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$H(\cdot,\cdot)$-Mixed Relaxed Co-η-Monotone Mapping with an Application for Solving a Resolvent Equation Problem

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Abstract. In this paper, we define an $H(\cdot,\cdot)$-mixed relaxed co-η-monotone mapping and we apply the same for solving a resolvent equation problem in Hilbert spaces. We also prove some of the properties of $H(\cdot,\cdot)$-mixed relaxed co-η-monotone mapping. A mann-type iterative algorithm is developed to approximate the solution of resolvent equation problem. Convergence of the iterative sequences is also demonstrated. In support of the concept of $H(\cdot,\cdot)$-mixed relaxed co-η-monotone mapping, we construct an example.

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1. Introduction

The ideas and techniques of variational inequalities are being used to interpret the basic principles of pure and applied sciences in the form of

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simplicity and elegance. It is worth-mentioning that the variational inequality theory constitutes an important and significant extension of the variational principles. Since the theory of variational inequalities (inclusions) is quite application oriented, thus developed a lot in recent past in many different directions. This theory provides us, a framework to understand and solve many problems occurring in economics, optimization, transportation, elasticity and applied sciences, etc., see [6, 7, 10, 11] and references therein.

Equally important is the concept of resolvent equations, which is mainly due to Noor [12] and he developed equivalence between variational inequalities and resolvent equations. The resolvent equations technique is quite general and flexible. This technique has been used to develop some numerical methods for solving variational inclusions. It is notable that the resolvent equations include the Wiener-Hopf equations as a special case. The Wiener-Hopf equations technique was used to develop various numerical methods for solving the variational inequalities and complementarity problems. Thus, the approach of resolvent equations is quite applicable in nature and depends upon some type of resolvent operators. In 2001, Huang and Fang [8] first introduced the concept of generalized m-accretive mapping and defined a resolvent operator for the generalized m-accretive mapping in Banach spaces. After that, Fang et al. [5], Lan et al. [9] and many others introduced and studied many generalized mappings. Very recently, Ahmad et al. [1] introduced and studied $H(\cdot,\cdot)$-co-monotone mapping and its resolvent operator in real Hilbert space. For related work, see also [2–4].

Motivated by the facts stated above, in this paper, we define a new generalized monotone mapping and we call it $H(\cdot,\cdot)$-mixed relaxed co-η-monotone mapping. We prove some of the properties of $H(\cdot,\cdot)$-mixed relaxed co-η-monotone mapping and develop a resolvent operator associated with $H(\cdot,\cdot)$-mixed relaxed co-η-monotone mapping. Finally, we solve a resolvent equation problem. In support of definition of $H(\cdot,\cdot)$-mixed relaxed co-η-monotone mapping, we construct an example.

2. Preliminaries

We begin this section by recalling required definitions and concepts to prove the main result of this paper. Their details can be traced in [2].

Definition 2.1. *Let $H : X \times X \longrightarrow X$, $\eta : X \times X \longrightarrow X$ and $A, B : X \longrightarrow X$ be the mappings. Then*

(i) *$H(A,\cdot)$ is said to be η-cocoercive with respect to A if for a fixed $u \in X$, there exists a constant $\nu > 0$ such that*

$$\langle H(Ax,u) - H(Ay,u), \eta(x,y)\rangle \geq \nu\|Ax - Ay\|^2, \quad \forall x, y \in X;$$

(ii) *$H(A,\cdot)$ is said to be relaxed η-cocoercive with respect to A if for a fixed $u \in X$, there exists a constant $\mu > 0$ such that*

$$\langle H(Ax,u) - H(Ay,u), \eta(x,y)\rangle \geq (-\mu)\|Ax - Ay\|^2, \quad \forall x, y \in X;$$

(iii) *$H(\cdot,B)$ is said to be α-ξ-relaxed η-cocoercive with respect to B if for a fixed $u \in X$, there exist constants $\alpha, \xi > 0$ such that*

$$\langle H(u,Bx) - H(u,By), \eta(x,y)\rangle \geq (-\alpha)\|Bx - By\|^2 + \xi\|x - y\|^2, \quad \forall x, y \in X;$$

(iv) *$H(A,\cdot)$ is said to be ρ-Lipschitz continuous with respect to A if for a fixed $u \in X$, there exists a constant $\rho > 0$ such that*

$$\|H(Ax,u) - H(Ay,u)\| \leq \rho\|x - y\|, \quad \forall x, y \in X.$$

Similarly, we can define the Lipschitz continuity of H with respect to B in the second argument.

Definition 2.2. *A multi-valued mapping $M : X \longrightarrow 2^X$ is said to be m-relaxed η-monotone if, there exists a constant $m > 0$ such that*

$$\langle u - v, \eta(x,y)\rangle \geq -m\|u - v\|^2, \quad \forall u, v \in X, x \in M(u), y \in M(v).$$

Definition 2.3. *Let $A, B : X \longrightarrow X$, $H, \eta : X \times X \longrightarrow X$ be the single valued mappings such that H is μ-relaxed η-cocercive with respect to A and α-ξ-relaxed η-cocoercive with respect to B. Then a multi-valued mapping $M : X \longrightarrow 2^X$ is said to be $H(\cdot,\cdot)$-mixed relaxed co-η-monotone mapping with respect to A and B, if M is m-relaxed η-monotone and $(H(A,B) + \lambda M)(X) = X$, for every $\lambda > 0$.*

In support of above definition of $H(\cdot,\cdot)$-mixed relaxed co-η-monotone mapping, we give the following example.

Example 2.4. Let $X = \mathbb{R}^2$ with usual inner product. Let $A, B : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by

$$\begin{aligned} A(x) &= (-2x_1, x_1 - 2x_2),\ \forall\ x = (x_1, x_2) \in \mathbb{R}^2; \\ B(y) &= (2y_1, y_1 + y_2,),\ \forall\ y = (y_1, y_2) \in \mathbb{R}^2. \end{aligned}$$

Let $\eta : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is defined by

$$\eta(x, y) = x - y, \quad \forall x, y \in \mathbb{R}^2.$$

Suppose $H(A, B) : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is defined by

$$H(A(x), B(y)) = A(x) + B(y), \quad \forall x, y \in \mathbb{R}^2.$$

Then, $H(A, B)$ is relaxed η-cocoercive with respect to A with constant $\frac{1}{4}$ and $(1, 2)$-relaxed η-cocoercive with respect to B.The verification is as follows:

$$\begin{aligned} & \langle H(A(x), u) - H(A(y), u), \eta(x, y)\rangle \\ = \ & \langle A(x) - A(y), x - y\rangle \\ = \ & \Big\langle (- 2(x_1 - y_1), (x_1 - y_1) - 2(x_2 - y_2)), ((x_1 - y_1), (x_2 - y_2)) \Big\rangle \\ = \ & -2(x_1 - y_1)^2 + (x_1 - y_1)(x_2 - y_2) - 2(x_2 - y_2)^2 \\ = \ & -[2(x_1 - y_1)^2 + 2(x_2 - y_2)^2 - (x_1 - y_1)(x_2 - y_2)], \end{aligned}$$

and

$$\begin{aligned}\|A(x)-A(y)\|^2 &= \langle A(x)-A(y), A(x)-A(y)\rangle \\ &= 4(x_1-y_1)^2+(x_1-y_1)^2+4(x_2-y_2)^2 \\ &\quad -4(x_1-y_1)(x_2-y_2) \\ &= 5(x_1-y_1)^2-4(x_1-y_1)(x_2-y_2)+4(x_2-y_2)^2 \\ &\leq 8(x_1-y_1)^2-4(x_1-y_1)(x_2-y_2)+8(x_2-y_2)^2 \\ &\leq 4\{2(x_1-y_1)^2-(x_1-y_1)(x_2-y_2)+2(x_2-y_2)^2\} \\ &= -4\langle H(A(x),u)-H(A(y),u), \eta(x,y)\rangle.\end{aligned}$$

Therefore,

$$\langle H(A(x),u)-H(A(y),u), \eta(x,y)\rangle \geq -\frac{1}{4}\|A(x)-A(y)\|^2,$$

which implies that $H(A,B)$ is relaxed η-cocoercive with respect to A with constant $\frac{1}{4}$.

Further,

$$\begin{aligned}&\langle H(u,B(x))-H(u,B(y)), \eta(x,y)\rangle \\ =\ & \langle B(x)-B(y), x-y\rangle \\ =\ & \Big\langle (2(x_1-y_1), (x_1-y_1)+(x_2-y_2)), ((x_1-y_1), (x_2-y_2))\Big\rangle \\ =\ & 2(x_1-y_1)^2+(x_1-y_1)(x_2-y_2)+(x_2-y_2)^2.\end{aligned}$$

Also

$$\begin{aligned}\|B(x)-B(y)\|^2 &= \langle B(x)-B(y), B(x)-B(y)\rangle \\ &= 4(x_1-y_1)^2+(x_1-y_1)^2+(x_2-y_2)^2 \\ &\quad +2(x_1-y_1)(x_2-y_2) \\ &= 5(x_1-y_1)^2+(x_2-y_2)^2+2(x_1-y_1)(x_2-y_2);\end{aligned}$$

and

$$\|x-y\|^2 = (x_1-y_1)^2+(x_2-y_2)^2.$$

Now,

$$\begin{aligned}
& - \quad \|B(x)-B(y)\|^2 + 2\|x-y\|^2 \\
&= -3(x_1-y_1)^2 + (x_2-y_2)^2 - 2(x_1-y_1)(x_2-y_2) \\
&\leq 2(x_1-y_1)^2 + (x_1-y_1)(x_2-y_2) + (x_2-y_2)^2 \\
&= \langle H(u,B(x)) - H(u,B(y)), \eta(x,y)\rangle;
\end{aligned}$$

i.e.,

$$\langle H(u,B(x)) - H(u,B(y)), \eta(x,y)\rangle \geq -\|B(x)-B(y)\|^2 + 2\|x-y\|^2.$$

Therefore, $H(A,B)$ is $(1,2)$-relaxed η-cocoercive with respect to B.

Suppose that $M : \mathbb{R}^2 \longrightarrow 2^{\mathbb{R}^2}$ is defined by

$$M(x) = (-3x_1, -2x_2), \quad \forall (x_1,x_2) \in \mathbb{R}^2.$$

Now,

$$\begin{aligned}
\langle M(x) - M(y), \eta(x,y)\rangle &= \Big\langle (-3(x_1-y_1), -2(x_2-y_2)), \\
&\qquad ((x_1-y_1),(x_2-y_2))\Big\rangle \\
&= -[3(x_1-y_1)^2 + 2(x_2-y_2)^2];
\end{aligned}$$

and

$$\begin{aligned}
\|x-y\|^2 &= \langle x-y, x-y\rangle \\
&= (x_1-y_1)^2 + (x_2-y_2)^2 \\
&\leq 3(x_1-y_1)^2 + 2(x_2-y_2)^2 \\
&= (-1)\langle M(x)-M(y), \eta(x,y)\rangle;
\end{aligned}$$

i.e.,

$$\langle M(x) - M(y), \eta(x,y)\rangle \geq (-1)\|x-y\|^2.$$

Therefore, M is 1-relaxed η-monotone. Also, for $\lambda > 0$, one can easily check that

$$(H(A,B) + \lambda M)\mathbb{R}^2 = \mathbb{R}^2,$$

which shows that M is $H(\cdot,\cdot)$-mixed relaxed co-η-monotone with respect to A and B.

Theorem 2.5. *Let $H(\cdot,\cdot)$ be a μ-relaxed η-cocoercive with respect to A and α-ξ-relaxed η-cocoercive with respect to B, A is β-Lipschitz continuous and B is γ-Lipschitz continuous. Let M be $H(\cdot,\cdot)$-mixed relaxed co-η-monotone with respect to A and B. Then the operator $(H(A,B)+\lambda M)^{-1}$ is single-valued for $0<\lambda<\dfrac{\xi-(\mu\beta^2+\alpha\gamma^2)}{m}$.*

Proof. For any given $u\in X$, let $x,\ y\in (H(A,B)+\lambda M)^{-1}(u)$. It follows that

$$\begin{aligned} -H(Ax,Bx)+u &\in \lambda Mx;\\ -H(Ay,By)+u &\in \lambda My. \end{aligned} \tag{1}$$

As M is m-relaxed η-monotone, we have

$$\begin{aligned} -m\|x-y\|^2 &\le \frac{1}{\lambda}\langle -H(Ax,Bx)+u-(-H(Ay,By)+u),\eta(x,y)\rangle\\ &= -\frac{1}{\lambda}\langle H(Ax,Bx)-H(Ay,By),\eta(x,y)\rangle\\ &= -\frac{1}{\lambda}\langle H(Ax,Bx)-H(Ay,Bx)+H(Ay,Bx)\\ &\quad -H(Ay,By),\eta(x,y)\rangle\\ &= -\frac{1}{\lambda}\langle H(Ax,Bx)-H(Ay,Bx),\eta(x,y)\rangle\\ &\quad -\frac{1}{\lambda}\langle H(Ay,Bx)-H(Ay,By),\eta(x,y)\rangle. \end{aligned} \tag{2}$$

Since H is μ-relaxed η- cocoercive with respect to A and α-ξ-relaxed η-cocoercive with respect to B, A is β-Lipschitz continuous and B is γ-Lipschitz continuous,thus (2) becomes

$$\begin{aligned} -m\lambda\|x-y\|^2 &\le \mu\beta^2\|x-y\|^2+\alpha\gamma^2\|x-y\|^2-\xi\|x-y\|^2\\ &= (\mu\beta^2+\alpha\gamma^2-\xi)\|x-y\|^2, \end{aligned}$$

which implies that

$$m\lambda\|x-y\|^2 \ge -(\mu\beta^2+\alpha\gamma^2-\xi)\|x-y\|^2. \tag{3}$$

If $x\neq y$, then $\lambda\ge\dfrac{\xi-\mu\beta^2-\alpha\gamma^2}{m}$, which contradicts that $0<\lambda<\dfrac{\xi-(\mu\beta^2+\alpha\gamma^2)}{m}$. Thus, we have $x=y$, i.e., $(H(A,B)+\lambda M)^{-1}$ is single-valued. □

Definition 2.6. *Let $H(A,B)$ be a μ-relaxed η- cocoercive with respect to A and α-ξ-relaxed η-cocoercive with respect to B, A is β-Lipschitz continuous and B is γ-Lipschitz continuous. Let M be an $H(\cdot,\cdot)$-mixed relaxed co-η-monotone mapping with respect to A and B. The resolvent operator $R_{\lambda,M}^{H(\cdot,\cdot)-\eta} : X \longrightarrow X$ associated with H and M is defined by*

$$R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(u) = [H(A,B) + \lambda M]^{-1}(u), \quad \forall u \in X. \tag{4}$$

Theorem 2.7. *Let $H(A,B)$ be a μ-relaxed η-cocoercive with respect to A and α-ξ-relaxed η-cocoercive with respect to B, $\eta : X \times X \longrightarrow X$ be σ-Lipschitz continuous. A is β-Lipschitz continuous and B is γ-Lipschitz continuous. Let M be an $H(\cdot,\cdot)$-mixed relaxed co-η-monotone mapping with respect to A and B. Then the resolvent operator defined by (4) is $\dfrac{\sigma}{[\xi - m\lambda - (\mu\beta^2 + \alpha\gamma^2)]}$-Lipschitz continuous for $0 < \lambda < \dfrac{\xi - (\mu\beta^2 + \alpha\gamma^2)}{m}$, i.e.,*

$$\|R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(u) - R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(v)\| \leq \frac{\sigma}{[\xi - m\lambda - (\mu\beta^2 + \alpha\gamma^2)]}\|u - v\|, \quad \forall u, v \in X. \tag{5}$$

Proof. Let u and v be any given points in X. It follows from (4) that

$$\begin{aligned} R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(u) &= [H(A,B) + \lambda M]^{-1}(u);\\ R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(v) &= [H(A,B) + \lambda M]^{-1}(v). \end{aligned} \tag{6}$$

For the sake of clarity, we denote

$$t_1 = R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(u); \quad t_2 = R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(v).$$

Then (1) implies that

$$\begin{aligned} &\frac{1}{\lambda}\Big(u - H(A(t_1), B(t_1))\Big) \in M(t_1);\\ &\frac{1}{\lambda}\Big(v - H(A(t_2), B(t_2))\Big) \in M(t_2). \end{aligned} \tag{7}$$

Since M is m-relaxed η-monotone, we have

$$\begin{aligned} -m\|t_1-t_2\|^2 &\leq \frac{1}{\lambda}\Big\langle \Big(u-H(A(t_1),B(t_1))\Big)-\Big(v-H(A(t_2),B(t_2))\Big), \\ &\quad \eta(t_1,t_2)\Big\rangle \\ &= \frac{1}{\lambda}\Big\langle u-v-H(A(t_1),B(t_1))+H(A(t_2),B(t_2)), \\ &\quad \eta(t_1,t_2)\Big\rangle, \end{aligned}$$

which implies that

$$\begin{aligned} -m\lambda\|t_1-t_2\|^2 \leq \langle u-v,\eta(t_1,t_2)\rangle + \Big\langle -H(A(t_1),B(t_1)) \\ +H(A(t_2),B(t_2)),\eta(t_1,t_2)\Big\rangle. \end{aligned} \tag{8}$$

By Cauchy-Schwartz inequality and (8), we have

$$\begin{aligned} \|u-v\|\|\eta(t_1,t_2)\| &\geq \langle u-v,\eta(t_1,t_2)\rangle \\ &\geq -\Big\langle -H(A(t_1),B(t_1))+H(A(t_2),B(t_2)), \\ &\quad \eta(t_1,t_2)\Big\rangle - m\lambda\|t_1-t_2\|^2 \\ &= \Big\langle H(A(t_1),B(t_1))-H(A(t_2),B(t_2)), \\ &\quad \eta(t_1,t_2)\Big\rangle - m\lambda\|t_1-t_2\|^2 \\ &= \Big\langle H(A(t_1),B(t_1))-H(A(t_2),B(t_1)),\eta(t_1,t_2)\Big\rangle - \\ &\quad m\lambda\|t_1-t_2\|^2 + \Big\langle H(A(t_2),B(t_1)) - \\ &\quad H(A(t_2),B(t_2)),\eta(t_1,t_2)\Big\rangle. \end{aligned} \tag{9}$$

As H is μ-relaxed η-cocoercive with respect to A, α-ξ-relaxed η-cocoercive with respect to B, $\eta : X\times X \longrightarrow X$ is σ-Lipschitz continuous, A is β-Lipschitz continuous and B is γ-Lipschitz continuous, we have

$$\sigma\|u-v\|\|t_1-t_2\| \geq [-\mu\beta^2-\alpha\gamma^2+\xi-m\lambda]\|t_1-t_2\|^2.$$

Thus, we have

$$\|t_1 - t_2\| \leq \frac{\sigma}{[\xi - m\lambda - (\mu\beta^2 + \alpha\gamma^2)]}\|u - v\|,$$

i.e.,

$$\|R^{H(\cdot,\cdot)-\eta}_{\lambda,M}(u) - R^{H(\cdot,\cdot)-\eta}_{\lambda,M}(v)\| \leq \theta_1\|u - v\|, \quad \forall u, v \in X,$$

where $\theta_1 = \dfrac{\sigma}{[\xi - m\lambda - (\mu\beta^2 + \alpha\gamma^2)]}$. This completes the proof. □

3. Resolvent Equation Problem and Existence Theory

In this section, we deal with a variational inclusion problem and its corresponding resolvent equation problem. We establish an equivalence relation between both the problems. An iterative algorithm is also defined to approximate the solution of the resolvent equation problem.

Let $T, F : X \to CB(X)$, $M : X \to 2^X$ be the multi-valued mappings, $S : X \times X \to X$ and $g : X \to X$ are the single valued mappings. Then we consider the problem of finding $u \in X$, $x \in T(u)$, $y \in F(u)$ such that

$$0 \in S(x, y) + M(g(u)). \tag{10}$$

Problem (10) is called variational inclusion problem, studied by many authors, see e.g. [12, 13].

In connection with the variational inclusion problem (10), we consider the following resolvent equation problem:

Find $z, u \in X$, $x \in T(u)$, $y \in F(u)$ such that

$$S(x, y) + \lambda^{-1} J^{H(\cdot,\cdot)-\eta}_{M,\lambda}(z) = 0, \tag{11}$$

where $\lambda > 0$ is a constant and $J^{H(\cdot,\cdot)-\eta}_{M,\lambda}(z) = [I - H(A(R^{H(\cdot,\cdot)-\eta}_{\lambda,M}(z)),$ $B(R^{H(\cdot,\cdot)-\eta}_{\lambda,M}(z)))]$, where I is the identity operator, $R^{H(\cdot,\cdot)-\eta}_{\lambda,M}$ is the resolvent operator and

$$\begin{aligned} &H[A(R^{H(\cdot,\cdot)-\eta}_{\lambda,M}(z)), B(R^{H(\cdot,\cdot)-\eta}_{\lambda,M}(z))] \\ &\quad = [H(A(R^{H(\cdot,\cdot)-\eta}_{\lambda,M}(\cdot)), B(R^{H(\cdot,\cdot)-\eta}_{\lambda,M}(\cdot)))](z). \end{aligned} \tag{12}$$

Lemma 3.1. *The triplet (u, x, y), where $u \in X, x \in T(u), y \in F(u)$, is a solution of variational inclusion problem (10) if and only if it satisfies the equation:*

$$g(u) = R_{\lambda,M}^{H(\cdot,\cdot)-\eta}[H(A(g(u)), B(g(u))) - \lambda S(x, y)], \tag{13}$$

where $\lambda > 0$ is a constant.

Proof. By using the definition of resolvent operator $R_{\lambda,M}^{H(\cdot,\cdot)-\eta}$, the conclusion follows directly. □

Based on Lemma (3.1), we prove the following lemma which may be treated as an equivalence lemma for variational inclusion problem (10) and resolvent equation problem (11)

Lemma 3.2. *The variational inclusion problem (10) has a solution (u, x, y), where $u \in X$, $x \in T(u)$, $y \in F(u)$, if and only if the resolvent equation problem (11) has a solution (z, u, x, y), where $z, u \in X$, $x \in T(u)$, $y \in F(u)$, where*

$$g(u) = R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(z), \tag{14}$$

and $z = H(A(g(u)), B(g(u))) - \lambda S(x, y)$.

Proof. Let (u, x, y) where $u \in X, x \in T(u), y \in F(u)$ is a solution of variational inclusion problem (10). Then by lemma (3.1), we have

$$g(u) = R_{\lambda,M}^{H(\cdot,\cdot)-\eta}[H(A(g(u)), B(g(u))) - \lambda S(x, y)].$$

Using the fact that $J_{M,\lambda}^{H(\cdot,\cdot)-\eta} = I - H(A(R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(\cdot)), B(R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(\cdot)))$ and equation (14), we obtain

$$\begin{aligned} & J_{M,\lambda}^{H(\cdot,\cdot)-\eta}[H(A(g(u)), B(g(u))) - \lambda S(x, y)] \\ = \; & H(A(g(u)), B(g(u))) - \lambda S(x, y) \\ - \; & H[(A(R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(H(A(g(u)), B(g(u))) - \lambda S(x, y))), \\ & B(R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(H(A(g(u)), B(g(u))) - \lambda S(x, y)))] \\ = \; & -\lambda S(x, y), \end{aligned}$$

which implies that $S(x,y)+\lambda^{-1}J_{\lambda,M}^{H(\cdot,\cdot)-\eta}(z)=0$, with $z=H(A(g(u)),B(g(u)))-\lambda S(x,y)$, that is (z,u,x,y) is a solution of resolvent equation problem (11).

Conversely, let (z,u,x,y) is a solution of the resolvent equation problem (11), then

$$S(x,y)+\lambda^{-1}J_{\lambda,M}^{H(\cdot,\cdot)-\eta}(z)=0,$$

i.e., $J_{\lambda,M}^{H(\cdot,\cdot)-\eta}(z)=-\lambda S(x,y)$.

Using the definition of $J_{\lambda,M}^{H(\cdot,\cdot)-\eta}$, we have

$$\begin{aligned}
[I-H(A(R_{\lambda,M}^{H(\cdot,\cdot)-\eta}),B(R_{\lambda,M}^{H(\cdot,\cdot)-\eta}))](z) &= -\lambda S(x,y),\\
z-H(A(R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(z)),B(R_{\lambda,M}^{H(\cdot,\cdot)-\eta})(z)) &= -\lambda S(x,y),\\
z-H(A(g(u)),B(g(u))) &= -\lambda S(x,y),\\
R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(z) = R_{\lambda,M}^{H(\cdot,\cdot)-\eta}[H(A(g(u)),B(g(u)))&-\lambda S(x,y)],\\
g(u) = R_{\lambda,M}^{H(\cdot,\cdot)-\eta}[H(A(g(u)),B(g(u)))&-\lambda S(x,y)],
\end{aligned}$$

that is , (u,x,y) is a solution of the variational inclusion problem (10). □

Based on above discussion, we now define the following algorithm for solving resolvent equation problem (11).

Algorithm 3.3. For any initial points (z_o,u_o,x_o,y_o), where $z_o,u_o\in X$, $x_o\in T(u_o)$, $y_o\in F(u_o)$, compute the sequences $\{z_n\}$, $\{u_n\}$, $\{x_n\}$ and $\{y_n\}$ by the iterative scheme:

(i) $g(u_n)=R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(z_n)$;

(ii) $\|x_n-x_{n+1}\|\leq D(T(u_n),T(u_{n+1}))$, $x_n\in T(u_n)$, $x_{n+1}\in T(u_{n+1})$;

(iii) $\|y_n-y_{n+1}\|\leq D(F(u_n),F(u_{n+1}))$, $y_n\in F(u_n)$, $y_{n+1}\in F(u_{n+1})$;

(iv) $z_{n+1}=H(A(g(u_n)),B(g(u_n)))-\lambda S(x_n,y_n)$, where $n=0,1,2,.....$, and $\lambda>0$ is a constant, $D(\cdot,\cdot)$ is the Hausdorff metric on $CB(X)$.

Theorem 3.4. *Let X be a real Hilbert space and $A, B, g : X \to X$, $S, H : X \times X \to X$, $\eta : X \times X \to X$ be the single valued mappings. Let $T, F : X \to CB(X)$ be the multi-valued mappings and $M : X \to 2^X$ be $H(\cdot,\cdot)$-mixed relaxed co-η-monotone mapping. Assume that*

(i) *$H(A, B)$ is μ-relaxed η-cocoercive with respect to A and α-ξ-relaxed η-cocorecive with respect to B, r_1-Lipschitz continuous with respect to A and r_2-Lipschitz continuous with respect to B;*

(ii) *T and F are D-Lipschitz continuous mappings with constant δ_T and δ_F, respectively;*

(iii) *A is β-Lipschitz continuous and B is γ-Lipschitz continuous;*

(iv) *g is λ_g-Lipschitz continuous and t-strongly monotone;*

(v) *S is λ_{s_1}-Lipschitz continuous in first argument and λ_{s_2} Lipschitz continuous in second argument.*

If for some $\lambda > 0$, the following condition is satisfied:

$$\sqrt{1 - 2t + \lambda_g^2} < [1 - \theta\theta_1], \tag{15}$$

where $\theta = \dfrac{\sigma}{[\xi - m\lambda - (\mu\beta^2 + \alpha\gamma^2)]}$ *and* $\theta_1 = [(r_1 + r_2)\lambda_g + \lambda(\lambda_{s_1}\delta_T + \lambda_{s_2}\delta_F)]$.

Then, there exist $z, u \in X, x \in T(u), y \in F(u)$ satisfying resolvent equation problem (10) and the iterative sequences $\{u_n\}$, $\{z_n\}$, $\{x_n\}$, $\{y_n\}$ generated by the Algorithm 3.1 strongly converge to u, z, x and y, respectively.

Proof. From (iv) of Algorithm 3.3, we have

$$\begin{aligned}
& \|z_{n+1}-z_n\| \\
= & \ \|[H(A(g(u_n)),B(g(u_n)))-\lambda S(x_n,y_n)]-[H(A(g(u_{n-1})), \\
& B(g(u_{n-1}))-\lambda S(x_{n-1},y_{n-1})]\| \\
\leq & \ \|H(A(g(u_n)),B(g(u_n)))-H(A(g(u_{n-1})),B(g(u_{n-1})))\| \\
& -\lambda\|S(x_n,y_n)-S(x_{n-1},y_{n-1})\| \\
\leq & \ \|H(A(g(u_n)),B(g(u_n)))-H(A(g(u_{n-1})),B(g(u_n)))\| \\
& +\|H(Ag(u_{n-1}),Bg(u_n))-H(Ag(u_{n-1}),Bg(u_{n-1}))\| \\
& +\lambda\|S(x_n,y_n)-S(x_{n-1},y_n)\| \\
& +\lambda\|S(x_{n-1},y_n)-S(x_{n-1},y_{n-1})\|. \qquad (16)
\end{aligned}$$

As $H(A,B)$ is r_1-Lipschitz continuous with respect to A and r_2-Lipschitz with respect to B and g is λ_g-Lipschitz continuous, we have

$$\begin{aligned}
& \|H(A(g(u_n)),B(g(u_n)))-H(A(g(u_{n-1})),B(g(u_{n-1})))\| \\
& +\|H(A(g(u_{n-1})),B(g(u_n))) \\
& -H(A(g(u_{n-1})),B(g(u_{n-1})))\| \\
\leq & \ r_1\lambda_g\|u_n-u_{n-1}\|+r_2\lambda_g\|u_n-u_{n-1}\|. \qquad (17)
\end{aligned}$$

Since S is λ_{s_1}-Lipschitz continuous in the first argument and λ_{s_2}-Lipschitz continuous in the second argument, T is D-Lipschitz continuous with constant δ_T and F is D-Lipschitz continuous with constant δ_F, we have

$$\begin{aligned}
& \|S(x_n,y_n)-S(x_{n-1},y_n)\|+\|S(x_{n-1},y_n) \\
& -S(x_{n-1},y_{n-1})\| \\
\leq & \ \lambda_{s_1}\|x_n-x_{n-1}\|+\lambda_{s_2}\|y_n-y_{n-1}\| \\
\leq & \ \lambda_{s_1}\delta_T\|u_n-u_{n-1}\| \\
& +\lambda_{s_2}\delta_F\|u_n-u_{n-1}\|. \qquad (18)
\end{aligned}$$

Using (17) and (18), (16) becomes

$$\begin{aligned}
\|z_{n+1}-z_n\| & \leq [r_1\lambda_g+r_2\lambda_g+\lambda\lambda_{s_1}\delta_T+\lambda\lambda_{s_2}\delta_F]\|u_n-u_{n-1}\| \\
& = \theta_1\|u_n-u_{n-1}\|, \qquad (19)
\end{aligned}$$

where θ_1=$[(r_1 + r_2)\lambda_g + \lambda(\lambda_{s_1}\delta_T + \lambda_{s_2}\delta_F)]$.
By (i) of Algorithm 3.3, we have

$$\begin{aligned}\|u_n - u_{n-1}\| &= \|u_n - u_{n-1} - g(u_n) + g(u_{n-1}) + R^{H(\cdot,\cdot)-\eta}_{\lambda,M}(z_n) - \\ &\quad R^{H(\cdot,\cdot)-\eta}_{\lambda,M}(z_{n-1})\| \\ &\leq \|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\| \\ &\quad + \|R^{H(\cdot,\cdot)-\eta}_{\lambda,M}(z_n) - R^{H(\cdot,\cdot)-\eta}_{\lambda,M}(z_{n-1})\|. \qquad (20)\end{aligned}$$

Since g is t-strongly monotone and λ_g-Lipschitz continuous,by using technique of Noor [12], it follows that

$$\|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\| \leq \sqrt{1 - 2t + \lambda_g^2}\|u_n - u_{n-1}\|. \quad (21)$$

Using equation (21) and Lipschitz continuity of the resolvent operator $R^{H(\cdot,\cdot)-\eta}_{\lambda,M}$, equation (20) becomes

$$\begin{aligned}\|u_n - u_{n-1}\| &\leq \sqrt{1 - 2t + \lambda_g^2}\|u_n - u_{n-1}\| + \theta\|z_n - z_{n-1}\|, \\ \|u_n - u_{n-1}\| &\leq \frac{\theta}{[1 - \sqrt{1 - 2t + \lambda_g^2}]}\|z_n - z_{n-1}\|. \qquad (22)\end{aligned}$$

Combining (22) with (19), we obtain

$$\begin{aligned}\|z_{n+1} - z_n\| &\leq \frac{\theta\theta_1}{[1 - \sqrt{1 - 2t + \lambda_g^2}]}\|z_n - z_{n-1}\|, \\ \|z_{n+1} - z_n\| &\leq P(\theta)\|z_n - z_{n-1}\|, \qquad (23)\end{aligned}$$

where $P(\theta) = \frac{\theta\theta_1}{1-\sqrt{1-2t+\lambda_g^2}}$, $\theta = \dfrac{\sigma}{[\xi - m\lambda - (\mu\beta^2 + \alpha\gamma^2)]}$
and θ_1=$[(r_1 + r_2)\lambda_g + \lambda(\lambda_{s_1}\delta_T + \lambda_{s_2}\delta_F)]$.

From (15), it follows that $P(\theta) < 1$. Consequently from (23) it follows that $\{z_n\}$ is a cauchy sequence in X and as X is complete, $z_n \longrightarrow z$ as $n \longrightarrow \infty$. From (22), it follows that $\{u_n\}$ is also a cauchy sequence in X such that $u_n \longrightarrow u$ as $n \longrightarrow \infty$. From the D-Lipschitz continuous of

T, F and (ii) and (iii) of Algorithm 3.3, we know that $\{x_n\}$ and $\{y_n\}$ are also cauchy sequences in X such that $x_n \longrightarrow x$ and $y_n \longrightarrow y$, as $n \longrightarrow \infty$. Further, we show that $x \in T(u)$, we have

$$\begin{aligned} d(x, T(u)) &\leq \|x - x_n\| + d(x_n, T(u)) \\ &\leq \|x - x_n\| + D(T(u_n), T(u)) \\ &\leq \|x - x_n\| + \delta_T \|u_n - u_{n-1}\| \longrightarrow 0, \quad as \quad n \longrightarrow \infty. \end{aligned}$$

which implies that $d(x, T(u)) = 0$, it follows that $x \in T(u)$. Similarly we can show that $y \in F(u)$.
Since H, A, B, g, T, F and S all are continuous and by (iv) of Algorithm 3.3, it follows that

$$\begin{aligned} z_{n+1} = H(A(g(u_n)), B(g(u_n))) - \lambda S(x_n, y_n), \\ \longrightarrow z = H(A(g(u)), B(g(u))) - \lambda S(x, y). \end{aligned} \tag{24}$$

Consequently,

$$R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(z_n) = g(u_n) \longrightarrow g(u) = R_{\lambda,M}^{H(\cdot,\cdot)-\eta}(z), as \quad n \longrightarrow \infty. \tag{25}$$

From (24), (25) and by Lemma 3.2, the result follows. □

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