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The General Solutions of Fuzzy Linear Matrix Equations

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Abstract. The main aim of this paper is to illustrate and discuss consistent Fuzzy Linear Matrix Equations (shown as **FLME**) of the form $\sum_{i=1}^{l}(A_iX_iB_i) = C$ for finding its fuzzy number solutions. Similar equation is studied in [12] based on Friedman's method [5]. Recently, Mikaeilvand et al. [13] proposed a novel method based on Ezzati's method [4] for solving fuzzy linear systems, which is better than Friedman's method numerically. In this article, we used this method for finding the general solutions of consistent FLME. The parametric form of fuzzy numbers is used. The embedding method is profited to transform fuzzy linear matrix equations to parametric linear equations, and a numerical procedure for calculating the solutions is designed.

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1. Introduction

Most problems extant in engineering and mathematical perspectives such as control theory, system theory, optimization, power systems, sig-

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nal processing, linear algebra, differential equations and etc. [3,6] usually leading to solve a system of fuzzy linear matrix equations(FLME). So it is an important subject in applied mathematics. In this paper, we investigate a class of the fuzzy linear matrix equation system and propose the new method for solving it, then we will extend this method for solving the general consistent fuzzy linear matrix equations. Let $R^{m\times n}$ denotes the vector space of real matrices of order $m \times n$ and $F^{m\times n}$ denotes the set of all $m \times n$ matrices which their elements are fuzzy numbers. For given matrices $\mathbf{A_i} \in R^{m\times n_i}$, $\mathbf{B_i} \in R^{p_i\times q}$ and $\mathbf{C} \in F^{m\times q}$, consider the problem of determining the solutions $\mathbf{X_i} \in F^{n_i\times p_i}(i = 1, 2, ..., l)$, of the matrix equation

$$A_1X_1B_1 + A_2X_2B_2 + ... + A_lX_lB_l = C. \tag{1}$$

A special case of this equation is when the $\forall \mathbf{i}, \mathbf{X_i}$ are similar ($\mathbf{X} \in F^{n\times p}$). An other case of Eq (3) where $l = 2$ ($\mathbf{A_1XB_1} + \mathbf{A_2XB_2} = \mathbf{C}$) occurs in theory of minimum norm quadratic unbiased estimation (MINQUE) theory of estimating covariance components in a covariance components model [11]. A familiar example occurs in the Lyapunov theory of stability [10] with $\mathbf{B_2} = \mathbf{A_1^t}$ and A_2 and B_1 are Identity matrices. Also, Fuzzy matrix equation in the form $AXB = C$ was introduced by Allahviranloo et al. [2] and fuzzy Sylvester matrix equation is investigated by Khujasteh Salkuyeh [8]. Recently, Mikaeilvand in [12] extended its method to solve a special case of FLME by using an embedding approach that based on Friedman et al. [5]. He firstly used the Kronecker matrix product then replaced the original fuzzy linear matrix (2) by $2mq \times 2np$ parametric-crisp function-linear matrix equations. In this work, we use the embedding method and replace the system of FLME (2) by two $mq\times np$ parametric-crisp function-linear matrix equations. It is clear that, in large matrix equations, solving a $mq \times np$ linear system of matrix equation is better than solving a $2mq \times 2np$ linear system of matrix equation. Finally, we extended our method for solving the general consistent fuzzy matrix equations.

This paper is organized as follows:

In Section 2, we discuss some basic definitions and results on fuzzy numbers and the fuzzy linear matrix equations **FLME**. In Section 3, the proposed model for solving the linear matrix equation and its limitations are discussed. Consequently, we extended the method for general consistent **FLME** (1) and it is illustrated by solving a numerical example in Section 4. The conclusion and future research are drawn in Section 5.

2. Basic Concepts

Here, some basic definitions and results that we will need in this paper are revisited.

Definition 2.1. [7]*An arbitrary fuzzy number* $\widetilde{u}$ *is represented by an ordered pair of functions* $(\underline{u}(r), \overline{u}(r))$; $0 \leqslant r \leqslant 1$ *and satisfies the following requirements:*

(i) $\underline{u}(r)$ *is a bounded monotonic increasing left continuous function;*

(ii) $\overline{u}(r)$ *is a bounded monotonic decreasing left continuous function;*

(iii) $\underline{u}(r) \leqslant \overline{u}(r),\ 0 \leqslant r \leqslant 1$.

For arbitrary $\widetilde{u} = (\underline{u}(r), \overline{u}(r))$, $\widetilde{v} = (\underline{v}(r), \overline{v}(r))$ and scalar k, addition $(\widetilde{u}+\widetilde{v})$ and scalar multiplication by k are defined as follows respectively:

addition:

$$(\underline{u+v})(r) = \underline{u}(r) + \underline{v}(r), \quad (\overline{u+v})(r) = \overline{u}(r) + \overline{v}(r),$$

scalar multiplication:

$$\widetilde{ku} = \begin{cases} (k\underline{u}(r), k\overline{u}(r)), & k \geqslant 0, \\ \\ (k\overline{u}(r), k\underline{u}(r)), & k < 0. \end{cases}$$

For two arbitrary fuzzy numbers $\widetilde{x} = (\underline{x}(r), \overline{x}(r))$ and $\widetilde{y} = (\underline{y}(r), \overline{y}(r))$, $\widetilde{x} = \widetilde{y}$ if and only if $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$.

Definition 2.2. [9,14] *Let* $\boldsymbol{A} = (a_{ij})$ *and* $\boldsymbol{B} = (b_{ij})$ *be* $m \times n$ *and* $p \times q$ *matrices, respectively, the Kronecker product*

$$\boldsymbol{A} \otimes \mathbf{B} = (a_{ij}\mathbf{B})$$

is a $mp \times nq$ matrix expressible as a partitioned matrix with $a_{ij}\mathbf{B}$ as the (i,j)th partition, $i = 1, \cdots, m; j = 1, \cdots, n$.

Finally in this section, the generalized inverses of matrix $\mathbf{\Gamma}$ in a special structure is discussed.

Theorem 2.3. [1,15] *Let*

$$\mathbf{\Gamma} = \begin{pmatrix} \mathbf{S_1} & \mathbf{S_2} \\ \mathbf{S_2} & \mathbf{S_1} \end{pmatrix},$$

then the matrix

$$\mathbf{\Gamma}^- = \frac{1}{2}\begin{pmatrix} (\mathbf{S_1}+\mathbf{S_2})^- + (\mathbf{S_1}-\mathbf{S_2})^- & (\mathbf{S_1}+\mathbf{S_2})^- - (\mathbf{S_1}-\mathbf{S_2})^- \\ (\mathbf{S_1}+\mathbf{S_2})^- - (\mathbf{S_1}-\mathbf{S_2})^- & (\mathbf{S_1}+\mathbf{S_2})^- + (\mathbf{S_1}-\mathbf{S_2})^- \end{pmatrix},$$

is a g-inverses of matrix $\mathbf{\Gamma}$, where $(\mathbf{S_1}+\mathbf{S_2})^-$ and $(\mathbf{S_1}-\mathbf{S_2})^-$ are g-inverses of matrices $\mathbf{S_1}+\mathbf{S_2}$ and $\mathbf{S_1}-\mathbf{S_2}$, respectively.

(For a detailed study of the g-inverse of a matrix reader is referred to [9,14]).

Since $\mathbf{S_1}-\mathbf{S_2} = \sum_{\mathbf{r=1}}^{\mathbf{l}} \mathbf{B_r^t} \otimes \mathbf{A_r}$ and $(\mathbf{B_r^t} \otimes \mathbf{A_r})^- = \mathbf{B_r^{t^-}} \otimes \mathbf{A_r^-}$, we obtain the following corollary.

Corollary 2.4. *Let $\mathbf{\Gamma}$ be in the form introduced in (5), then the matrix*

$$\mathbf{\Gamma}^- = \frac{1}{2}\begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

is a g-inverse of the matrix $\mathbf{\Gamma}$. where

$$a = \sum_{r=1}^{l}((\mathbf{B_{r1}}+\mathbf{B_{r2}})^{\mathbf{t}^-} \otimes (\mathbf{A_{r1}}+\mathbf{A_{r2}})^- + \mathbf{B_r^{t^-}} \otimes \mathbf{A_r^-}),$$

$$b = \sum_{r=1}^{l}((\mathbf{B_{r1}}+\mathbf{B_{r2}})^{\mathbf{t}^-} \otimes (\mathbf{A_{r1}}+\mathbf{A_{r2}})^- - \mathbf{A_r^-} \otimes \mathbf{B_r^{t^-}}).$$

Ezzati [4] first solved the system: $A(\underline{x}+\overline{x}) = \underline{y}+\overline{y}$ and considered the solution of this system is

$$\mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} = \underline{\mathbf{X}} + \overline{\mathbf{X}} = \begin{pmatrix} \underline{x}_1 + \overline{x}_1 \\ \underline{x}_2 + \overline{x}_2 \\ \vdots \\ \underline{x}_n + \overline{x}_n \end{pmatrix},$$

They obtained the following solution by using the parametric form of the Eq ($A\tilde{x} = \tilde{y}$)

$$\underline{x}(r) = A^{-1}(\underline{y}(r) + Cd),$$
$$\overline{x}(r) = A^{-1}(\overline{y}(r) - Cd),$$

which is the solution of the fuzzy linear system ($A\tilde{x} = \tilde{b}$), $A \in R^{n\times n}$ and $\tilde{x}, \tilde{y}$ are fuzzy vectors.

Theorem 2.5. [14] *A general solution of a consistent system equations $SX = Y$ is $X = S^{-}Y + (I - H)Z$ where $H = S^{-}S$, S^{-} is g-inverse of matrix S and Z is an arbitrary vector.*

3. Fuzzy Linear Matrix Equations

Definition 3.1. *The linear matrix equation* (1) *is called a fuzzy linear matrix equation,* **FLME**, *if the left coefficient matrices $\mathbf{A_r} = (a_{rij})$ $(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n_k,\ 1 \leqslant r \leqslant l)$ and the right coefficient matrices $\mathbf{B_r} = (b_{rij})$ $(1 \leqslant i \leqslant p_k,\ 1 \leqslant j \leqslant q,\ 1 \leqslant r \leqslant l, (k = 1, 2, ..., l))$ are crisp matrices and the right-hand side matrix $\mathbf{C} = (\widetilde{c}_{ij})$ $(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant q)$ is a fuzzy number matrix. Also, $\mathbf{X_k} \in F^{n_k\times p_k}, (k = 1, 2, ..., l)$ are its unknown matrix.*

Firstly, we investigate a special case of consistent FLME, which has one unknown matrix, namely in this case for each iX_i are invariant. Then we have the following system:

$$\mathbf{A_1XB_1} + \mathbf{A_2XB_2} + ... + \mathbf{A_lXB_l} = \mathbf{C}. \tag{2}$$

The ij-th equation of this system is:

$$\sum_{r=1}^{l}\sum_{t=1}^{p}\sum_{k=1}^{n} a_{rik}x_{kt}b_{rtj} = \widetilde{c}_{ij},\ 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant q.$$

A basic method to express (2) in an equivalent vector form is as follows. Let $\mathbf{\Gamma}$ denote the matrix of order $mq \times np$,

$$\mathbf{\Gamma} = \mathbf{B_1^t} \otimes \mathbf{A_1} + \mathbf{B_2^t} \otimes \mathbf{A_2} + ... + \mathbf{B_l^t} \otimes \mathbf{A_l},$$

where $\otimes$ denotes Kronecker product [11], then (2) is equivalent to the fuzzy equation $\mathbf{\Gamma X} = \mathbf{C}$.

Definition 3.2. *A fuzzy number matrix* $\mathbf{X} = (x_{ij})$ $(1 \leqslant i \leqslant m, \;\; 1 \leqslant j \leqslant q)$ *given by* $x_{ij} = (\underline{x_{ij}}(r), \overline{x_{ij}}(r))$ $(1 \leqslant i \leqslant m, \; 1 \leqslant j \leqslant q)$ *is called a solution of the fuzzy linear matrix equation* **FLME** *if:*

$$\underline{\sum_{r=1}^{l} \sum_{t=1}^{p} \sum_{k=1}^{n} a_{rik} x_{kt} b_{rtj}}(r) = \sum_{r=1}^{l} \sum_{t=1}^{p} \sum_{k=1}^{n} \underline{a_{rik} x_{kt} b_{rtj}}(r) = \underline{c}_{ij}(r),$$

$$\overline{\sum_{r=1}^{l} \sum_{t=1}^{p} \sum_{k=1}^{n} a_{rik} x_{kt} b_{rtj}}(r) = \sum_{r=1}^{l} \sum_{t=1}^{r} \sum_{k=1}^{n} \overline{a_{rik} x_{kt} b_{rtj}}(r) = \overline{c}_{ij}(r).$$

In particular, if $\mathbf{A_1}, ..., \mathbf{A_l}$ and $\mathbf{B_1}, ..., \mathbf{B_l}$ are nonnegative matrices, simultaneously, or negative matrices, simultaneously, we simply get

$$\underline{\sum_{r=1}^{l} \sum_{t=1}^{p} \sum_{k=1}^{n} a_{rik} x_{kt} b_{rtj}}(r) = \sum_{r=1}^{l} \sum_{t=1}^{p} \sum_{k=1}^{n} a_{rik} \underline{x_{kt}}(r) b_{rtj} = \underline{c}_{ij}(r),$$

$$\overline{\sum_{r=1}^{l} \sum_{t=1}^{p} \sum_{k=1}^{n} a_{rik} x_{kt} b_{rtj}}(r) = \sum_{r=1}^{l} \sum_{t=1}^{p} \sum_{k=1}^{n} a_{rik} \overline{x_{kt}}(r) b_{rtj} = \overline{c}_{ij}(r). \tag{3}$$

In general, however, an arbitrary equation for either $\underline{c}_{ij}(r)$ or $\overline{c}_{ij}(r)$ may include a linear combination of $\underline{x}_{ij}(r)$'s and $\overline{x}_{ij}(r)$'s. Consequently, in order to solve the system given by (3), one must solve a crisp linear system where the right hand side vector is the function vector $(\underline{c}_{11}, ..., \underline{c}_{nr}, \overline{c}_{11}, ..., \overline{c}_{np})^t$. Now, Let us rearrange the linear equation

$$\mathbf{\Gamma X} = \mathbf{C}, \tag{4}$$

such that the unknowns are $\underline{x}_{ij}, \;\; -\overline{x}_{ij}$ $(1 \leqslant i \leqslant n, \;\; 1 \leqslant j \leqslant p)$. The linear matrix equation (4) is now a general fuzzy function system of linear equations and can be solved for $\mathbf{X}$.

Before resolvent the Eq (4), we state the following theorem:

Theorem 3.3. [13] *Suppose that* $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)^T$ *is an arbitrary fuzzy solution of Eq* (4). *Then the general solution of the consistent system* $\Gamma(\underline{x} + \overline{x}) = \underline{c} + \overline{c}$ *can be expressed by:*

$$d = \underline{x} + \overline{x} = \Gamma^{-}(\underline{c} + \overline{c}) + (I_n - \Gamma^{-}\Gamma) z(r),$$

where $\underline{c}+\overline{c} = (\underline{c}_1+\overline{c}_1, \underline{c}_2+\overline{c}_2, \ldots, \underline{c}_n+\overline{c}_n)^T$, Γ^- *is a g-inverse of matrix* Γ, I_n *is an* n *order unit matrix and z(r) is an arbitrary vector with parameter r.*

Proof. According to Theorem 2.5, it is obvious. □

Now, we propose a new method for solving the fuzzy linear system (4). Assume that the system (4) is consistent. Firstly, we solve the following system:

$$\Gamma(\underline{x}+\overline{x}) = \underline{c}+\overline{c}\ , \tag{5}$$

according the Theorem 3.3, the general solutions of this system is

$$d = \underline{x}+\overline{x} = \Gamma^-(\underline{c}+\overline{c}) + (I_n - \Gamma^-\Gamma)z(r).$$

For continuance, let matrices E and F contain the positive entries and the absolute values of the negative entries of Γ, respectively. Using Eq (4), we get $\Gamma\tilde{x} = \tilde{c}$ or $(E-F)\tilde{x} = \tilde{c}$ and in parametric form $(E-F)(\underline{x}(r), \overline{x}(r)) = (\underline{c}(r), \overline{c}(r))$. We can write this system as follows:

$$\begin{cases} E\underline{x}(r) - F\overline{x}(r) = \underline{c}(r), \\ E\overline{x}(r) - F\underline{x}(r) = \overline{c}(r), \end{cases} \tag{6}$$

By substituting of $\overline{x}(r) = d - \underline{x}(r)$ and $\underline{x}(r) = d - \overline{x}(r)$ in the first and second equation of above system, respectively, we have $(E+F)\underline{x}(r) = \underline{c}(r) + Fd$ and $(E+F)\overline{x}(r) = \overline{c}(r) - Fd$ hence

$$\begin{cases} \overline{x}(r) = (E+F)^-(\overline{c}(r)+Fd) + (I_n - (E+F)^-(E+F))z_1'(r), \\ \underline{x}(r) = d - \overline{x}(r), \end{cases} \tag{7}$$

where $(E+F)^-$ is a g-inverse of matrix $(E+F)$, I_n is an n order unit matrix and $z_1'(r)$ is a vector with parameter r. Therefore, we can solve consistent fuzzy linear matrix equation (1) by solving Eqs (5) and (6).

Remark 3.4. *Finally, we have the general fuzzy solutions for the consistent FLME, we should choose the special kind of* $z_1'(r)$ *and we know that the matrix* X *is fuzzy solution, whenever any coefficient of this matrix be fuzzy number. Then, we just admit* $z_1^0(r)$*, that will be satisfied in the conditions Definition 1.*

Corollary 3.5. *When $mq = np$, if a crisp linear system does not have a unique solution, the associated fuzzy matrix linear system does not have one either.*

Theorem 3.6. *Assume that the maximum numbers of multiplications which are required to calculate general solutions for the Eq (2) by Friedman's method and proposed method are denoted by $F_{mq\times np}$ and $E_{mq\times np}$, respectively. Then $F_{mq,np} \geqslant E_{mq,np}$ and $F_{mq,np} - E_{mq,np} = 7np \times (mq)^2 + 2np \times mq$.*

Proof. Assume that N is $n \times n$ matrix and denote by $h_{mq\times np}(N)$ the number of multiplication operations that the required to calculate N^{-1}. It is clear that: since $\widetilde{x} \in E^1; \underline{x}$ or $\overline{x}$, in the simplest case is line

$$\tilde{x} \in E^1 \Rightarrow (\underline{x} = \alpha_1 + \beta_1 r,\ \overline{x} = \alpha_2 + \beta_2 r,\ \ \alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2 \in R), \quad (8)$$

hence

$$F_{mq,np} = 2h_{mq,np}(\Gamma) + 8np \times (mq)^2 + 8np \times mq.$$

For computing the general solution of Eq (5) and the general solution for $\overline{x}$ from Eq (7) and according to relation (8), the maximum number of multiplication operations are $h_{mq,np}(\Gamma) + 2np \times mq$ and $h_{mq,np}(E+F) + np \times (mq)^2 + 4np \times mq$, respectively. Clearly $h_{mq,np}(\Gamma) = h_{mq,np}(E + F)$, so

$$E_{mq,np} = 2h_{mq,np}(\Gamma) + np \times (mq)^2 + 6np \times mq,$$

and hence

$$F_{mq,np} - E_{mq,np} = 7np \times (mq)^2 + 2np \times mq. \quad \square$$

Example 3.7. Consider the fuzzy linear matrix equation $\mathbf{A_1XB_1} + \mathbf{A_2XB_2} = \mathbf{C}$ where

$$\mathbf{A_1} = \begin{pmatrix} 1 & -1 \end{pmatrix}, \qquad \mathbf{A_2} = \begin{pmatrix} 2 & 3 \end{pmatrix}, \qquad \mathbf{B_1} = \mathbf{B_2} = \begin{pmatrix} 0 & 1 \end{pmatrix},$$
$$\mathbf{C} = \begin{pmatrix} (1+r, 3-r) & (-1+2r, 4-3r) \end{pmatrix}.$$

For solving this system, first we transform it to $[(\mathbf{B_1^t} \otimes \mathbf{A_1}) + (\mathbf{B_2^t} \otimes \mathbf{A_2})]\mathbf{X} = \mathbf{C}$, where

$$\Gamma = [(\mathbf{B_1^t} \otimes \mathbf{A_1}) + (\mathbf{B_2^t} \otimes \mathbf{A_2})] = \begin{pmatrix} 0 & 0 \\ 3 & 2 \end{pmatrix}.$$

So, we have

$$\begin{pmatrix} 0 & 0 \\ 3 & 2 \end{pmatrix} \mathcal{X} = \begin{pmatrix} (1+r, 3-r) \\ (-1+2r, 4-3r) \end{pmatrix}.$$

Consider a arbitrary g-inverse of matrices Γ and $(E+F)$:

$$\Gamma^- = (E+F)^- = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$$

from Eq(5), we have the following solutions:

$$d = \underline{x} + \overline{x} = \begin{pmatrix} 3-2r \\ 3-2r \end{pmatrix} + \begin{pmatrix} -2 & -2 \\ -3 & -1 \end{pmatrix} z(r),$$

and hence, from Eqs(6)

$$\overline{x}(r) = \begin{pmatrix} 4-3r \\ 4-3r \end{pmatrix} + \begin{pmatrix} -2 & -2 \\ -3 & -1 \end{pmatrix} z'(r), \quad \underline{x}(r) = \begin{pmatrix} r-1 \\ r-1 \end{pmatrix} + \begin{pmatrix} -2 & -2 \\ -3 & -1 \end{pmatrix} z''(r),$$

where $z'(r)$ and $z''(r)$ will be choosen so that, we have $\underline{x_1} \leqslant \overline{x_1}$, $\underline{x_2} \leqslant \overline{x_2}$ and $\underline{x_1}$, $\underline{x_2}$ are monotonically increasing functions and $\overline{x_1}$, $\overline{x_2}$ are monotonically decreasing functions. Therefore, the general solutions will be the vector of fuzzy number.

In the next section, we extende our method for solving the system of fuzzy matrix equations (1).

4. Extend Proposed Method for General Consistent FLME

In this section, solving the general consistent fuzzy linear matrix equation is discussed, with the following form.

$$\mathbf{A_1 X_1 B_1} + \mathbf{A_2 X_2 B_2} + \dots + \mathbf{A_l X_l B_l} = \mathbf{C}, \tag{9}$$

where $\mathbf{A_i} \in R^{m \times n_i}$, $\mathbf{B_i} \in R^{p_i \times q}$, $\mathbf{C} \in F^{m \times q}$ and $\mathbf{X_i} \in F^{n_i \times p_i} (i = 1, 2, \dots, l)$.

It is obvious that, our proposed method in past section for solving the fuzzy matrix equations is numerically better than the methods based on Friedman et al. [12].

According to the proposed method, at first we employ the Kronecker matrix product then replace the original fuzzy linear matrix (9) by $(mq \times \sum_{i=1}^{l} n_i p_i)$ parametric - fuzzy function - linear matrix equations. Then, we have following fuzzy linear matrix equation instead of Eq(1):

$$(\mathbf{B_1^t} \otimes \mathbf{A_1})\mathbf{x_1} + (\mathbf{B_2^t} \otimes \mathbf{A_2})\mathbf{x_2} + \ldots + (\mathbf{B_l^t} \otimes \mathbf{A_l})\mathbf{x_l} = \mathbf{C},$$

where $\forall \mathbf{i}, \mathbf{x_i}$ and vector $\mathbf{C}$ defined in Section 3 and

$$\mathbf{\Gamma}' \mathcal{X} = \mathbf{C}, \tag{10}$$

where Γ' is the $(mq \times \sum_{i=1}^{l} n_i p_i)$ matrix, made by matrices $B_i^t \otimes A_i$. The columns of Γ' are the columns of $B_i^t \otimes A_i$, which earned by locate columns of matrices $B_i^t \otimes A_i$ in corresponding each another stead the columns of Γ', respectively.As the same way, $\mathcal{X}$ is the column vector obtained by writing the column of $\forall i, x_i$ one below the other in the natural order.

Finally, we solve the fuzzy linear system equation (10) with Eqs (5) and (7). Therefore, we will find the general fuzzy solutions for the consistent FLME (1).

Example 4.1. Consider the consistent general fuzzy linear matrix equation $\mathbf{A_1X_1B_1} + \mathbf{A_2X_2B_2} = \mathbf{C}$ where

$$\mathbf{A_1} = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}, \ \mathbf{A_2} = \begin{pmatrix} 1 & 1 \end{pmatrix}, \ \mathbf{B_1} = \begin{pmatrix} 0 & 1 \end{pmatrix}, \ \mathbf{B_2} = \begin{pmatrix} 2 & 1 \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} (4r, 1-4r) & (6r-4, 8-5r) \end{pmatrix}.$$

For solving this system, first we transform it to $(\mathbf{B_1^t} \otimes \mathbf{A_1})\underline{\mathbf{X_1}} + (\mathbf{B_2^t} \otimes \mathbf{A_2})\underline{\mathbf{X_2}} = \mathbf{C}$, where

$$(\mathbf{B_1^t} \otimes \mathbf{A_1}) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \qquad (\mathbf{B_2^t} \otimes \mathbf{A_2}) = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}.$$

Then, the fuzzy linear system substituting of fuzzy matrix equation is

$$\begin{pmatrix} 0 & 0 & 0 & 2 & 2 \\ 1 & 0 & -1 & 1 & 1 \end{pmatrix} \mathcal{X} = \begin{pmatrix} (4r, 1-4r) \\ (6r-4, 8-5r) \end{pmatrix},$$

where $\mathcal{X} = (\underline{x}(r), \overline{x}(r)) = \begin{pmatrix} x_{11}(r) \\ x_{12}(r) \\ x_{13}(r) \\ x_{21}(r) \\ x_{22}(r) \end{pmatrix}$ is a vector of fuzzy numbers. Now, consider the arbitrary g-inverse of matrix Γ and $(E+F)$

$$\Gamma^{-} = (E+F)^{-} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

At the first step, we have the following solutions by solving the Eq (5)

$$d = \overline{x} + \underline{x} = \begin{pmatrix} 10 \\ 0 \\ r+4 \\ 10 \\ r+4 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & -2 & -2 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & -2 \\ -1 & 0 & 1 & -1 & 0 \end{pmatrix} z(r),$$

where z(r) is an arbitrary vector with parameter r. Finally, using (7), we enter the following solutions:

$$\overline{x}(r) = \begin{pmatrix} -4r+10 \\ 0 \\ -4r+12 \\ -4r+10 \\ -4r+12 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & -1 \end{pmatrix} z(r) + \begin{pmatrix} 1 & 0 & 0 & -2 & -2 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & -2 \\ -1 & 0 & 1 & -1 & 0 \end{pmatrix} z'(r),$$

$$\underline{x}(r) = \begin{pmatrix} 4r \\ 0 \\ 5r-8 \\ 4r \\ 5r-8 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & -2 & -2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} z(r) + \begin{pmatrix} 1 & 0 & 0 & -2 & -2 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & -2 \\ -1 & 0 & 1 & -1 & 0 \end{pmatrix} z'(r),$$

where z(r) and $z'(r)$ are arbitrary vectors with parameter r. With the appropriate choice of z(r) and $z'(r)$, we have the fuzzy solutions for the above FLME, so that conditions expressed in remark 2 is confirmed.

5. Conclusion and Future Research

Linear matrix equation has many applications in various areas of science [10,11]. In this paper, the fuzzy linear matrix equation of the form $\sum_{\mathbf{i=1}}^{\mathbf{l}} \mathbf{A_i X_i B_i} = \mathbf{C}$ is introduced. We found a fuzzy solution of fuzzy linear matrix equations by an analytic approach. For this end, the original system is transformed to a parametric system using the Kronecker product and the embedding approach based on Ezzati's method [4]. Then, we proofed that our proposed method is numerically better than the method based on Friedman's model [12]. For future work, we try to extend our method to solve a inconsistent fuzzy linear matrix equations.

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