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Original Research Paper

Evaluating DMUs Efficiency with Uncertain Data and Minimized Undesirable Outputs via Belief Degree

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Abstract. Assessing decision-making units frequently involves managing undesirable outputs that impair performance. While reducing these outputs improves efficiency, their total removal is infeasible in practice. This research presents two models: the first determines the minimum unavoidable undesirable outputs while keeping other inputs and outputs constant, and a second one that measures efficiency under uncertainty. Most current models depend on exact data, yet actual situations often include uncertainties when information is only accessible through expert opinions. Recognizing that such information inherently contains inexactness, Liu's uncertainty theory effectively manages this imprecise data. Using this axiomatic foundation, employing the directional distance function with individual proportion weak disposability, we suggest an uncertain framework for decision-making units. Unlike earlier models, constrained to either radial or non-radial forms, our technique flexibly allows both radial and non-radial efficiency calculations, contracting inputs and undesirable outputs while expanding desirable outputs. We

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transform uncertain models into a deterministic counterpart using belief degrees as a solution method. Our examination of the impact of desirable and undesirable outputs on inefficiency delivers superior detail, exceeding prior research. The results indicate that greater belief degrees increase efficiency, differing from deterministic outcomes. We confirm our method by evaluating the environmental efficiency of renewable energy in 20 OECD countries in 2020. This framework performs well in handling data imprecision, assisting policymakers in optimizing sustainability and resource utilization.

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Keywords and Phrases: Uncertainty theory, Belief degree, Efficiency, Weak disposability, Minimum undesirable outputs.

1 Introduction

Continuous performance improvement is vital for organizational growth and requires reliable evaluation process. Data Envelopment Analysis (DEA) is a widely recognized technique for evaluating the efficiency with which Decision-Making Units (DMUs) convert inputs into outputs [1].

DEA has been studied from different points of view, including inverse DEA models with ratio data [30], extending Free Disposal Hull (FDH) techniques for non-convex cases, and optimizing input-output estimation while preserving efficiency [24]. Applications include efficiency assessments in Taiwanese medical centers, rail transportation ranking [26]. However, DMUs often produce undesirable outputs alongside desirable ones, which may have side effects such as environmental damage, ecological degradation, and negative environmental impact. For example, environmental pollution resulting from energy generation through waste incineration is one such undesirable output [10]. Additionally, some undesirable outputs may harm human health by chemical pollutants, particulate matter in air pollution, and certain naturally occurring elements. These issues are thereby considered as efficiency reducing reasons.

To address this issue, Shephard introduced the weak disposability concept, proposing that undesirable outputs should be contracted in proportion to desirable outputs [17]. This concept led to two models. The first one, so called the common proportion, introduced by Fare and Grosskopf in 2003 [12] that applies a single contraction factor to all

DMUs. Kuosmanen criticized this approach, arguing that it failed to target abatement efforts towards businesses with lower costs, and consequently proposed another approach known as individual proportion model [15]. Later, Kuosmanen and Podinovski highlighted the limitations of the common proportion model, and demonstrated that Kuosmanen's production possibility set offers a more reliable framework by ensuring that the smallest convex hull satisfies production principles [36].

Although undesirable outputs can be reduced, their complete elimination is not feasible in practice. Following the above-mentioned approaches, Kao and Huang employed the Slack-Based measure Model (SBM) with common proportion weak disposability for efficiency assessment. This model utilizes a uniform contraction factor across all DMUs, disregarding input variations [28]. They also proposed an individual-proportion weak disposability model; however, the non-radial SBM model led to intricate, time-intensive solutions and positive shadow prices [33]. In addition to these challenges, Jian-Xin Ma et al. [18] highlighted further limitations of the SBM model. Additional researchers, such as Kuosmanen [15] and Maghbouli et al. [19], have attempted to minimize undesirable outputs; however, these obstacles still remain in their approaches.

In addition to the prior approaches for evaluating DMUs with minimum undesirable outputs, pioneered by Chambers et al. [2], researchers have employed the directional distance function (DDF) along with individual-proportion weak disposability to simultaneously expand desirable outputs and contract undesirable ones. Its linear objective function, flexible optimization directions, independent contraction, and expansion factors for each input and output enable both radial and non-radial analyses, permitting DMU efficiency assessments while minimizing unavoidable undesirable outputs. However, earlier models, such as those by Karagiannis and Kourtzidis [16], were hindered by radial structures, uniform abatement and expansion factors, which reduced their flexibility in complex scenarios. To address these challenges, Pourmahmoud and Radfar proposed an innovative DDF model with individual-proportion weak disposability [25]. Their approach permits efficiency evaluation of DMUs with minimum unavoidable levels of undesirable outputs and supporting both input-oriented and output-oriented analyses while preserving non-radial properties [25]. It is important to emphasize that despite these advancements, their models depend on precise data and overlook the uncertainties inherent in real-world applications.

Traditional DEA models assume precise DMU data, but real-world uncertainties require resilient approaches. Stochastic DEA [23] uses probabilistic distributions for efficiency evaluation under randomness, though accurate distributional assumptions are challenging in data-scarce settings. Interval DEA, proposed by Cooper et al. [27], uses ranges for imprecise data but cannot capture value likelihoods, limiting its effectiveness. Sengupta's fuzzy DEA [32] employs fuzzy set theory for flexibility. Wen and Li [34] integrated fuzzy simulation and genetic algorithms, while Khoshfetrat and Daneshvar [11] enhanced fuzzy CCR models with lower bounds for inputs and outputs. However, Soleimani-Damaneh et al. [31] noted inconsistent results due to fuzzy number ambiguity. Resilient DEA models using intervals or ellipsoids optimize worst-case scenarios but face high computational complexity and conservative scores.

Data uncertainty extends beyond fuzzy, interval, and probabilistic classification. In many cases, reliable probability distributions cannot be estimated because of limited historical data, high costs, or impractical data collection. Such uncertainty poses significant challenges, as illustrated during the COVID-19 pandemic, when healthcare professionals and authorities struggled with lack of precise data. The sudden onset of the disease provided no historical data for analysis, rendering traditional DEA models inadequate for addressing real-world decision-making needs. This kind of events are prevalent globally; crises like earthquakes, volcanoes, pandemics, and wars. In such scenarios, constructing probability distributions becomes infeasible, and expert opinions based on available evidence are often utilized. Liu's uncertainty theory [22] offers a mathematical framework in dealing with these cases, applying expert opinions and available evidence to tackle uncertainty effectively. This approach provides a critical alternative when probabilistic, interval-based, or fuzzy methods are impractical or infeasible.

Uncertainty theory has been applied in various fields; however, few studies have addressed DEA evaluation under uncertainty. Wen et al. in 2014 introduced an additive model using the maximum belief degree to convert uncertain data into a deterministic form [33]. Mohammadnejad and Ghafari-Hadigheh employed the maximum belief degree and set bounds for the objective to convert the uncertain model into its deterministic counterpart [8]. Lio and Liu proposed a model that assesses DMU efficiency using the expected value to transform uncertainty into a deterministic framework [20]. However, prior studies, such as Pourmahmoud and Radfar [25], relied on deterministic data for efficiency evaluation, limiting their applicability in uncertain environments.

The contribution of the proposed method lies in integrating our novel approach to the DDF model with individual proportion weak disposability and Liu's uncertainty theory. We developed this approach to employ the maximum belief degree method, which effectively addresses data variability in uncertain environments. This method enables the use of standard optimization techniques to evaluate DMUs while minimizing undesirable outputs. It also identifies inefficiencies in both desirable and undesirable outputs and provides targeted strategies for improvement, leveraging higher belief degrees to enhance efficiency assessments, which yields results distinct from deterministic evaluations. Consequently, it equips policymakers with effective tools to optimize resource allocation and select technologies in complex and uncertain environments.

The remainder of this paper is organized as follows: Section 2 covers key concepts, including uncertainty theory, minimum undesirable outputs, and the impact of desirable and undesirable outputs on DMU inefficiency. Section 3 applies the DDF model with individual-proportion weak disposability to uncertain data and converts it into a deterministic counterpart using the belief degrees. This section also examines how variations in belief influence the efficiency of the DMU. Section 4 provides an example for model validation. Section 5 applies the proposed models to a real-world scenario. Finally, Section 6 concludes the study.

2 Preliminary Concepts

2.1 Uncertainty theory

An uncertainty space is defined by the triple $(\Gamma, L, \mathcal{M})$, where $\mathcal{M}$ is a function on the σ-algebra L over a non-empty universal set Γ [15]. The measure $\mathcal{M}$ must satisfy the following four axioms. In 2009, Liu added the fourth axiom known as a product uncertainty measure [16]. The axioms are as follows:

Axioms 1 (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.

Axioms 2 (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.

Axioms 3 (Subadditivity Axiom) $\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\left\{\Lambda_i\right\}$, for each countable sequence of events $\Lambda_1,\ \Lambda_2, \ldots$.

Axioms 4 (Product Axiom) Assume that $(\Gamma_k, L_k, \mathcal{M}_k)$ $k = 1, 2, \ldots$ are uncertainty spaces. The product uncertainty measure $\mathcal{M}$ is an uncertain measure that satisfies $\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_i\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}\left\{\Lambda_i\right\}$, where Λ_k is an event in L_k, $k = 1, 2, \ldots$.

Each member of L is referred to as an uncertain event. The fundamental definitions and concepts of uncertainty theory, necessary for our study, are as follows [15].

An uncertain variable ξ is a measurable function ξ from the uncertainty space $(\Gamma, L, \mathcal{M})$ to the set of real numbers, such that for any Borel set of real numbers, the set $\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$ represents an event. Information on uncertain variables is provided using an uncertainty distribution. For any real number x, the uncertainty distribution Φ for an uncertain variable ξ is defined as $\Phi(x) = \mathcal{M}\{\xi \leq x\}$. To determine the inverse of the uncertainty distribution function $\Phi^{-1}(\alpha)$, the uncertainty distribution must be regular. A regular uncertainty distribution $\Phi(x)$ is a continuous and strictly increasing function with respect to x, that satisfies [17]:

$$\lim_{x \to -\infty} \Phi(x) = 0, \quad \lim_{x \to +\infty} \Phi(x) = 1, \quad 0 < \Phi(x) < 1.$$

A set of uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ is considered independent if, for any collection of Borel sets $B_1, B_2, \ldots, B_n$ of real numbers, the following equality holds [16]:

$$\mathcal{M}\left\{\bigcap_{i=1}^{n} \Lambda_i^*\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\left\{\Lambda_i^*\right\},$$

where Λ_i^* is arbitrarily chosen from $\{\Lambda_i, \Lambda_i^c, \Gamma\}$, and Γ is the universal set.

In practical applications, uncertain variables may have different distribution functions like linear, zigzag, normal, and empirical. This study specifically considers uncertain variables having linear uncertain distribution functions, which are defined as follows. Suppose a and b are real numbers with $a < b$. The linear uncertain variable, denoted by $L(a,b)$, is defined with the following uncertainty distribution:

$$\Phi(x) = \begin{cases} 0 & x \le a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ 1 & x \ge b. \end{cases}$$

The inverse uncertainty distribution of $L(a,b)$ is $\Phi^{-1}(\alpha) = (1-\alpha)a + \alpha b$.

Based on specific assumptions about the type of function and properties of these independent uncertain variables, the inverse uncertainty distribution of independent uncertain variables is given by the following theorem.

Theorem 2.1 ([17]). *Assume that $\Phi_1, \Phi_2, \ldots, \Phi_n$ are regular uncertainty distributions of independent uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$. If the function $f(\xi_1, \xi_2, \ldots, \xi_n)$ strictly increases with respect to $\xi_1, \xi_2, \ldots,$ ξ_m and strictly decreases with respect to $\xi_{m+1}, \xi_{m+2}, \ldots, \xi_n$, then the uncertain variable $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has the following inverse uncertainty distribution,*

$$\Psi^{-1}(\alpha) = f\left(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)\right).$$

Theorem 2.2 ([15]). *Assume that $\Phi_1, \Phi_2, \ldots, \Phi_n$ are regular uncertainty distributions of independent uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$. If the function $f(\xi_1, \xi_2, \ldots, \xi_n)$ strictly increases with respect to $\xi_1, \xi_2, \ldots,$ ξ_m and strictly decreases with respect to $\xi_{m+1}, \xi_{m+2}, \ldots, \xi_n$, then the chance constraint*

$$\mathcal{M}\{f(\xi_1, \xi_2, \ldots, \xi_n) \le 0\} \ge \alpha,$$

holds if and only if

$$f\left({\Phi_1}^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), {\Phi_{m+1}}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)\right) \le 0.$$

Theorem 2.3 ([33]). *Assume that $\Phi_{1i},\ \Phi_{2i},\ldots,\Phi_{ni}$ are uncertainty distributions of independent uncertain inputs $\tilde{x}_{1i},\tilde{x}_{2i},\ldots,\tilde{x}_{ni}$ for $i=1,2,\ldots,p$, and $\Psi_{1j},\Psi_{2j},\ldots,\Psi_{nj}$ are the uncertainty distribution of independent uncertain outputs $\tilde{y}_{1i},\tilde{y}_{2i},\ldots,\tilde{y}_{ni}$ for $j=1,2,\ldots,q$. Then*

$$\mathcal{M}\left\{\sum_{k=1}^{n}\tilde{x}_{ki}\lambda_k \le \tilde{x}_{oi}-{s_i}^{-}\right\}\ge\alpha, \qquad i=1,\ldots,p,$$
$$\mathcal{M}\left\{\sum_{k=1}^{n}\tilde{y}_{kj}\lambda_k \ge \tilde{y}_{oj}+{s_j}^{+}\right\}\ge\alpha, \qquad j=1,\ldots,q,$$

holds if and only if

$$\sum_{k=1,k\neq o}^{n}\lambda_k\ \Phi_{ki}^{-1}(\alpha)+\lambda_o\Phi_{oi}^{-1}(1-\alpha)\le\Phi_{oi}^{-1}(1-\alpha)-{s_i}^{-};\quad i=1,\ldots,p,$$
$$\sum_{k=1,k\neq o}^{n}\lambda_k\Psi_{kj}^{-1}(1-\alpha)+\ \lambda_o\Psi_{oj}^{-1}(\alpha)\ge\Psi_{oj}^{-1}(\alpha)+{s_j}^{+};\qquad j=1,\ldots,q.$$

2.2 Minimum undesirable outputs with certain data

As mentioned earlier, Kuosmanen introduced the individual-proportion weak disposability production possibility set T^{IP} under variable returns to scale and reformulated it through an appropriate variable transformation, as follows:

$$T^{IP}=\left\{(\boldsymbol{x},\ \boldsymbol{y},\boldsymbol{u})\ \middle|\ \begin{array}{ll}\sum_{j=1}^{n}(\mu_j+\theta_j)\ x_{ij}\ \le\ x_i; & i=1,\ldots,m,\\ \sum_{j=1}^{n}\mu_j\ y_{rj}\ge\ y_r; & r=1,\ldots,s,\\ \sum_{j=1}^{n}\mu_j\ u_{fj}=\ u_f; & f=1,\ldots,h,\\ \sum_{j=1}^{n}(\mu_j+\theta_j)=1, & \\ \mu_j\ge 0\ ,\quad \theta_j\ge 0\ ; & j=1,\ldots,n.\end{array}\right\}. \tag{1}$$

Here, $\boldsymbol{x} = (x_1, x_2, \ldots, x_m)$, $\boldsymbol{y} = (y_1, y_2, \ldots, y_s)$, and $\boldsymbol{u} = (u_1, u_2, \ldots, u_h)$ represent the inputs, desirable outputs, and undesirable outputs, respectively. To calculate the contraction coefficient for undesirable outputs while maintaining constant levels of inputs and desirable outputs, Kuosmanen employed (1) to propose the following model for DMU_k [12].

$$\begin{array}{lll} \min & \rho & \\ \text{s.t.} & \sum_{j=1}^{n} (\mu_j + \theta_j) x_{ij} \ \leq x_{ik}; & i = 1, \ldots, m, \\ & \sum_{j=1}^{n} \mu_j \ y_{rj} \geq \ y_{rk}; & r = 1, \ldots, s, \\ & \sum_{j=1}^{n} \mu_j \ u_{fj} = \ \rho u_{fk}; & f = 1, \ldots, h, \\ & \sum_{j=1}^{n} (\mu_j + \theta_j) = \ 1, & \\ & \mu_j \geq 0, \quad \theta_j \geq 0 \ ; & j = 1, \ldots, n. \\ & \rho \geq 0. & \end{array} \tag{2}$$

Let ρ^* denote the optimal value of model (2) which represents the maximum contraction coefficient of undesirable outputs while maintaining constant levels of inputs and desirable outputs. The minimum unavoidable level of undesirable outputs in the production of desirable outputs can be defined as: $u^*_{fk} = \rho^* u_{fk}; \quad f = 1, \ldots, h$. It is obvious that $u_{fk} \geq u^*_{fk}$.

2.3 Impact of desirable and undesirable outputs on DMU inefficiency

The evaluation of DMUs indicates that outputs significantly influence performance; higher desirable outputs improve efficiency, whereas higher undesirable outputs worsen it. To analyze the impact of outputs on DMU inefficiency, Kao and Hwang [8] decomposed inefficiency $1 - E_k$ as $(1 - E_k) = (1 - E^*_k) + (E^*_k - E_k)$, where E_k and E^*_k denote the efficiencies of the evaluated DMU_k with undesirable and minimum undesirable outputs, respectively. By dividing both sides to $1 - E_k$, they derived the following equality.

$$1 = \frac{(1-E_k^*)}{(1-E_k)} + \frac{(E_k^* - E_k)}{(1-E_k)}. \tag{3}$$

The terms $\frac{(1-E_k^*)}{(1-E_k)}$ and $\frac{(E_k^*-E_k)}{(1-E_k)}$ represent the impact of inefficiency due to shortfall of desirable outputs and additional undesirable outputs, respectively. By analyzing these components, decision-makers can analyze how each type of output changes a DMU's performance.

3 Proposed Models

3.1 Efficiency with uncertain undesirable outputs

In this section, we address the inherent uncertainty in the data related to the DMUs. By considering inputs, desirable and undesirable outputs as uncertain variables, we aim to evaluate the DMU efficiencies under variable returns to scale. Assume that $\tilde{x}_{ij}, i = 1, \dots, m,\ \tilde{y}_{rj},\ r = 1, \dots, s$, and $\tilde{u}_{fj},\ f = 1, \dots, h$, represent the inputs, desirable and undesirable outputs of $\text{DMU}_j,\ j = 1, \cdots n$, respectively. To evaluate these DMUs, we propose a DDF model incorporating individual proportion weak disposability. Our proposed model is presented as follows:

$$\begin{aligned}
\max \quad & \frac{1}{m+s+h}\Big(\sum_{i=1}^{m} \zeta_i + \sum_{r=1}^{s} \beta_r + \sum_{f=1}^{h} \delta_f \Big) & \\
\text{s.t.} \quad & \sum_{j=1}^{n} (\mu_j + \theta_j)\tilde{x}_{ij} \leq \tilde{x}_{ik} - \zeta_i d_i^x; & i = 1, \dots, m, \\
& \sum_{j=1}^{n} \mu_j\ \tilde{y}_{rj} \geq \tilde{y}_{rk} + \beta_r d_r^y; & r = 1, \dots, s, \\
& \sum_{j=1}^{n} \mu_j\ \tilde{u}_{fj} \leq \tilde{u}_{fk} - \delta_f d_f^u\ ; & f = 1, \dots, h, \\
& \sum_{j=1}^{n} (\mu_j + \theta_j) = 1, & \\
& \mu_j \geq 0,\ \theta_j \geq 0; & j = 1, \dots, n,
\end{aligned} \tag{4}$$

where $\boldsymbol{d^x} = [d_1^x,\ d_2^x, \ldots, d_m^x]^T$, $\boldsymbol{d^y} = [d_1^y,\ d_2^y, \ldots, d_s^y]^T$, and $\boldsymbol{d^u} = [d_1^u,\ d_2^u, \ldots, d_h^u]^T$ represent the predetermined directions for the input, desirable, and undesirable outputs of the evaluated DMU, respectively. In this model, due to the presence of variables $\zeta_i;\ i = 1, \ldots, m,\ \ \beta_r;\ r = 1, \ldots, s$ and $\delta_f;\ f = 1, \ldots, h$, desirable outputs are independently expanded, whereas inputs and undesirable outputs are separately contracted.

When simultaneous changes to the inputs and outputs are impossible, the model must be either input-oriented or output-oriented. If the evaluation focuses on inputs, an input-oriented model is used by setting d_r^y and d_f^u to zero in (4). Conversely, for an output-oriented approach, setting d_i^x to zero transforms the model accordingly.

To calculate the efficiency of the described units using model (4), we apply the following definition.

Definition 3.1. *Suppose $\zeta_i^*;\ i = 1, \ldots, m,\ \ \beta_r^*;\ r = 1, \ldots, s$ and $\delta_f^*;\ f = 1, \ldots, h$, represent the optimal solution of model (4). Efficiency of* DMU_k *is defined as*

$$E_k = \frac{1 - \dfrac{1}{m+h}\Big(\sum\limits_{i=1}^{m} \zeta_i^* + \sum\limits_{f=1}^{h} \delta_f^*\Big)}{1 + \frac{1}{s}\sum\limits_{r=1}^{s} \beta_r^*}. \tag{5}$$

In the evaluation of DMU_k*, two scenarios may be observed. (a)* DMU_k *is efficient if the optimal value of (4) is zero. In this case, $E_k = 1$. (b)* DMU_k *is inefficient if the optimal value of (4) is nonzero and $0 < E_k < 1$.*

To assess DMUs using any proposed model, it is crucial to extract the variable values from the model solution, as they are pivotal indicators of DMU's performance. Ensuring the feasibility and boundedness of the suggested model is necessary to determine the values of these variables. In Theorem 3.3, these properties will be proved for model (4). This ensures that the problem has an optimal solution.

3.2 Efficiency with minimum uncertain undesirable outputs

In scenarios involving uncertain data, undesirable outputs would have a significant impact on efficiency, similar to deterministic cases. To address this challenge, we aim to minimize undesirable outputs under variable returns to scale. To achieve this goal, we employ the DDF model with individual proportion weak disposability and propose the following model.

$$
\begin{aligned}
\min \quad & \frac{1}{h}\sum_{f=1}^{h}\delta_f \\
\text{s.t.} \quad & \sum_{j=1}^{n}(\mu_j+\theta_j)\tilde{x}_{ij} \leq \tilde{x}_{ik}; && i=1,\dots,m, \\
& \sum_{j=1}^{n}\mu_j\tilde{y}_{rj} \geq \tilde{y}_{rk}; && r=1,\dots,s, \\
& \sum_{j=1}^{n}\mu_j\tilde{u}_{fj} \leq \tilde{u}_{fk}-\delta_f d_f^u; && f=1,\dots,h, \\
& \sum_{j=1}^{n}(\mu_j+\theta_j)=1, \\
& \mu_j \geq 0, \qquad \theta_j \geq 0; && j=1,\dots,n, \\
& \delta_f \geq 0\ ; && f=1,\dots,h,
\end{aligned}
\tag{6}
$$

where $\delta_f^*;\ f=1,\dots,h$ represent the optimal solution corresponding to the f-th undesirable output of each DMU. According to this solution, relation ($\tilde{u}_{fk}^* = \tilde{u}_{fk} - \delta_f^* d_f^u$) is employed to determine the minimum undesirable output of each DMU, with $\delta_f^* d_f^u$ indicating the additional undesirable output values. For $f=1,\dots,h$, it is clear that $\tilde{u}_{fk} \geq \tilde{u}_{fk}^*$.

To evaluate the efficiency of DMU_k with minimum undesirable outputs (E_k^*), we modified model (4) by substituting $\tilde{u}_{fj}^*;\ f=1,\dots,h$, $j=1,\dots,n$ with $\tilde{u}_{fj};\ f=1,\dots,h,\ j=1,\dots,n$. This modification resulted in the following updated model, optimized for computing efficiency with minimum undesirable outputs.

$$
\begin{aligned}
\max \quad & \frac{1}{m+s+h}\left(\sum_{i=1}^{m}\zeta_i+\sum_{r=1}^{s}\beta_r+\sum_{f=1}^{h}\delta_f\right) \\
\text{s.t.} \quad & \sum_{j=1}^{n}(\mu_j+\theta_j)\tilde{x}_{ij}\leq \tilde{x}_{ik}-\zeta_i d_i^x; && i=1,\dots,m, \\
& \sum_{j=1}^{n}\ \mu_j\ \tilde{y}_{rj}\geq \tilde{y}_{rk}+\beta_r d_r^y; && r=1,\dots,s, \\
& \sum_{j=1}^{n}\ \mu_j\ \tilde{u}^*_{fj}\leq \tilde{u}^*_{fk}-\delta_f d_f^u\ ; && f=1,\dots,h, \\
& \sum_{j=1}^{n}(\mu_j+\theta_j)=\ 1, \\
& \mu_j\geq 0,\ \theta_j\geq 0; && j=1,\dots,n.
\end{aligned}
\tag{7}
$$

Model (7) ensures that the minimum contractions and maximum expansions are respectively applied to each input and output individually. Let $\zeta_i^{**};\ i=1,\dots,m,\ \ \beta_r^{**};\ r=1,\dots,s$ and $\delta_f^{**};\ f=1,\dots,h$, represent an optimal solution of (7). By utilizing this optimal solutions and applying the efficiency definition of DMU$_k$ from (5), E_k^* can be computed, where it is evident that $E_k^*\geq E_k$. The influence of undesirable outputs on the inefficiency of DMU$_k$ can be analyzed using (3) in conjunction with (4) and (7). In summary, model (4) valuates the DMUs with all undesirable outputs. model (6) minimizes undesirable outputs by focusing solely on δ_f maintaining inputs and desirable outputs at current levels through its constraints, and achieving minimum undesirable output values, $\tilde{u}^*_{fj}$. Model (7) extends model (4) by incorporating $\tilde{u}^*_{fj}$ to assess DMUs with minimum undesirable outputs.

When uncertain inputs and outputs cannot be modified simultaneously, the non-radial model (7) is not applicable, requiring the use of input- oriented or output-oriented models. Analogous models can be generated similar to the transformation of model (4).

The proposed models involve elements of uncertainty which are unsolvable directly. To overcome this limitation, equivalent models were formulated. These models must meet specific conditions, including the linearity of the objective function, constraints associated with uncertain parameters, independence, feasibility and boundedness, and regularity of uncertain variables distribution, as outlined in the following theorems.

Theorem 3.2. *let for DMU_j, $j = 1, \cdots, n$, $\tilde{x}_{ij}$, $i = 1, \ldots, m$, $\tilde{y}_{rj}, r = 1, \ldots, s$, and $\tilde{u}_{fj}$, $f = 1, \ldots, h$, represent the independent uncertain inputs, desirable outputs, and undesirable outputs, with regular uncertainty distributions Φ_{ij}, Ψ_{rj}, and Ψ'_{fj}, respectively. The equivalent crisp forms of models (6) and (7) are as follows.*

$$
\begin{aligned}
\min \quad & \frac{1}{h}\sum_{f=1}^{h}\delta_f \\
\text{s.t.} \quad & \sum_{j=1, j\neq k}^{n} (\mu_j+\theta_j)\Phi_{ij}^{-1}(\alpha) + (\mu_k+\theta_k)\,\Phi_{ik}^{-1}(1-\alpha) \le \Phi_{ik}^{-1}(1-\alpha)\,; \\
& \quad i = 1, \ldots, m, \\
& \sum_{j=1, j\neq k}^{n} \mu_j \Psi_{rj}^{-1}(1-\alpha) + \ \mu_k \Psi_{rk}^{-1}(\alpha) \ge \Psi_{rk}^{-1}(\alpha)\,; \\
& \quad r = 1, \ldots, s, \\
& \sum_{j=1, j\neq k}^{n} \mu_j \ {\Psi'}_{fj}^{-1}(\alpha) + \mu_k {\Psi'}_{fk}^{-1}(1-\alpha) \le {\Psi'}_{fk}^{-1}(1-\alpha) - \delta_f d_f^u; \\
& \quad f = 1, \ldots, h, \\
& \sum_{j=1}^{n} (\mu_j+\theta_j) = \ 1, \\
& \mu_j \ge 0, \ \theta_j \ge 0; \qquad j = 1, \ldots, n, \\
& \delta_f \ge 0\ ; \qquad f = 1, \ldots, h,
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
\max \quad & \frac{1}{m+s+h}\left(\sum_{i=1}^{m} \zeta_i + \sum_{r=1}^{s} \beta_r + \sum_{f=1}^{h} \delta_f \right) \\
\text{s.t.} \quad & \sum_{j=1, j\neq k}^{n} \left(\mu_j + \theta_j\right) \Phi_{ij}^{-1}(\alpha) + \left(\mu_k + \theta_k\right) \Phi_{ik}^{-1}(1-\alpha) \leq \Phi_{ik}^{-1}(1-\alpha) \\
& \qquad -\zeta_i d_i^x; i = 1, \ldots, m, \\
& \quad \sum_{j=1, j\neq k}^{n} \mu_j \Psi_{rj}^{-1}(1-\alpha) + \mu_k \Psi_{rk}^{-1}(\alpha) \geq \Psi_{rk}^{-1}(\alpha) + \beta_r d_r^y; \\
& \qquad r = 1, \ldots, s, \\
& \quad \sum_{j=1, j\neq k}^{n} \mu_j \, {\Psi'}_{fj}^{-1}(\alpha) + \mu_k {\Psi'}_{fk}^{-1}(1-\alpha) \leq {\Psi'}_{fk}^{-1}(1-\alpha) - \delta_f d_f^u \; ; \\
& \qquad f = 1, \ldots, h, \\
& \quad \sum_{j=1}^{n} \left(\mu_j + \theta_j\right) = 1, \\
& \quad \mu_j \geq 0, \; \theta_j \geq 0; \quad j = 1, \ldots, n.
\end{aligned} \tag{9}
$$

Proof. Given the uncertainty in the data for model (9), it is crucial to determine the inverse distribution function of the uncertain variables to derive an equivalent model. To do this, suppose

$$
\begin{aligned}
f_i^1\left(\mu, \theta, \tilde{x}_i, \tilde{x}_{ik}, \zeta_i, d_i^x\right) &= \sum_{j=1}^{n} \left(\mu_j + \theta_j\right) \tilde{x}_{ij} - \tilde{x}_{ik} + \zeta_i d_i^x; \quad i = 1, \ldots, m, \\
f_r^2\left(\mu, \tilde{y}_r, \tilde{y}_{rk}, \beta_r, d_r^y\right) &= -\sum_{j=1}^{n} \mu_j \tilde{y}_{rj} + \tilde{y}_{rk} - \beta_r d_r^y; \qquad r = 1, \ldots, s, \\
f_f^3\left(\mu, \tilde{u}_f^*, \tilde{u}_{fk}^*, \delta_f, d_f^u\right) &= \sum_{j=1}^{n} \mu_j \, \tilde{u}_{fj}^* - \tilde{u}_{fk}^* + \delta_f d_f^u; \qquad f = 1, \ldots, h.
\end{aligned}
$$

The function f_i^1 is increasing with respect to $\tilde{x}_{ij}$ and decreasing with respect to $\tilde{x}_{ik}$. Function f_r^2 is decreasing with respect to $\tilde{y}_{rj}$ and decreasing with respect to $\tilde{y}_{rk}$. Analogously, function f_f^3 is increasing with respect to $\tilde{u}_{fj}^*$ and decreasing with respect to $\tilde{u}_{fk}^*$. According to Theorem 2.3, inverse distribution of these functions are as follows.

$$\left[\boldsymbol{F}_i^1\right]^{-1}(\boldsymbol{\alpha}) = \sum_{j=1, j\neq k}^{n} (\mu_j + \theta_j)\ \Phi_{ij}^{-1}(\alpha) + (\mu_k + \theta_k)\,\Phi_{ik}^{-1}(1-\alpha) - \Phi_{ik}^{-1}(1-\alpha) + \zeta_i d_i^x, \tag{10}$$

$$\left[\boldsymbol{F_r^2}\right]^{-1}(\boldsymbol{\alpha}) = -\sum_{j=1, j\neq k}^{n} \ \mu_j \Psi_{rj}^{-1}(1-\alpha) - \ \mu_k \Psi_{rk}^{-1}(\alpha) + \Psi_{rk}^{-1}(\alpha) - \beta_r d_r^y, \tag{11}$$

$$\left[\boldsymbol{F_f^3}\right]^{-1}(\boldsymbol{\alpha}) = \sum_{j=1, j\neq k}^{n} \ \mu_j \ \Psi'^{-1}_{fj}(\alpha) + \mu_k \Psi'^{-1}_{fk}(1-\alpha) - \Psi'^{-1}_{fk}(1-\alpha) + \delta_f d_f^u. \tag{12}$$

According to Theorem 2, model (9) can be derived using (10)-(12). Sine the objective function is deterministic and unaffected by uncertain data directly, it remains unchanged. A similar proof can be applied to derive model (8). □

Theorem 3.3. *Models* (8) *and* (9) *are feasible and bounded.*

Proof. Feasibility: Consider the evaluated DMU_k, and the hypothetical improvement direction $\overrightarrow{\boldsymbol{d}} = (d_i^x, d_r^y, d_f^u)$, where d_i^x, d_r^y, d_f^u represents arbitrary directions with positive values. To prove the feasibility of (9), it suffices to provide a solution for the variables that satisfy all the constraints of the model. Considering

$$\zeta_i = \beta_r = \delta_f = 0\ , \qquad \theta_k = 0, \ \ \mu_j = 0, \ \ j \neq k, \ \ \ \mu_k = 1,$$

they satisfy the constraints of model (9), demonstrating its feasibility.

Boundedness: To prove the boundedness of model (9), it is necessary to ensure that the variables (ζ, β, δ) and objective function remain within reasonable bounds under uncertainty. From the last constraint of model (9), we have $0 \leq \mu_j + \theta_j \leq 1$, and can be rewritten as

$$-1 \leq \mu_k + \theta_k - 1 \leq 0.$$

On the other hand, inverse uncertainty distribution exists in the open interval $(0, 1)$, as established in [15]. Assume that there exist finite numbers M and M' in $(0, 1]$ such that, for all $j \neq k$ and specific confidence level $(\alpha \neq 0, 1)$, the following condition holds:

$$M' \leq \Phi_{ij}^{-1}(\alpha) \leq M, \tag{13}$$

and for $j = k$

$$M' \leq \Phi_{ik}^{-1}(1-\alpha) \leq M. \tag{14}$$

According to the fourth constraint of model (4), given by $\sum_{j=1}^{n}(\mu_j + \theta_j) = 1$, three distinct cases are considered to examine the model's boundedness.

Case 1: $\sum_{j=1, j\neq k}^{n}(\mu_j + \theta_j) = 1$. In this case $(\mu_k + \theta_k - 1) = -1$. Therefore

$$M' \leq \sum_{j=1, j\neq k}^{n} (\mu_j + \theta_j)\Phi_{ij}^{-1}(\alpha) \leq M, \tag{15}$$

and

$$-M' \leq (\mu_k + \theta_k - 1)\Phi_{ik}^{-1}(1-\alpha) \leq -M'. \tag{16}$$

Summing up (15) and (16), we have

$$M' - M \leq \sum_{j=1, j\neq k}^{n} (\mu_j + \theta_j)\phi_{ij}^{-1}(\alpha) + (\mu_k + \theta_k - 1)\phi_{ik}^{-1}(1-\alpha) \leq M - M', \tag{17}$$

for ; $i = 1, \ldots, m$. Recall that the first constraint of model (9) can be rewritten as

$$M' - M \leq \sum_{j=1, j\neq k}^{n} (\mu_j + \theta_j)\Phi_{ij}^{-1}(\alpha) + (\mu_k + \theta_k - 1)\Phi_{ik}^{-1}(1-\alpha) \leq -\zeta_i d_i^x, \tag{18}$$

for $i = 1, \ldots, m$. From (17), it follows that the left-hand side of (18) is confined to the interval $(M' - M, M - M')$, and given that d_i^x is positive, the variable ζ is bounded from above. Although ζ is a free variable and could potentially be negative, the objective function is maximization; therefore, in optimality, it cannot be negative.

Case 2: $\sum_{j=1,j\neq k}^{n}(\mu_j+\theta_j)=0$. In this case, $(\mu_k+\theta_k-1)=0$. Consequently, the left-hand side of (18) will be at most zero, revealing that ζ is bounded above. Analogous to Case 1, ζ cannot be negative in the optimality.

Case 3: $0<\sum_{j=1,j\neq k}^{n}(\mu_j+\theta_j)<1$, and $-1<\mu_k+\theta_k-1<0$. Multiplying both sides of (13) by $\sum_{j=1,j\neq k}^{n}(\mu_j+\theta_j)$, we have

$$0 \le \sum_{j=1,j\neq k}^{n}(\mu_j+\theta_j)\phi_{ij}^{-1}(\alpha) \le M. \tag{19}$$

Similarly, multiplying both sides of (14) by $\mu_k+\theta_k-1$, we have

$$-M \le (\mu_k+\theta_k-1)\phi_{ik}^{-1}(1-\alpha) \le 0. \tag{20}$$

Using (19) and (20), it follows that the right-hand side of (18) is confined to the interval $(-M,M)$. Because M is finite, it follows that the variable ζ is bounded.

To prove the boundedness of β, consider the rewritten form of the second constraint of model (9) as

$$\beta_r d_r^y \le \sum_{j=1,j\neq k}^{n}\mu_j\psi_{rj}^{-1}(1-\alpha)+(\mu_k-1)\psi_{rk}^{-1}(\alpha); \quad r=1,\dots,s. \tag{21}$$

From the final constraint of model (9), we concluded that for all $j\neq k$, we have

$$0 \le \mu_j \le 1.$$

This relation for evaluating DMU can be rewritten as

$$-1 \le \mu_k - 1 \le 0.$$

Similar to (13) and (14), the following relations also hold for the inverse uncertainty distribution ψ^{-1}.

$$\forall j\neq k; \qquad M' \le \psi_{rj}^{-1}(1-\alpha) \le M, \tag{22}$$

and, for $j=k$, we have

$$M' \le \psi_{rk}^{-1}(\alpha) \le M. \tag{23}$$

When both sides of (22) are multiplied by μ_j, we have

$$0 \leq \mu_j \psi_{rj}^{-1}(1-\alpha) \leq M.$$

Consequently,

$$0 \leq \sum_{j=1, j\neq k}^{n} \mu_j \psi_{rj}^{-1}(1-\alpha) \leq M(n-1). \tag{24}$$

On the other hand, when both sides of (23) are multiplied by $\mu_k - 1$, we have

$$-M \leq (\mu_k - 1)\psi_{rk}^{-1}(\alpha) \leq 0. \tag{25}$$

From (24) and (25), it follows that the right-hand side of (21) is at most $M(n-1)$. Positivity of d_r^y proves the boundedness of β from above. Analogous to the first constraint in (9), β is positive in optimality. The proof for boundedness of δ is similar to ζ.

Model (9) is feasible, as demonstrated by the provided solution, and bounded, as the variables ζ, β, and δ are confined within finite intervals due to the constraints and finite bounds M' and M. The proof is complete. □

Lemma 3.4. *Models (6) and (7) are feasible and bounded.*

Proof. As established in Theorem 3.2, models (6) and (7), are equivalent to their deterministic counterparts, models (8) and (9), respectively. Consequently, models (6) and (7) inherit the properties of feasibility and boundedness from their deterministic equivalents. □

Theorem 3.5. *Let α_1 and α_2 be confidence levels in an uncertain environment with $\alpha_1 < \alpha_2$. Consider $E_{\alpha_1}^k$ and $E_{\alpha_2}^k$ as the efficiency values of model (9) at confidence levels α_1 and α_2, respectively. Then $E_{\alpha_1}^k < E_{\alpha_2}^k$.*

Proof. Consider uncertain variables $\tilde{u}_{fj}, f = 1, \ldots, h, j = 1, \ldots, n$ with uncertainty distributions ψ'_{fj}. According to [14], the inverse uncertainty distribution is an increasing function. Thus, for $\alpha_1 < \alpha_2$, we have

$$\psi'^{-1}_{fj}(\alpha_1) < \psi'^{-1}_{fj}(\alpha_2), \qquad f = 1, \ldots, h, \quad j = 1, \ldots, n, \tag{26}$$

and since $1-\alpha_2 < 1-\alpha_1$, it follows that

$$\psi'^{-1}_{fk}(1-\alpha_2) < \psi'^{-1}_{fk}(1-\alpha_1), \qquad f=1,\ldots,h, \quad k=1,\ldots,n. \tag{27}$$

When both sides of (26) are multiplied by $\sum_{j=1,j\neq k}^{n}\mu_j > 0$ (since $\mu_j > 0$), we have

$$\sum_{j=1,j\neq k}^{n}\mu_j\psi'^{-1}_{fj}(\alpha_1) < \sum_{j=1,j\neq k}^{n}\mu_j\psi'^{-1}_{fj}(\alpha_2), \quad f=1,\ldots,h. \tag{28}$$

Analogouesly, when both sides of (27) are multiplied by $\mu_k - 1 < 0$, we have

$$(\mu_k-1)\psi'^{-1}_{fk}(1-\alpha_1) < (\mu_k-1)\psi'^{-1}_{fk}(1-\alpha_2), \quad f=1,\ldots,h,\ k=1,\ldots,n. \tag{29}$$

Thus, from (28) and (29), we have

$$\sum_{j=1,j\neq k}^{n}\mu_j\psi'^{-1}_{fj}(\alpha_1) + (\mu_k-1)\psi'^{-1}_{fk}(1-\alpha_1) < \sum_{j=1,j\neq k}^{n}\mu_j\psi'^{-1}_{fj}(\alpha_2) + (\mu_k-1)\psi'^{-1}_{fk}(1-\alpha_2), \tag{30}$$

for $k=1,\ldots,n,\ f=1,\ldots,h$. If we simplify the third constraint of model (9) and replace α with α_1 and α_2, for $k=1,\ldots,n$ and $f=1,\ldots,h$, we have

$$\sum_{j=1,j\neq k}^{n}\mu_j\psi'^{-1}_{fj}(\alpha_1) + (\mu_k-1)\psi'^{-1}_{fk}(1-\alpha_1) \leq -\delta_f^1 d_f^u, \tag{31}$$

$$\sum_{j=1,j\neq k}^{n}\mu_j\psi'^{-1}_{fj}(\alpha_2) + (\mu_k-1)\psi'^{-1}_{fk}(1-\alpha_2) \leq -\delta_f^2 d_f^u, \tag{32}$$

where $d_f^u > 0, f=1,\ldots,h$. Comparing (30), (31) and (32), we have

$$-\delta_f^1 d_f^u < -\delta_f^2 d_f^u, \quad f=1,\ldots,h.$$

Dividing both sides by $-d_f^u$, we have

$$\delta_f^1 > \delta_f^2, \qquad f=1,\ldots,h. \tag{33}$$

Summing up (33) over $f = 1, \dots, h$, we have

$$\sum_{f=1}^{h} \delta_f^1 > \sum_{f=1}^{h} \delta_f^2. \tag{34}$$

A similar argument applies to the first and second constraints of model (9), which involve parameters ζ_i and β_r. Therefore, we have

$$\sum_{i=1}^{m} \zeta_i^1 > \sum_{i=1}^{m} \zeta_i^2, \tag{35}$$

$$\sum_{i=1}^{m} \beta_r^1 > \sum_{i=1}^{m} \beta_r^2. \tag{36}$$

Inequalities (34) and (35) result in

$$\sum_{i=1}^{m} \zeta_i^1 + \sum_{f=1}^{h} \delta_f^1 > \sum_{i=1}^{m} \zeta_i^2 + \sum_{f=1}^{h} \delta_f^2.$$

Consequently, we have

$$1 - \frac{1}{m+h}\Big(\sum_{i=1}^{m} \zeta_i^1 + \sum_{f=1}^{h} \delta_f^1\Big) < 1 - \frac{1}{m+h}\Big(\sum_{i=1}^{m} \zeta_i^2 + \sum_{f=1}^{h} \delta_f^2\Big). \tag{37}$$

On the other hand, from (36) we have

$$1 + \frac{1}{s}\sum_{i=1}^{m} \beta_r^1 > 1 + \frac{1}{s}\sum_{i=1}^{m} \beta_r^2,$$

and consequently,

$$\frac{1}{1 + \frac{1}{s}\sum_{i=1}^{m} \beta_r^1} > \frac{1}{1 + \frac{1}{s}\sum_{i=1}^{m} \beta_r^2}. \tag{38}$$

Combining (37) with (38), we have

$$\begin{aligned} E_k^{\alpha_1} &= \frac{1 - \frac{1}{m+h}(\sum_{i=1}^{m} \zeta_i^1 + \sum_{f=1}^{h} \delta_f^1)}{1 + \frac{1}{s}\sum_{i=1}^{m} \beta_r^1} < \frac{1 - \frac{1}{m+h}(\sum_{i=1}^{m} \zeta_i^2 + \sum_{f=1}^{h} \delta_f^2)}{1 + \frac{1}{s}\sum_{i=1}^{m} \beta_r^2} \\ &= E_k^{\alpha_2}. \end{aligned}$$

The proof is complete. □

4 Special Case with Uncertain Linear Variables

Recall that the uncertainty distribution describes the mathematical patterns and characteristics of uncertain events. Models (8) and (9) provide a general framework without assuming a specific distribution. However, in practical applications, the variables typically follow a specific distribution. The linear distribution, which is commonly used in uncertainty theory owing to its mathematical simplicity and limited expert data, is assumed for uncertain inputs, desirable and undesirable outputs. Using this assumption, the inverse distribution of the uncertain data can be derived, allowing models (8) and (9) to be rewritten accordingly, as follows.

$$
\begin{aligned}
\min \quad & \frac{1}{h}\sum_{f=1}^{h}\delta_f \\
\text{s.t.} \quad & \sum_{j=1,j\neq k}^{n}(\mu_j+\theta_j)\ \left[(1-\alpha)\,L_{ij}+\alpha U_{ij}\right]+(\mu_k+\theta_k)\left[(1-\alpha)\,U_{ik}+\alpha L_{ik}\right] \\
& \quad \le (1-\alpha)\,U_{ik}+\alpha L_{ik}; \qquad i=1,\ldots,m, \\
& \sum_{j=1,j\neq k}^{n}\ \mu_j\left[(1-\alpha)\,\bar{U}_{rj}+\alpha\bar{L}_{rj}\right]+\ \mu_k\left[(1-\alpha)\,\bar{L}_{rk}+\alpha\bar{U}_{rk}\right] \\
& \quad \ge (1-\alpha)\,\bar{L}_{rk}+\alpha\bar{U}_{rk}; \qquad r=1,\ldots,s, \\
& \sum_{j=1,j\neq k}^{n}\ \mu_j\ \left[(1-\alpha)\,\tilde{L}_{fj}+\alpha\tilde{U}_{fj}\right]+\mu_k\left[(1-\alpha)\,\tilde{U}_{fk}+\alpha\tilde{L}_{fk}\right] \\
& \quad \le (1-\alpha)\,\tilde{U}_{fk}+\alpha\tilde{L}_{fk}-\delta_f d_f^u; \qquad f=1,\ldots,h, \\
& \sum_{j=1}^{n}(\mu_j+\theta_j)=1, \\
& \mu_j\ge 0,\ \theta_j\ge 0; \qquad j=1,\ldots,n, \\
& \delta_f\ge 0\ ; \qquad f=1,\ldots,h,
\end{aligned}
$$

$$
\begin{aligned}
\max \quad & \frac{1}{m+s+h}\left(\sum_{i=1}^{m}\zeta_i+\sum_{r=1}^{s}\beta_r+\sum_{f=1}^{h}\delta_f\right)\\
\text{s.t.} \quad & \sum_{j=1,j\neq k}^{n}(\mu_j+\theta_j)\ \left[(1-\alpha)\,L_{ij}+\alpha U_{ij}\right]+(\mu_k+\theta_k)\\
& \quad \left[(1-\alpha)\,U_{ik}+\alpha L_{ik}\right]\leq(1-\alpha)\,U_{ik}+\alpha L_{ik}-\zeta_i d_i^x;\\
& \quad i=1,\dots,m,\\
& \sum_{j=1,j\neq k}^{n}\mu_j\left[(1-\alpha)\,\bar{U}_{rj}+\alpha\bar{L}_{rj}\right]+\ \mu_k\left[(1-\alpha)\,\bar{L}_{rk}+\alpha\bar{U}_{rk}\right]\\
& \quad \geq(1-\alpha)\,\bar{L}_{rk}+\alpha\bar{U}_{rk}+\beta_r d_r^y;\qquad r=1,\dots,s,\\
& \sum_{j=1,\ j\neq k}^{n}\mu_j\ \left[(1-\alpha)\,\tilde{L}^*_{fj}+\alpha\tilde{U}^*_{fj}\right]+\mu_k\left[(1-\alpha)\,\tilde{U}^*_{fk}+\alpha\tilde{L}^*_{fk}\right]\\
& \quad \leq(1-\alpha)\,\tilde{U}^*_{fk}+\alpha\tilde{L}^*_{fk}-\delta_f d_f^u\ ;\qquad f=1,\dots,h,\\
& \sum_{j=1}^{n}(\mu_j+\theta_j)=\ 1,\\
& \mu_j\geq 0,\ \theta_j\geq 0;\ j=1,\dots,n.
\end{aligned}
\tag{39}
$$

where $L_{ij}, U_{ij}, \bar{L}_{rj}, \bar{U}_{rj}$ represent the lower and upper bounds of the i-th input, r-th desirable output, respectively. Similarly, $\tilde{L}_{fj}$ and $\tilde{U}_{fj}$ correspond to the lower and upper bounds of the f-th undesirable output, respectively, while $\tilde{L}^*_{fj}$, $\tilde{U}^*_{fj}$ represent the lower and upper bounds of the f-th minimum undesirable output from DMU_j, respectively.

Lemma 4.1. *For any confidence level $\alpha \in (0,1]$, let E^k_α and E^k denote the efficiency of model (9) in an uncertain environment at confidence level α and the efficiency in a deterministic environment, respectively. In the case where $\alpha = 1$, the efficiency values E^k_α and E^k are not always equal.*

Proof. As demonstrated, model (9) is transformed into an equivalent model (39) under the assumption of linear distributions of uncertain variables, as outlined in Theorem 2.2. In the specific case of a deterministic environment where the lower and upper bounds are identical, model (39) is reformulated as model (40).

$$
\begin{array}{ll}
\max & \dfrac{1}{m+s+h}\left(\displaystyle\sum_{i=1}^{m} \zeta_i + \sum_{r=1}^{s} \beta_r + \sum_{f=1}^{h} \delta_f \right) \\
\text{s.t.} & \displaystyle\sum_{j=1, j\neq k}^{n} (\mu_j + \theta_j)\ [U_{ij}] + (\mu_k + \theta_k)\,[L_{ik}] \leq L_{ik} - \zeta_i d_i^x; \\
& \qquad i = 1, \ldots, m \\
& \displaystyle\sum_{j=1, j\neq k}^{n} \mu_j\,[\bar{L}_{rj}] + \ \mu_k\,[\bar{U}_{rk}] \geq \bar{U}_{rk} + \beta_r d_r^y; \\
& \qquad r = 1, \ldots, s, \\
& \displaystyle\sum_{j=1,\ j\neq k}^{n} \mu_j\ \left[\tilde{U}_{fj}^*\right] + \mu_k \left[\tilde{L}_{fk}^*\right] \leq \tilde{L}_{fk}^* - \delta_f d_f^u\ ; \\
& \qquad f = 1, \ldots, h, \\
& \displaystyle\sum_{j=1}^{n} (\mu_j + \theta_j) = \ 1, \\
& \mu_j \geq 0,\ \theta_j \geq 0; \qquad j = 1, \ldots, n.
\end{array}
\tag{40}
$$

Considering the values of the parameters influencing the models, it can be inferred that model (40) and the deterministic model proposed by Pourmahmoud and Radfar [25, Model(13)] are not always identical. □

5 Case Study in OECD Countries

Here, we apply the proposed model on a real-world case. Since 1960, 38 nations have been registered with the OECD[3] [9]. However, owing to restricted access and unavailability of data, we opted to assess only 20 OECD members for this study. To evaluate the environmental efficiency of renewable energy use in these countries in 2020, we applied the inputs and outputs from Wang et al. [32]. Inputs include labor force, gross capital formation, total renewable energy capacity, and share of renewable energy. Desirable output is GDP, and CO_2 emissions is considered as undesirable output.

[3]Organization for Economic Co-operation and Development

The variables used are defined below, and Table 1 lists their values and names of relevant countries [32].

Table 1: Country names, input and output values

DMU	Countries	I1	I2	I3	I4	Y	U
1	Australia	13587001	319789	34536	22.6	1350534	370.48
2	Belgium	5167188	122655	11277	27.4	462150	87.24
3	Canada	20482633	388410	100582	67.1	1556509	521.00
4	Czech Republic	5375292	57656	10151	14.3	188033	89.09
5	France	30379167	606702	55365	24.4	2438208	273.59
6	Germany	43501190	771775	131739	44.8	3356236	619.29
7	Italy	25126337	353309	55299	42.4	1835899	280.47
8	Japan	68898380	1243302	186259	20.3	4444931	1029.82
9	Korea, Rep.	28597159	524326	27405	6.4	1465773	609.20
10	Mexico	53137902	265449	28358	19.8	171868	369.00
11	Netherlands	9502134	171292	17678	26.6	765265	147.04
12	New Zealand	2832047	49384	7425	80.4	178064	33.00
13	Norway	2893601	110890	37212	98.2	385802	37.82
14	Poland	18245536	104835	12220	18.4	477812	286.17
15	Portugal	5166305	38039	14274	59.6	199314	37.87
16	Spain	22838137	254150	59108	44.5	1195119	200.80
17	Sweden	5569519	139176	32883	68.5	505104	34.21
18	Turkey	31361351	263662	49398	41.8	864317	405.11
19	United Kingdom	34633314	492004	47387	43.9	2956574	306.50
20	United States	165649358	4049754	291680	19.9	18238301	4686.08

The input for the **labor force** (I1) consists of individuals aged 15 years and older who contribute labor to the production of goods and services over a specified period. The number of people serves as the input for each DMU [35]. The **Gross capital formation** (I2), previously known as gross domestic investment, refers to net changes in inventories and expenditures on the growth of a country's fixed assets. The input for this unit is measured in million USD [35]. The **total renewable energy capacity** (I3) is measured in megawatts (MW). This input represents the total net generating capacity of power plants and other facilities that produce electricity from renewable energy sources [6]. The **share of renewable energy** (I4) indicates the proportion of electricity generated from renewable sources such as solar, geothermal, wind, and hydro. This input is expressed as a percentage (%) [4]. The **Gross domestic product** (Y), known as GDP, represents the total gross value added by all resident producers, plus any commodity taxes in the economy. The output for this unit is measured in millions of USD [35]. Finally, **CO_2 emissions** (U) includes only the emissions from the burning of fuels, such as coal, oil, and gas. The output of this unit is measured in million metric tons of CO_2 ($MtCO_2$) [4].

Observe that presented data in Table 1 are in crisp format. Assuming that they are usually imprecise, we transformed the data in Table 1 into uncertain data. Using MATLAB, we generated 1000 independent random datasets based on Table 1. Each deterministic value was assigned an interval of ±10% of its original value, from which two random numbers were selected: the larger as the upper bound and the smaller as the lower bound. The mean of the upper and lower bounds was calculated as the final bounds.

Table 2: The values of E_k for different values of α and ratios for $\alpha = 0.5$.

DMU	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 0.95$	$\alpha = 1$	E_k^*	$\frac{1-E_k^*}{1-E_k}$	$\frac{E_k^*-E_k}{1-E_k}$
1	0.972	0.986	0.994	0.997	0.999	0.999	1	0.985	0.539	0.461
2	1	1	1	1	1	1	1	1	0	0
3	0.550	0.567	0.584	0.604	0.628	0.643	0.662	0.780	0.489	0.511
4	1	1	1	1	1	1	1	1	0	0
5	1	1	1	1	1	1	1	1	0	0
6	0.735	0.757	0.782	0.811	0.845	0.863	0.880	0.802	0.748	0.252
7	0.792	0.819	0.851	0.887	0.921	0.935	0.947	0.926	0.356	0.644
8	0.785	0.823	0.866	0.910	0.941	0.951	0.962	0.837	0.760	0.240
9	1	1	1	1	1	1	1	1	0	0
10	0.107	0.110	0.114	0.117	0.121	0.122	0.124	0.809	0.213	0.787
11	0.953	0.972	0.986	0.994	0.998	0.999	0.999	0.992	0.180	0.820
12	1	1	1	1	1	1	1	1	0	0
13	1	1	1	1	1	1	1	1	0	0
14	0.996	0.999	1	1	1	1	1	0.998	0.474	0.526
15	1	1	1	1	1	1	1	1	0	0
16	0.662	0.680	0.701	0.728	0.763	0.780	0.800	0.850	0.443	0.557
17	1	1	1	1	1	1	1	1	0	0
18	0.456	0.469	0.483	0.497	0.510	0.517	0.524	0.823	0.325	0.675
19	1	1	1	1	1	1	1	1	0	0
20	1	1	1	1	1	1	1	1	0	0
Ave	0.85	0.859	0.868	0.877	0.886	0.891	0.895	0.940	0.226	0.274

To evaluate the performance of the proposed model, efficiency scores (E_k) for each DMU were computed using a linear distribution of the uncertainty parameter α within the range $[0.5, 1]$, as presented in columns two–eight of Table 2. These scores were derived by applying the optimal values from model (7) to Definition 3.1. These values quantify the environmental efficiency of renewable energy use across 20 OECD countries in 2020. To calculate the efficiency scores with minimum undesirable outputs (E_k^*), presented in column nine of Table 2, model (8) was first employed to determine the contraction coefficient for undesirable outputs (δ_f). Subsequently, the relation $\tilde{u}_{fj}^* = \tilde{u}_{fj} - \delta_f^* d_f^u$ was used to compute the minimum undesirable output (CO2 emissions) for each DMU. These values were then substituted into model (9), and the resulting optimal values were applied in Definition 3.1 to derive E_k^*. To ensure robustness under data uncertainty, 1000 previously described random datasets were generated, with the mean of the results reported in Table 2, providing a realistic depiction of the model's performance. The last row of Table 2 provides the average values for each column, offering a comprehensive overview of the model outcomes.

The analysis in Table 2 reveals significant insights into the environmental efficiency of the selected OECD countries. As α increases from 0.5 to 1, both the efficiency scores and the number of efficient DMUs generally increase, indicating that higher confidence levels in the uncertain data lead to better efficiency outcomes. This trend is consistent with Theorem 3.5, which posit that efficiency scores improve with higher confidence levels.

The average efficiency score across all DMUs (E_k) is 0.895 at $\alpha = 1$, improving to 0.940 for E_k^*, highlighting the potential for significant efficiency gains when undesirable outputs are minimized. Furthermore, a comparison of the efficiency scores in column eight of Table 2 with those from the deterministic model [25] validates Lemma 4.1, confirming that the proposed model's results align with deterministic outcomes at $\alpha = 1$, while providing additional insights into performance under uncertainty.

Columns ten and eleven of Table 2 present the ratios $\frac{1-E_k^*}{1-E_k}$ and $\frac{E_k^*-E_k}{1-E_k}$ at $\alpha = 0.5$, which quantify the contributions of desirable (GDP) and undesirable (CO2 emissions) outputs to inefficiency, as defined in (3). These metrics enable researchers and policymakers to evaluate the influence of each output type on DMU efficiency. To elucidate these concepts, we analyze DMU_6 (Germany) from Table 2 as follows.

Germany's efficiency score improves from $E_6 = 0.735$ at $\alpha = 0.5$ to $E_6^* = 0.802$ when CO2 emissions are minimized, thereby emphasizing their critical role in enhancing efficiency. To quantify the impact of desirable (GDP) and undesirable (CO2) outputs on inefficiency, assertion (3) is applied resulting to

$$\frac{(1-E_6^*)}{(1-E_6)}=(\frac{0.198}{0.265})\times 100 = 74.83\%, \frac{(E_6^*-E_6)}{(1-E_6)}=(\frac{0.067}{0.265})\times 100 = 25.16\%.$$

These results indicate that 74.83% of Germany's inefficiency is attributed to insufficient GDP, while 25.16% stems from excess CO2 emissions. The substantial difference between these percentages demonstrates that prioritizing strategies to enhance GDP has a greater impact on improving Germany's efficiency than focusing on CO2 reduction. This analysis identifies the key factors influencing operational performance, guiding targeted interventions and strategic improvements. By decomposing inefficiency into components due to excess undesirable outputs and shortfalls in desirable outputs, decision-makers can identify specific areas for intervention. Similar inefficiency patterns can be analyzed across other DMUs to develop tailored strategies for enhancing performance.

The average values presented in Table 2 indicate that excess undesirable outputs (CO2 emissions) account for 27.4% of inefficiency, while shortfalls in desirable outputs (GDP) contribute 22.6%. These findings underscore the fact that reducing CO2 emissions is a critical lever for enhancing DMUs efficiency across the OECD countries. Such insights are vital not only for the DMUs analyzed but also for reflecting broader operational trends that can guide targeted strategies for performance improvement.

6 Conclusion

This study presents a robust framework for evaluating the efficiency of DMUs while minimizing undesirable outputs, which negatively impact performance and cannot be fully eliminated in practical applications. By integrating Liu's uncertainty theory with the DDF model featuring individual-proportion weak disposability, we surpassed the limitations of prior models, which rely on precise data and are constrained by radial structures. We transformed uncertain models into deterministic equivalents using the maximum belief degree method. This transformation enabled the application of standard optimization techniques, ensuring robust efficiency evaluation under uncertainty.

We applied the model in a real-world problem. The significance of these findings lies in their ability to provide policymakers with actionable strategies in uncertain environments, outperforming deterministic models that may overestimate their efficiency. Future research research direction would be adapting for complex scenarios, including negative or zero data, variable returns to scale, or additional factors such as operational costs. Developing network-based DDF models for multistage DMU structures would result in better outcome.

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