×5	5×5	5×6	6×6	6×7	7×7	7×8
$\eta(r,s) = 1 - 0.3(r^2+s^2)$	1.179	1.1161	0.8146	0.7305	0.5633	0.4031	0.2601
$\eta(r,s) = 1 - 0.2e^{((rs))}$	1.1326	1.060	0.767	0.7009	0.5412	0.5303	0.2284

Figure 6: Approximate solution $u(r,s)$ is presented for $\eta(r,s) = 1$, $m = 7$, $n = 8$, in Example 2



tional derivative

$$\frac{\partial^{\eta(r,s)} z(r,s)}{\partial s^{\eta(r,s)}} = (\frac{\partial^2 z(r,s)}{\partial r^2} + \frac{2}{r}\frac{\partial z(r,s)}{\partial r}) + u(r,s),$$

and the conditions:

$$z(1,s) = 0,\ \ s > 0, \qquad z(r,0) = sin(2\pi r), \quad 0 < r < 1.$$

The proposed method employs an approximation for the state function, given by:

$$z(r,s) \simeq (r-1)s\Phi_m(r)^T E\Phi_n(s) + sin(2\pi r).$$

The problem is numerically solved subject to the specified boundary conditions.

$$in \quad J(u) = \tfrac{1}{2}\int_0^1\int_0^1 r(((r-1)s\Phi_m(r)^T E\Phi_n(s) + 1 - r^2)^2(r,s)$$

$$+(\tfrac{\partial^{\eta(r,s)} z(r,s)}{\partial s^{\eta(r,s)}} - (\tfrac{\partial^2 z(r,s)}{\partial r^2} + \tfrac{2}{r}\tfrac{\partial z(r,s)}{\partial r}))^2(r,s))drds.$$

Table 4: : A comparison of J values is conducted with $\eta(r,s) = 1$, using both the method presented in [19] and the proposed method for Example 2.

$m \times n$	4× 5	5× 5	5× 6	6×6	6×7	7× 7	7× 8
Proposed method	0.7287	0.650	0.4193	0.3586	0.2508	0.2472	0.09675
Method in [19]	2.72722	1.92027	1.27424	0.91850	0.55287	0.54935	0.36868

Figure 7: Approximate solution $z(r,s)$ is presented for $\eta(r,s) = 1$, $m = 7$, $n = 8$, in Example 2

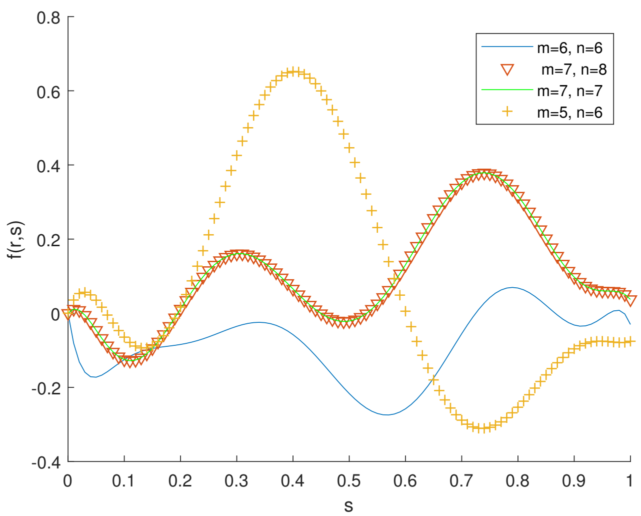
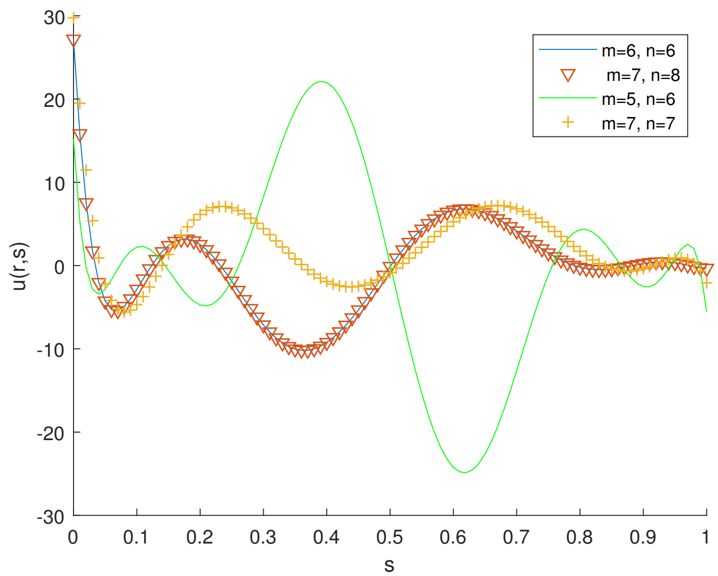


Table 3 presents the computed values of the cost functional J, obtained using the proposed numerical method for various combinations of the parameters n, m and the variable-order function $\eta(r,s)$. The state and control functions for $\eta(r,s) = 1$, $m = 7$ and $n = 8$ are plotted in Figure 5 and 6. Figure 7 and 8 display the plot of the approximate, state and control functions fo $\eta(r,s) = 1$ and $r = 0.5$. Table 4 presents the approximate state and control functions for various values of n, m. As evidenced by the results, the proposed method consistently outperforms

Figure 8: Approximate solution $u(r,s)$ is presented for $\eta(r,s) = 1$, $m = 7$, $n = 8$, in Example 2



the integer-order approach introduced in [20].

8 Conclusions

The article presents a valuable contribution to the field of fractional optimal control by introducing the Ritz method based on shifted Gegenbauer polynomials for solving 2D-VOFOCPs. This method effectively transforms the complex 2D-VOFOCPs into an unconstrained optimization problem, significantly simplifying its solution. The method's effectiveness is demonstrated through numerical examples. Furthermore, the authors identify a promising avenue for future research by suggesting the inclusion of time-delay in these problems.

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Farzaneh Soufivand
Department of Mathematics, Do.C.
Islamic Azad University
Doroud, Iran
soufivand2019@gmail.com