Journal of Mathematical Extension
Vol. 20, No. 3 (2026) (5) 1-15
ISSN: 1735-8299
URL: http://doi.org/10.30495/JME.2026.3332
Original Research Paper

E-K-frames in Hilbert Spaces

Morteza Rahmani
Ilk.C., Islamic Azad University

Esmaeil Alizadeh*
Mara.C., Islamic Azad University

Abstract. In this paper, we introduce E-K-frames for a separable Hilbert space $\mathcal{H}$, where E is an infinite matrix of real or complex numbers, and K is a bounded operator on $\mathcal{H}$. Actually, E-K-frames are a kind of generalization of K frames by infinite matrices. Assuming an (invertible) infinite matrix E as a mapping on the Hilbert space $\bigoplus_{n=1}^{\infty} \mathcal{H}_n$, we aim to study the properties of E-K-frames, define the notion of E-atomic system for an operator K on $\mathcal{H}$ and address its relationship by E-K-frames.

AMS Subject Classification: 42C15; 46C05; 15A60
Keywords and Phrases: Infinite matrix, E-K-frame, E-atomic system

1 Introduction

A frame for a Hilbert space $\mathcal{H}$ is a sequence of elements in $\mathcal{H}$, in which each element of $\mathcal{H}$ can be represented as a linear combination of these elements that are not necessarily linearly independent. Indeed, a frame can be considered a basis that probably has some more elements.

Received: April 2025; Accepted: May 2026
*Corresponding Author

Gavruta proposed the K-frames in Hilbert spaces to study atomic decomposition systems, stating some properties of them [6]. Later, Hua and Hung introduced K-g-frames [10], studying some new results and characterizations of K-g-frames [5, 7, 9]. Moreover, Alizadeh and et al. proposed the concept of continuous K-g-frames [1].

In this paper, by assuming an (invertible) infinite matrix E as a mapping on the Hilbert space $\bigoplus_{n=1}^{\infty}\mathcal{H}_n$, where $\{H_n\}_{n=1}^{\infty}$ is a sequence of separable Hilbert spaces, we introduce the notion of E-K-frames for a Hilbert space $\mathcal{H}$ and investigate some properties of them. Also, we present the definition of E-atomic system for an operator K on $\mathcal{H}$, addressing their relationships by E-K-frames.

Across this paper, $\mathcal{H}$ is a separable Hilbert space, $\mathfrak{B}(\mathcal{H})$ is the set of all bounded linear operators on $\mathcal{H}$ and $\mathfrak{B}(\mathcal{H}_1,\mathcal{H}_2)$ is the set of all bounded linear operators between Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. For an operator $U\in\mathfrak{B}(\mathcal{H}_1,\mathcal{H}_2)$, its range is denoted by $\mathcal{R}(U)$ and its null space by $\mathcal{N}(K)$.

At first, we will review some required results. Also, some other results about operator theory are given [4, 8].

Lemma 1.1. ([2]) *Assume that $U\in\mathfrak{B}(\mathcal{H}_1,\mathcal{H}_2)$ and U has closed range. Then the following holds:*

1. *U^* has closed range and $(U^*)^\dagger=(U^\dagger)^*$.*
2. *The orthogonal projection of $\mathcal{H}_2$ onto $\mathcal{R}(U)$ is given by $UU^\dagger$.*
3. *The orthogonal projection of $\mathcal{H}_1$ onto $\mathcal{R}(U^\dagger)$ is given by $U^\dagger U$.*
4. *$\mathcal{N}(U^\dagger)=\mathcal{R}^\perp(U)$ and $\mathcal{R}(U^\dagger)=\mathcal{N}^\perp(U)$.*

Theorem 1.2. ([3]) *(Douglas' factorization theorem) Let $L_1,L_2\in\mathfrak{B}(\mathcal{H})$. Then the following assertions are equivalent:*

(i) $\mathcal{R}(L_1)\subseteq\mathcal{R}(L_2)$.

(ii) $L_1L_1^*\leq\lambda L_2L_2^*$ *for some* $\lambda>0$.

(iii) There exists an operator $U\in\mathfrak{B}(\mathcal{H})$ such that $L_1=L_2U$.

Moreover, if (i), (ii) and (iii) are valid, then there exists a unique operator U such that

1. $\|U\|^2 = inf\{\mu : L_1L_1^* \leq \mu L_2L_2^*\}$,
2. $\mathcal{N}(L_1) = \mathcal{N}(U)$,
3. $\mathcal{R}(U) \subseteq \overline{\mathcal{R}(L_2)^*}$.

Theorem 1.3. ([11]) *Suppose that $L_1, L_2, L_3 \in \mathfrak{B}(\mathcal{H})$. Then the following statements are equivalent:*

(i) $\mathcal{R}(L_1) \subseteq \mathcal{R}(L_2) + \mathcal{R}(L_3)$.

(ii) $L_1L_1^* \leq \lambda(L_2L_2^* + L_3L_3^*)$ *for some* $\lambda > 0$.

(iii) There exist operators $U, V \in \mathfrak{B}(\mathcal{H})$ such that $L_1 = L_2U + L_3V$.

Assume that X and Y are two sequence spaces and $E = (E_{n,i})_{n,i\geq 1}$ is an infinite matrix of real or complex numbers. We say that E defines a matrix mapping from X into Y, if for each sequence $x = \{x_n\}_{n=1}^{\infty}$ in X, the sequence $Ex = \{(Ex)_n\}_{n=1}^{\infty}$ is in Y, where

$$(Ex)_n = \sum_{i=1}^{\infty} E_{n,i}x_i, \quad n = 1, 2, \dots .$$

If X and Y are normed spaces, then we say E is bounded if there exists $M > 0$ such that for each $x \in X$, $\|Ex\|_Y \leq M\|x\|_X$. Also, we say that E is bounded below if there exists $M > 0$ such that for each $x \in X$, $\|Ex\|_Y \geq M\|x\|_X$.

For a sequence $\{\mathcal{H}_n\}_{n=1}^{\infty}$ of Hilbert spaces, define

$$\bigoplus_{n=1}^{\infty} \mathcal{H}_n := \Big\{\{f_n\}_{n=1}^{\infty} \mid f_n \in \mathcal{H}_n, \quad \text{and} \quad \sum_{n=1}^{\infty} \|f_n\|^2 < \infty\Big\}.$$

It is easily seen that with pointwise operations and with the inner product defined by

$$\big\langle \{f_n\}_{n=1}^{\infty}, \{g_n\}_{n=1}^{\infty}\big\rangle = \sum_{n=1}^{\infty} \langle f_n, g_n\rangle,$$

$\bigoplus_{n=1}^{\infty} \mathcal{H}_n$ is a Hilbert space.

Suppose that $\{\mathcal{H}_n\}_{n=1}^{\infty}$ is a sequences of Hilbert spaces such that for each $n \in \mathbb{N}$, $\mathcal{H}_n \subseteq \mathbb{H}$, where $\mathbb{H}$ is a Hilbert space. If E is an infinite matrix mapping on $\bigoplus_{n=1}^{\infty} \mathcal{H}_n$, then for each $\{f_i\}_{i=1}^{\infty} \in \bigoplus_{n=1}^{\infty} \mathcal{H}_n$,

$$E\{f_i\}_{i=1}^{\infty} = \Big\{ \sum_{i=1}^{\infty} E_{n,i} f_i \Big\}_{n=1}^{\infty}.$$

If $E = (E_{n,i})_{n,i\geq 1}$ is an infinite matrix of complex numbers, then $\overline{E} = (\overline{E}_{n,i})_{n,i\geq 1}$, where $\overline{E}_{n,i}$'s are complex conjugates of $\overline{E}_{n,i}$'s.

Definition 1.4. ([12]) *The sequence $\{f_i\}_{i=1}^{\infty}$ is called an E-frame for $\mathcal{H}$ if there exist constants $0 < A \leq B < \infty$ such that*

$$A\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, (E\{f_i\}_{i=1}^{\infty})_n\rangle|^2 \leq B\|f\|^2, \quad f \in \mathcal{H} \tag{1}$$

If only the right inequality in (1) holds, then $\{f_i\}_{i=1}^{\infty}$ is called an E-Bessel sequence for $\mathcal{H}$ with E-Bessel bound B.

The synthesis operator of E-Bessel sequence $\{f_i\}_{i=1}^{\infty}$ is defined by

$$T_E : l^2(\mathbb{N}) \longrightarrow \mathcal{H}, \quad T_E\{c_n\}_{n=1}^{\infty} = \sum_{n=1}^{\infty} c_n (E\{f_i\}_{i=1}^{\infty})_n. \tag{2}$$

It can easily be checked that the sequence $\{f_i\}_{i=1}^{\infty}$ is an E-Bessel sequence with bound B if and only if the operator T_E is a bounded operator from $l^2(\mathbb{N})$ in to $\mathcal{H}$ with $\|T_E\| \leq \sqrt{B}$. The adjoint of T_E is called the analysis operator of E-Bessel sequence $\{f_i\}_{i=1}^{\infty}$ and is defined

$$T_E^* : \mathcal{H} \longrightarrow l^2(\mathbb{N}), \quad T_E^* f = \Big\{ \langle f, (E\{f_i\}_{i=1}^{\infty})_n\rangle \Big\}_{n=1}^{\infty}. \tag{3}$$

The frame operator of an E-Bessel sequence $\{f_i\}_{i=1}^{\infty}$ is given by the composition of the synthesis and analysis operators of $\{f_i\}_{i=1}^{\infty}$ as below

$$S_E : \mathcal{H} \longrightarrow \mathcal{H}, \quad S_E f = \sum_{n=1}^{\infty} \langle f, (E\{f_i\}_{i=1}^{\infty})_n\rangle (E\{f_i\}_{i=1}^{\infty})_n. \tag{4}$$

If $\{f_i\}_{i=1}^{\infty}$ is an E-Bessel sequence for $\mathcal{H}$, it can be easily could be checked that S_E is bounded, self-adjoint and positive.

2 E-K-frames

In this section, we introduce E-K-frames and verify their properties. Suppose that E is an infinite matrix mapping on $\bigoplus_{n=1}^{\infty} \mathcal{H}_n$, then we define E-K-frames as below.

Definition 2.1. *Let $K \in \mathfrak{B}(\mathcal{H})$. The sequence $\{f_i\}_{i=1}^{\infty}$ is called an E-K-frame for $\mathcal{H}$ if there exist constants $0 < A \leq B < \infty$ such that*

$$A\|K^* f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, (E\{f_i\}_{i=1}^{\infty})_n\rangle|^2 \leq B\|f\|^2, \quad f \in \mathcal{H}. \tag{5}$$

Now, we give some examples of E-K-frames.

Example 2.2. *Suppose that $E = (E_{n,i})_{n,i\geq 1}$ is an infinite matrix which is defied as below:*

$$E_{n,i} = \begin{cases} 1 & \text{if } n \leq i \leq n+1 \\ 0 & \text{if } 1 \leq i < n \text{ or } i > n+1 \end{cases}, \quad (n \in \mathbb{N}) \tag{6}$$

indeed

$$E = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For each sequence $\{x_i\}_{i=1}^{\infty}$ in $\mathcal{H}$ we have

$$(E\{x_i\}_{i=1}^{\infty})_n = \sum_{i=1}^{\infty} E_{n,i} x_i = x_n + x_{n+1}.$$

Let $\{f_i\}_{i=1}^{\infty}$ be any K-frame in $\mathcal{H}$ and define $\{g_i\}_{i=1}^{\infty}$ as below:

$$g_i = \begin{cases} f_{\frac{i+1}{2}} & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even} \end{cases}, \tag{7}$$

that is,

$$\{g_i\}_{i=1}^{\infty} = \{f_1, 0, f_2, 0, f_3, 0, \cdots\}.$$

Then $E\{g_i\}_{i=1}^{\infty} = \{f_i\}_{i=1}^{\infty}$ *is a* K*-frame for* $\mathcal{H}$*.*

Also we can find another infinite matrix that its non zero entries are infinite. For example we can define the matrix E *as below:*

$$E_{n,i} = \begin{cases} 1 & \text{if } 2n-1 \leq i \leq 2n \text{ or } i > 2n \text{ and } i \text{ is even} \\ 0 & \text{if } i \leq 2n-1 \text{ or } i > 2n \text{ and } i \text{ is odd} \end{cases}, \quad (n \in \mathbb{N})$$

that is,

$$E = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1\cdots \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1\cdots \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1\cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\ddots \end{pmatrix}.$$

Then $E\{g_i\}_{i=1}^{\infty} = \{f_i\}_{i=1}^{\infty}$ *is a* K*-frame for* $\mathcal{H}$*, where* $\{g_i\}_{i=1}^{\infty}$ *is defined by (7).*

In the following example, we show that even if a sequence $\{f_i\}_{i=1}^{\infty}$ is not a K-frame, it can be converted to a K-frame with a suitable infinite matrix.

Example 2.3. *Let* E *be the infinite matrix which is defined by (6) in Example 2.2 and* $\{e_i\}_{i=1}^{\infty}$ *be an orthonormal basis of* $\mathcal{H}$*. We define* $\{f_i\}_{i=1}^{\infty}$ *as below:*

$$f_i = \begin{cases} e_{\frac{i+1}{2}} + e_{\frac{i+1}{2}+1} & \text{if } i \text{ is odd} \\ -e_{\frac{i}{2}+1} & \text{if } i \text{ is even} \end{cases},$$

that is,

$$\{f_i\}_{i=1}^{\infty} = \{e_1 + e_2, -e_2, e_2 + e_3, -e_3, \cdots\}.$$

We show that for any invertible operator $K \in \mathfrak{B}(\mathcal{H})$ *the sequence* $\{f_i\}_{i=1}^{\infty}$ *is not a* K*-frame for* $\mathcal{H}$*. Let*

$$g_j := \sum_{n=1}^{2j} f_n, \quad j \in \mathbb{N},$$

then

$$g_j = e_1 + e_2 + ...e_j,$$

and

$$\|g_j\|^2 = j, \quad j \in \mathbb{N}.$$

Considering a fixed $j \in \mathbb{N}$*, we see that*

$$\langle g_j, f_{2n}\rangle = \begin{cases} 0 & \textit{if } j < n+1 \\ -1 & \textit{if } 1 \leq n \leq j-1 \end{cases},$$

$$\langle g_j, f_{2n-1}\rangle = \begin{cases} 0 & \textit{if } j < n \\ 2 & \textit{if } 1 \leq n \leq j-1 \\ 1 & \textit{if } n = j \end{cases}.$$

Therefor for each $j \in \mathbb{N}$*,*

$$\sum_{n=1}^{\infty} |\langle g_j, f_n\rangle|^2 \geq 1 = \frac{1}{j}\|g_j\|^2,$$

so

$$\sum_{n=1}^{\infty} |\langle g_j, f_n\rangle|^2 \geq 1 = \frac{1}{j}\|g_j\|^2 = \frac{1}{j}\|(K^{-1})^*K^*g_j\|^2 \geq \frac{1}{j\|(K^{-1})^*\|^2}\|K^*g_j\|^2.$$

Therefore $\{f_i\}_{i=1}^{\infty}$ *does not meet the lower* K*-frame condition. If* E *is the infinite matrix which is defined by (6) in Example 2.2, then*

$$E\{f_i\}_{i=1}^{\infty} = \big\{e_1, e_3, e_2, e_4, e_3, e_5, e_4, e_6, e_5, e_7, e_6, e_8, e_7, \cdots\big\},$$

where all e_n*'s are repeated twice except* e_1 *and* e_2*. So for each* $f \in \mathcal{H}$*,*

$$\frac{1}{\|(K^{-1})^*\|^2}\|K^*f\|^2 \leq \|(K^{-1})^*K^*f\|^2 = \|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, (E\{f_i\}_{i=1}^{\infty})_n\rangle|^2 \leq 2\|f\|^2,$$

that is, $E\{f_i\}_{i=1}^{\infty}$ *is a* K*-frame for* $\mathcal{H}$*.*

The following gives a characterization of E-K-frames.

Theorem 2.4. *Let $\mathcal{H}$ be a separable Hilbert space and $K \in \mathfrak{B}(\mathcal{H})$. Then $\{f_i\}_{i=1}^{\infty}$ is an E-K-frame for $\mathcal{H}$ if and only if there exists a bounded linear operator $L : l^2(\mathbb{N}) \longrightarrow \mathcal{H}$ such that $Le_n = (E\{f_i\}_{i=1}^{\infty})_n$, $n = 1, 2, ...$ and $\mathcal{R}(K) \subseteq \mathcal{R}(L)$, where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis for $l^2(\mathbb{N})$.*

Proof. Assume that $\{f_i\}_{i=1}^{\infty}$ is an E-K-frame for $\mathcal{H}$ and $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis for $l^2(\mathbb{N})$. We define the mapping $\lambda : \mathcal{H} \longrightarrow l^2(\mathbb{N})$ as

$$\lambda(h) = \sum_{n=1}^{\infty} \langle h, (E\{f_i\}_{i=1}^{\infty})_n \rangle e_n, \quad h \in \mathcal{H}.$$

Obviously, λ is a bounded linear operator with $\|\lambda\| \leq B$. For each $h \in \mathcal{H}$ and $j \in \mathbb{N}$, we have

$$\langle \lambda h, e_j \rangle = \Big\langle \sum_{n=1}^{\infty} \langle h, (E\{f_i\}_{i=1}^{\infty})_n \rangle e_n, e_j \Big\rangle = \langle h, (E\{f_i\}_{i=1}^{\infty})_j \rangle.$$

So

$$\lambda^* e_j = (E\{f_i\}_{i=1}^{\infty})_j, \quad j = 1, 2, ... \tag{8}$$

Then by assumption,

$$A\|K^* f\| \leq \|\lambda h\|^2, \quad h \in \mathcal{H},$$

which implies that $AKK^* \leq \lambda^* \lambda$. Set $L = \lambda^*$ then $AKK^* \leq LL^*$ and Theorem 1.2 implies that $\mathcal{R}(K) \subseteq \mathcal{R}(L)$.

Conversely, suppose that there exists $L \in B(l^2(\mathbb{N}), \mathcal{H})$ such that $\mathcal{R}(K) \subseteq \mathcal{R}(L)$ and $Le_n = (E\{f_i\}_{i=1}^{\infty})_n$, $n = 1, 2, ...$. For each $f \in \mathcal{H}$ and $g \in l^2(\mathbb{N})$, we have

$$\begin{aligned}
\langle L^* f, g \rangle = \langle L^* f, \sum_{n=1}^{\infty} \langle g, e_n \rangle e_n \rangle &= \sum_{n=1}^{\infty} \overline{\langle g, e_n \rangle} \langle L^* f, e_n \rangle \\
&= \sum_{n=1}^{\infty} \overline{\langle g, e_n \rangle} \langle f, (E\{f_i\}_{i=1}^{\infty})_n \rangle \\
&= \sum_{n=1}^{\infty} \langle e_n, g \rangle \langle f, (E\{f_i\}_{i=1}^{\infty})_n \rangle \\
&= \Big\langle \sum_{n=1}^{\infty} \langle f, (E\{f_i\}_{i=1}^{\infty})_n \rangle e_n, g \Big\rangle.
\end{aligned}$$

So for each $f \in \mathcal{H}$, $L^*f = \sum_{n=1}^{\infty}\langle f, (E\{f_i\}_{i=1}^{\infty})_n\rangle e_n$, which implies

$$\sum_{n=1}^{\infty} |\langle f, (E\{f_i\}_{i=1}^{\infty})_n\rangle|^2 = \sum_{n=1}^{\infty} |\langle L^*f, e_n\rangle|^2 = \|L^*f\|^2 \leq \|L^*\|^2\|f\|^2.$$

So $\{f_i\}_{i=1}^{\infty}$ is an E-Bessel sequence for $\mathcal{H}$. Given the hypothesis $\mathcal{R}(K) \subseteq \mathcal{R}(L)$ and Theorem 1.2, we can draw the conclusion that there exists $A > 0$ such that $AKK^* \leq LL^*$. Therefore for each $f \in \mathcal{H}$,

$$A\|K^*f\|^2 \leq \|L^*f\|^2 = \sum_{n=1}^{\infty} |\langle f, (E\{f_i\}_{i=1}^{\infty})_n\rangle|^2.$$

□

Definition 2.5. *Let $E = (E_{n,k})_{n,k\geq 1}$ be an infinite invertible complex matrix which defines a bounded operator on $l^2(\mathbb{N})$. Also suppose that $\{e_n\}_{n=1}$ is an orthonormal basis for separable Hilbert space $\mathcal{H}$ and $K \in \mathfrak{B}(\mathcal{H})$. we say that $E\{f_i\}_{i=1}$ is an E-atomic system for K if the following statements hold:*

1. *the series $\sum_{n=1}^{\infty} c_n(E\{f_i\}_{i=1}^{\infty})_n$ converges for all $\{c_n\}_{n=1}^{\infty} \in l^2(\mathbb{N})$.*
2. *there exists $C > 0$ such that for each $f \in \mathcal{H}$ there exists $a_f = \{a_n\}_{n=1}^{\infty} \in l^2(\mathbb{N})$ such that $\|a_f\|_{l^2} \leq C\|f\|$ and*

$$Kf = \sum_{n=1}^{\infty} a_n(E\{f_i\}_{i=1}^{\infty})_n.$$

Theorem 2.6. *Let $\mathcal{H}$ be a separable Hilbert space and $K \in \mathfrak{B}(\mathcal{H})$. Then there exists an E-atomic system for K.*

Proof. Suppose that $E = (E_{n,k})_{n,k\geq 1}$ is an infinite invertible complex matrix and $\{e_n\}_{n=1}$ is an orthonormal basis for $\mathcal{H}$. Then for each $f \in \mathcal{H}$ we have

$$f = \sum_{n=1}^{\infty}\langle f, e_n\rangle e_n,$$

$$Kf = \sum_{n=1}^{\infty}\langle f, e_n\rangle Ke_n.$$

Let

$$f_n := (E^{-1}\{Ke_i\}_{i=1}^{\infty})_n, \quad a_n := \langle f, e_n\rangle, \quad n = 1, 2, \ldots .$$

So $\{f_n\}_{n=1}^{\infty} = E^{-1}\{Ke_i\}_{i=1}^{\infty}$. To show the first part of Definition 2.5, it is enough to prove that $E\{f_i\}_{i=1}^{\infty}$ is an E-Bessel sequence for $\mathcal{H}$ (by Corollary 3.1.5 of [2]). For each $f \in \mathcal{H}$,

$$\begin{aligned}
\sum_{n=1}^{\infty} |\langle f, (E\{f_i\}_{i=1}^{\infty})_n\rangle|^2 &= \sum_{n=1}^{\infty} |\langle f, (EE^{-1}\{Ke_i\}_{i=1}^{\infty})_n\rangle|^2 \\
&= \sum_{n=1}^{\infty} |\langle f, Ke_n\rangle|^2 = \sum_{n=1}^{\infty} |\langle K^* f, e_n\rangle|^2 \\
&= \|K^* f\|^2 \leq \|K^*\|^2 \|f\|^2.
\end{aligned}$$

Hence $\{f_i\}_{i=1}^{\infty}$ is an E-Bessel sequence for $\mathcal{H}$. Now, we show that the second part of Definition 2.5 holds. It is obvious that $a = \{a_n\}_{n=1}^{\infty} \in l^2(\mathbb{N})$ and $\sum_{n=1}^{\infty} |a_n|^2 = \|f\|^2$. Also

$$\sum_{n=1}^{\infty} a_n (E\{f_i\}_{i=1}^{\infty})_n = \sum_{n=1}^{\infty} \langle f, e_n\rangle (EE^{-1}\{Ke_i\}_{i=1}^{\infty})_n = \sum_{n=1}^{\infty} \langle h, e_n\rangle Ke_n = Kf.$$

□

Theorem 2.7. *Suppose that $\{f_i\}_{i=1}^{\infty} \subseteq H$ and E is an infinite invertible matrix. Then the following statements are equivalent:*

(i) $\{f_i\}_{i=1}^{\infty}$ *is an E-atomic system for K.*

(ii) There exists $A, B > 0$ such that

$$A\|K^* f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, (E\{f_i\}_{i=1}^{\infty})_n\rangle|^2 \leq B\|f\|^2, \quad f \in \mathcal{H}.$$

(iii) $E\{f_i\}_{i=1}^{\infty}$ *is an E-Bessel sequence for $\mathcal{H}$ and there exists an E-Bessel sequence $\{g_i\}_{i=1}^{\infty}$ such that*

$$Kf = \sum_{n=1}^{\infty} \langle f, (E\{g_i\}_{i=1}^{\infty})_n\rangle (E\{f_i\}_{i=1}^{\infty})_n, \quad f \in \mathcal{H}.$$

Proof. $(i) \Rightarrow (ii)$: Since $\{f_i\}_{i=1}^{\infty}$ is an E-atomic system for K, so there exists $C > 0$ such that for each $g \in \mathcal{H}$ there exists $\{a_n\}_{n=1}^{\infty} \in l^2(\mathbb{N})$ such that $(\sum_{n=1}^{\infty} |a_n|^2)^{\frac{1}{2}} \leq C\|g\|$ and

$$Kg = \sum_{n=1}^{\infty} a_n (E\{f_i\}_{i=1}^{\infty})_n.$$

Therefore for each $f \in \mathcal{H}$,

$$\begin{aligned}
\|K^* f\| &= \sup_{\|g\|=1} |\langle K^* f, g\rangle| = \sup_{\|g\|=1} |\langle f, Kg\rangle| \\
&= \sup_{\|g\|=1} |\langle f, \sum_{n=1}^{\infty} a_n (E\{f_i\}_{i=1}^{\infty})_n\rangle| \\
&= \sup_{\|g\|=1} |\sum_{n=1}^{\infty} \langle f, a_n (E\{f_i\}_{i=1}^{\infty})_n\rangle| \\
&\leq \sup_{\|g\|=1} (\sum_{n=1}^{\infty} |a_n|^2)^{\frac{1}{2}} (\sum_{n=1}^{\infty} |\langle f, (E\{f_i\}_{i=1}^{\infty})_n\rangle|^2)^{\frac{1}{2}} \\
&\leq C \sup_{\|g\|=1} (\sum_{n=1}^{\infty} |\langle f, (E\{f_i\}_{i=1}^{\infty})_n\rangle|^2)^{\frac{1}{2}},
\end{aligned}$$

so

$$\frac{1}{C^2}\|K^* f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, (E\{f_i\}_{i=1}^{\infty})_n\rangle|^2.$$

$(ii) \Rightarrow (iii)$: By assumption, $\{f_i\}_{i=1}^{\infty}$ is an E-Bessel sequence for $\mathcal{H}$. Let $L = \lambda^* : l^2(\mathbb{N}) \longrightarrow \mathcal{H}$ be the operator defined by (8). So

$$Le_j = (E\{f_i\}_{i=1}^{\infty})_j, \quad j = 1, 2, \dots ,$$

and $AKK^* \leq LL^*$. By Theorem 2.4, there exists a bounded linear operator $M : \mathcal{H} \longrightarrow l^2(\mathbb{N})$ such that $K = LM$. Consider the mapping $F_n : \mathcal{H} \longrightarrow \mathbb{C}$ as $F_n(h) = (Mh)_n$. If $a_n(h) = (Mh)_n$, $n = 1, 2, \dots$. Then $\{a_n(h)\}_{n=1}^{\infty} = Mh$ and we have

$$|a_n(h)|^2 \leq (\sum_{n=1}^{\infty} |a_n|^2)^{\frac{1}{2}} = \|Mh\|_{l^2} \leq \|M\|\|h\|.$$

By the Riesz representation theorem, for each $n = 1, 2, \dots$, there exists $x_n \in \mathcal{H}$ such that $a_n(h) = \langle h, x_n \rangle$. Then

$$Kh = LMh = L\{a_n(h)\}_{n=1}^{\infty} = \sum_{n=1}^{\infty} a_n(h)(E\{f_i\}_{i=1}^{\infty})_n = \sum_{n=1}^{\infty} \langle h, x_n \rangle (E\{f_i\}_{i=1}^{\infty})_n.$$

Let $\{g_n\}_{n=1}^{\infty} = E^{-1}\{x_i\}_{i=1}^{\infty}$, then

$$Kf = \sum_{n=1}^{\infty} \langle f, (E\{g_i\}_{i=1}^{\infty})_n \rangle (E\{f_i\}_{i=1}^{\infty})_n, \quad f \in \mathcal{H}.$$

$\{g_i\}_{i=1}^{\infty}$ is an E-Bessel sequence for $\mathcal{H}$, since for each $h \in \mathcal{H}$, we have

$$\sum_{n=1}^{\infty} |\langle h, (E\{g_i\}_{i=1}^{\infty})_n \rangle|^2 = \sum_{n=1}^{\infty} |\langle h, x_n \rangle|^2 = \|a_n(h)\|_{l^2}^2 \leq \|M\|^2 \|h\|^2.$$

$(iii) \Rightarrow (i)$: By Theorem 3.1.5 in [2], the first statement of Definition 1.4 holds. Also, clearly the second statement is satisfied. □

Theorem 2.8. *Consider that $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ are E-K-frames for $\mathcal{H}$ and T_f and T_g are corresponding synthesis operators, respectively. Then $\{f_i + g_i\}_{i=1}^{\infty}$ is an E-K-frame for $\mathcal{H}$.*

Proof. From the proof of Theorem 2.4, there exist bounded operators $L_1, L_2 : l^2(\mathbb{N}) \longrightarrow \mathcal{H}$ such that

$$L_1 e_n = (E\{f_i\}_{i=1}^{\infty})_n, \quad n = 1, 2, \dots,$$
$$L_2 e_n = (E\{g_i\}_{i=1}^{\infty})_n, \quad n = 1, 2, \dots,$$

where $\{e_n\}_{n=1}$ is supposed to be the canonical orthonormal basis of $l^2(\mathbb{N})$ and $\mathcal{R}(K) \subseteq \mathcal{R}(L_1)$, $\mathcal{R}(K) \subseteq \mathcal{R}(L_2)$. So $L_1 = T_f$, $L_2 = T_g$ and $\mathcal{R}(K) \subseteq \mathcal{R}(L_1) + \mathcal{R}(L_2)$. By Theorem 1.2, there is $\alpha > 0$ such that

$$KK^* \leq \alpha(L_1 L_1^* + L_2 L_2^*).$$

Now, for each $f \in \mathcal{H}$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} |\langle f, (E\{f_i + g_i\}_{i=1}^{\infty})_n \rangle|^2 = \sum_{n=1}^{\infty} |\langle f, L_1 e_n + L_2 e_n \rangle|^2 = \|(L_1^* + L_2^*)f\|^2 \\
&= \langle L_1 L_1^* f, f \rangle + \langle L_1 L_2^* f, f \rangle + \langle L_2 L_1^* f, f \rangle + \langle L_2 L_2^* f, f \rangle \\
&\geq \langle (L_1 L_1^* + L_2 L_2^*) f, f \rangle \geq \frac{1}{\alpha^2} \langle KK^* f, f \rangle = \frac{1}{\alpha^2} \|K^* f\|^2.
\end{aligned}$$

By the Minkowski's inequality, for each $f \in \mathcal{H}$, we have

$$\begin{aligned}\Big(\sum_{n=1}^{\infty}|\langle f,(E\{f_i+g_i\}_{i=1}^{\infty})_n\rangle|^2\Big)^{\frac{1}{2}} &\leq \Big(\sum_{n=1}^{\infty}|\langle f,(E\{f_i\}_{i=1}^{\infty})_n\rangle|^2\Big)^{\frac{1}{2}}\\ &\quad+\Big(\sum_{n=1}^{\infty}|\langle f,(E\{g_i\}_{i=1}^{\infty})_n\rangle|^2\Big)^{\frac{1}{2}}\\ &\leq B_1\|f\|+B_2\|f\|.\end{aligned}$$

Therefore $\{f_i+g_i\}_{i=1}^{\infty}$ is an E-K-frame for $\mathcal{H}$. $\square$

Lemma 2.9. *Let $\{f_i\}_{i=1}^{\infty}$ be an E-K-Bessel sequence for $\mathcal{H}$ and $K \in \mathfrak{B}(\mathcal{H})$. Then $\{f_i\}_{i=1}^{\infty}$ is an E-K-frame for $\mathcal{H}$ if and only if there is $\alpha > 0$ such that $S \geq \alpha KK^*$, where S is the frame operator of $\{f_i\}_{i=1}^{\infty}$.*

Proof. For each $f \in \mathcal{H}$,

$$\langle Sf, f\rangle = \sum_{n=1}^{\infty}|\langle f,(E\{f_i\}_{i=1}^{\infty})_n\rangle|^2.$$

So the condition $S \geq \alpha KK^*$ is equivalent to the lower frame bound condition for an E-K-frame. $\square$

Theorem 2.10. *Let E, F be infinite matrices and $\{f_i\}_{i=1}^{\infty}$ be an E-K-frame for $\mathcal{H}$ with bounds A and B. If $\bar{F}$ is bounded below and $EF = FE$, then $F\{f_i\}_{i=1}^{\infty}$ is an E-K-frame for $\mathcal{H}$.*

Proof. For each $f \in \mathcal{H}$,

$$\begin{aligned}\sum_{n=1}^{\infty}|\langle f,(E(F\{f_i\}_{i=1}^{\infty}))_n\rangle|^2 &= \sum_{n=1}^{\infty}|\langle f,(EF\{f_i\}_{i=1}^{\infty})_n\rangle|^2\\ &= \sum_{n=1}^{\infty}|\langle f,(FE\{f_i\}_{i=1}^{\infty})_n\rangle|^2 = \|\bar{F}T_E^*f\|^2.\end{aligned}$$

Since $T_{EF}^* = \bar{F}T_E^*$, so for some $\alpha > 0$ and each $f \in \mathcal{H}$,

$$\|T_{EF}^*f\|^2 = \|\bar{F}T_E^*f\|^2 \geq \alpha\|T_E^*f\|^2 \geq \alpha A\|K^*f\|^2.$$

Also for each $f \in \mathcal{H}$,

$$\|\bar{F}T_E^*f\|^2 \leq \|\bar{F}\|^2\|T_E^*f\|^2 \leq B\|F\|^2\|f\|^2.$$

Hence $F\{f_i\}_{i=1}^{\infty}$ is an E-K-frame for $\mathcal{H}$. $\square$

Proposition 2.11. *Let E, F be infinite matrices, $\bar{F}$ be bounded below and $\{f_i\}_{i=1}^{\infty}$ be an E-K-frame for $\mathcal{H}$. Then $E\{f_i\}_{i=1}^{\infty}$ is a F-K-frame for $\mathcal{H}$.*

Proof. For each $f \in \mathcal{H}$,

$$A\|K^*f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, (E\{f_i\}_{i=1}^{\infty})_n\rangle|^2 \leq B\|f\|^2,$$

also

$$\sum_{n=1}^{\infty} |\langle f, (F(E\{f_i\}_{i=1}^{\infty}))_n\rangle|^2 = \sum_{n=1}^{\infty} |\langle f, (FE\{f_i\}_{i=1}^{\infty})_n\rangle|^2 = \|\bar{F}T_E^*f\|^2.$$

So for some $\alpha > 0$ and each $f \in \mathcal{H}$,

$$\alpha A\|K^*f\|^2 \leq \alpha\|T_E^*f\|^2 \leq \|\bar{F}T_E^*f\|^2 \leq B\|\bar{F}\|^2\|f\|^2.$$

□

By above proposition, we have the following result.

Corollary 2.12. *Let E be a bounded below infinite matrix and $\{f_i\}_{i=1}^{\infty}$ be an K-frame for $\mathcal{H}$. Then $\{f_i\}_{i=1}^{\infty}$ is an E-K-frame for $\mathcal{H}$.*

References

[1] E. Alizadeh, A. Rahimi, E. Osgooei and M. Rahmani, Continuous K-g-frames in Hilbert spaces, *Bull. Iran. Math. Soc.*, 45(4) (2019), 1091-1104.

[2] O. Christensen, *Frames and Bases: An Introductory Course*, Birkhäuser, Boston (2008).

[3] R. G. Douglas, On majorization, factorization and range inclusion of operators on Hilbert space, *Pro. Amer. Math. Sco.*, 17(2) (1966), 413-415.

[4] R. G. Douglas, *Banacch Algebra Techniques in Operator Theory*, 2nd edn. Academic Press, New York (1998).

[5] D. Du and Y. C. Zhu, Constructions of K-g-frames and Tight K-g-frames in Hilbert Spaces, *Bull. Malays. Math. Sci. Soc.*, 43 (2020), 4107-4122.

[6] L. Gavruta, Frames for operators. *Appl. Comput. Harmon. Anal.*, 32 (2012), 139-144.

[7] Y. Huang and S. Shi, New Constructions of K-g-frames, *Results Math.*, 73(4) (2018), 1-13.

[8] H. Pourmahmood-Aghababa, M. H. Sattari and H. Shafie-Asl, Bounded Pseudo-Amenability and Contractibility of Certain Banach Algebras, *Filomat.*, 34(5) (2020), 1701-1712.

[9] Z. Q. Xiang, Some new results on the construction and stability of K-g-frames in Hilbert Spaces, *Int. J. Wavelets. Multiresolut. Inf. Process.*, 18(5) (2020), 2050034:1-2050034:19.

[10] D. Hua and Y. Hung., K-g-frames and stability of K-g-frames in Hilbert spaces, *J. Korean Math. Soc.*, 53(6) (2016), 1331-1345.

[11] P. A. Fillmore, J. P. Williams. On operator ranges, *Adv. Math.*, 7(3) (1971), 254-281.

[12] G. Talebi, M. A. Dehghan, On E-frames in separable Hilbert spaces, *Banach J. Math. Anal.*, 9(3) (2015), 43-74.

Morteza Rahmani
Young Researchers and Elite Club
Ilk.C., Islamic Azad University
Ilkhchi, Iran
E-mail: morteza.rahmany@gmail.com

Esmaeil Alizadeh
Department of Mathematics
Mara.C., Islamic Azad University
Marand, Iran
E-mail: esmaeil.alizadeh@iau.ac.ir