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Approximation Solution for a class of Fractional Partial Equations Using Fuzzy Transformations

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Abstract. This research focuses on obtaining continuous approximate solutions for real fractional partial differential equations. While numerous numerical methods exist for fractional partial differential equations, this article employs the inverse fuzzy transformation method a novel technique utilizing real data to derive approximation functions. The inverse fuzzy transformation method is chosen for its demonstrated convergence and its capability to approximate functions over entire intervals, leveraging all data points with a specified degree of accuracy, unlike point-specific methods. Specifically, we discretize the space-fractional derivative modeled using the Caputo derivative of order $1 < \alpha < 2$ via

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an explicit finite difference scheme. The efficiency of the proposed inverse fuzzy transformation method based approach is validated through two test problems for various time values t. Numerical results, presented in tables and graphs, demonstrate excellent agreement with the analytical solutions.

AMS Subject Classification: 65Nxx; 26A33; 03E72;
Keywords and Phrases: Time Fractional Partial Differential Equation; Explicit finite difference; Caputo derivative; Fuzzy transformation.

1 Introduction

Fractional differential equations (FDEs) have earned a great interests of scientists and engineers due to their applications in solving distinct problems of biology, fluid dynamics, electrical networks, viscoelasticity, finance, diffusive transport, signal processing, control theory, health, medicine, social sciences[12, 33, 3]. Fractional calculus is considered as the generalization of the classical (or integer order) calculus with a history of at least three hundred years. Although fractional calculus has such a long history, its research still stay in the realm of theory due to the lack of proper mathematical analysis methods and real applications. Caputo derivative in comparison with other fractional derivatives the is more general in nature and has the advantage over its counterpart Riemann Liouville, that it does not require the condition at origin. Most of the newly evolve definitions are special case of these two derivatives,[30]. Kumar et al. [19] have examined vibration equation containing a fractional derivative. Bhatter et al. [6] have analyzed the Drinfeld-Sokolov-Wilson equation of fractional order. Veeresha et al. [37] have suggested an effective numerical algorithm for solving nonlinear Klein-Fock-Gordon model of fractional order. Ghanbari and Kumar [16] have applied a new numerical algorithm for examining fractional order predator-prey model with Beddington-De Angelis functional response. Veeresha et al. [38] have projected a new fractional extension of differential equation interpreting the water transport in unsaturated porous media. Gill et al. [17] have studied extended fractional advection-dispersion equation by using integral transform scheme. In view of diverse applications of FDEs in present investigation, we focused to compute the numerical solution of space fractional partial differential equations(SFPDEs) for $1 < \alpha < 2$ ([11])

$$\frac{\partial u(x,t)}{\partial t} = a(x)\frac{\partial u(x,t)}{\partial x} + c(x)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + f(x,t), \tag{1}$$

with BCs

$$\begin{aligned} u(a,t) &= \vartheta_1(t), \\ u(b,t) &= \vartheta_2(t), \end{aligned} \tag{2}$$

and the IC

$$u(x,0) = \vartheta_0(x), \tag{3}$$

Where α is the space fractional parameter, $u(x,t)$ be a function of space and time, $f(x,t)$ is a source term, $\vartheta_1(t)$, $\vartheta_2(t)$ are BCs and $\vartheta_0(t)$ is the IC. To solve SFPDEs, different methods have been proposed. Meerschaert et al.[25] used the Riemann-Liouville derivatives to solve the space fractional advection dispersion equation. The space fractional KdV-Burgers-Kuramoto equation has solved by Safari et al.[32] using variational iteration method. Takale et al.[36] have solved the SFPDEs by using Adomian decomposition method. Krishnarajulu et al.[18] suggested a fractional Euler polynomial method for solving SFPDEs in which they used fractional order Euler function. In this research, $f(x,t)$ is source term and $\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}$ is the Caputo fractional derivative with repect to x defined as,

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \frac{1}{\Gamma(2-\alpha)}\int_0^x \frac{\partial^2 u(\zeta,t)}{\partial \zeta^2}\frac{d\zeta}{(x-\zeta)^{\alpha-1}}, \qquad 1 < \alpha < 2, \tag{4}$$

The method of separation of variables, Sumudu transform approach, decomposition scheme, Finite difference method, radial point interpolation method, B-spline collocation approach, Sinc-Chebyshev scheme introduce by Chen et al. [7], Darzi et al. [9], Ray [31], Huang et al.[15], Hosseini et al. [14], Esen [13], Mao [24] respectively are some of numerical schemes for solving the SFDWEs problems. In the past decades, a number of meshless approaches have been developed. Some of meshless methods that use nodal interpolation techniques and can therefor be widely employed in various fields can be found in [10, 34, 20, 21].
Fuzzy transform (F-transform for short) was proposed by Perfilieva in [26] and studied in several papers [27, 28, 29, 39, 35, 8], etc. We use

the inverse fuzzy transformation method because, firstly, it is convergent, and secondly, to approximate a function in certain intervals, unlike other methods that only use a limited number of points, in this method, all points are used with a degree of accuracy. You can see the use of inverse fuzzy transformation method in solving some real differential equations in [5, 1, 4, 2]. In this research, SFPDEs has been discretized using explicit finite difference method using the space caputo derivative with order $1 < \alpha < 2$. Then, we provide an approximation of the solution of the SFPDEs using fuzzy transformation method. The efficiency of the proposed method is shown by two test problems for various values of t. Tables and graphs are used to describe the results, which ensures that results are in an excellent agreement with analytical solution.

2 Basic Definitions And Notations

In this section, approximation and discretization of fractional derivatives as well as fuzzy transformations and their properties has been shown which is required in the next parts.

2.1 Discretization of space fractional derivative

The fractional derivative for $1 < \alpha < 2$ is defined by

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \frac{1}{\Gamma(2-\alpha)} \int_0^x \frac{\partial^2 u(\zeta,t)}{\partial \zeta^2} \frac{d\zeta}{(x-\zeta)^{\alpha-1}} \tag{5}$$

Let $z\,(x,t) = \frac{\partial u(x,t)}{\partial x}$ and $\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = w(x,t)$ then Eq. (5) can be written as

$$w(x,t) = \frac{1}{\Gamma(2-\alpha)} \int_0^x \frac{\partial z(\zeta,t)}{\partial \zeta} \frac{d\zeta}{(x-\zeta)^{\alpha-1}} \tag{6}$$

where with $x_i = i\Delta x$, $i = 1, 2, \ldots, m$ and Δx is space interval, $w(x,t)$ can be obtained at $x = x_{i+1}$ as follow

$$w(x_{i+1},t) = \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{i} \int_{x_j}^{x_{j+1}} \frac{\partial z(\zeta,t)}{\partial \zeta} \frac{d\zeta}{(x_{i+1}-\zeta)^{\alpha-1}}$$
$$= \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{i} \left[\frac{z(x_{j+1},t) - z(x_j,t)}{\Delta x} + O(\Delta x) \right] \times \int_{x_j}^{x_{j+1}} (x_{i+1}-\zeta)^{1-\alpha} d\zeta \tag{7}$$

By defining $b_j = (j+1)^{2-\alpha} - j^{2-\alpha}$, then Eq. (7) becomes as follow

$$w(x_{i+1},t) = \frac{(\Delta x)^{1-\alpha}}{\Gamma(3-\alpha)} \sum_{j=0}^{i} b_{i-j} \left[z(x_{j+1},t) - z(x_j,t) \right] + O((\Delta x)^{3-\alpha}) \tag{8}$$

By definition of $z(x,t)$, we have

$$\frac{\partial u(x_{j+1},t)}{\partial x} = z(x_{j+1},t) \tag{9}$$

and

$$\frac{\partial u(x_j,t)}{\partial x} = z(x_j,t) \tag{10}$$

By Taylor's expansion

$$u(x_{j-1},t) = u(x_j,t) - \Delta x \frac{\partial u(x_j,t)}{\partial x} + \frac{\Delta x^2}{2!} \frac{\partial^2 u(x_j,t)}{\partial x^2} - \ldots \tag{11}$$

$$u(x_j,t) = u(x_{j+1},t) - \Delta x \frac{\partial u(x_{j+1},t)}{\partial x} + \frac{\Delta x^2}{2!} \frac{\partial^2 u(x_{j+1},t)}{\partial x^2} - \ldots \tag{12}$$

by substraction Eq. (12) of Eq. (11), we have

$$\frac{\partial u(x_{j+1},t)}{\partial x} - \frac{\partial u(x_j,t)}{\partial x} = \frac{u(x_{j+1},t) - 2u(x_j,t) + u(x_{j-1},t)}{\Delta x} + O(\Delta x) \tag{13}$$

or

$$z(x_{j+1},t) - z(x_j,t) = \frac{u(x_{j+1},t) - 2u(x_j,t) + u(x_{j-1},t)}{\Delta x} + O(\Delta x) \tag{14}$$

Substitution Eq. (14) into Eq. (8) yields

$$w(x_{i+1},t)=\frac{(\Delta x)^{-\alpha}}{\Gamma(3-\alpha)}\sum_{j=0}^{i}b_{i-j}\left[u(x_{j+1},t)-2u(x_j,t)+u(x_{j-1},t)+O(\Delta x)\right]+O((\Delta x)^{3-\alpha}) \tag{15}$$

or

$$w(x_{i+1},t)=\frac{(\Delta x)^{-\alpha}}{\Gamma(3-\alpha)}\sum_{j=0}^{i}b_{j}\left[u(x_{i-j+1},t)-2u(x_{i-j},t)+u(x_{i-j-1},t)\right]+O((\Delta x)^{1-\alpha})+O((\Delta x)^{3-\alpha}) \tag{16}$$

Now by omitting truncation error terms, we have

$$\frac{\partial^\alpha u(x_{i+1},t)}{\partial x^\alpha}=w(x_{i+1},t)\approx\frac{(\Delta x)^{-\alpha}}{\Gamma(3-\alpha)}\sum_{j=0}^{i}b_{j}\left[u(x_{i-j+1},t)-2u(x_{i-j},t)+u(x_{i-j-1},t)\right] \tag{17}$$

Therefore, we use of equation (17) to approximate the fractional caputo derivative with order $1<\alpha<2$ in x_{i+1}.

2.2 Direct and inverse fuzzy (F-)transforms

Various kinds of transforms such as Fourier, Laplace, integral, wavelet are used as powerful methods for construction of approximation models and for solution of differential and integral-differential equations. F-transform was proposed by Perfilieva in [26] and studied in several papers [27, 28, 35, 8], etc. The F-transform has two phases: direct and inverse. The F-transform of a one-variable function establishes a correspondence between a set of continuous functions on an interval of real numbers and the set of n-dimensional real vectors. The formula inverse F-transform of one-variable converts an n-dimensional vector into another continuous function which approximates the original one. Before introducing F-transform and inverse F-transform, we need to introduce fuzzy partition and basic functions.

Definition 2.1. [26] Let $[a,b]$ be a real close interval and $P=\{x_0,x_1,x_2,...,x_m\}$ be a set of $m+1$ points in $[a,b]$, called nodes, such that $m\geq 2$ and $a=x_0<x_1<x_2<...<x_m=b$.
A family of fuzzy sets $\{A_0(x),...,A_m(x)\}$, called basic functions (BFs)

,where $A_i : [a,b] \longrightarrow [0,1]$, is a fuzzy partition of $[a,b]$ if, for each $i = 0,1,...,m$ and $x_{-1} = a, x_{m+1} = b$:
1. $A_i(x) = 0$ if $x \notin (x_{i-1}, x_{i+1})$(locality)
2. $A_i(x_i) = 1$ and $A_i(x)$ ¿0 if $x \in (x_{i-1}, x_{i+1})$ (positivity)
3. $A_i(x)$ is continuous in $[a,b]$ (continuity)
4. $A_i(x)$ is strictly increasing (decreasing) on(x_{i-1}, x_i) $((x_i, x_{i+1}))$ for $i = 2,...,m-1$
5. $\sum_{i=0}^{m} A_i(x) = 1$ for every $x \in [a,b]$ (Ruspini condition).
In particular, the fuzzy partition $\{A_0(x),...,A_m(x)\}$ from an h-uniform fuzzy partition if , given $h = \frac{(b-a)}{m}$.
6. $x_i = a + h.(i-1)$ (equidistance between consecutive nodes)
7. $A_i(x_i - x) = A_i(x_i + x)$ for every $x \in [0,h]$;
8. $A_{i+1}(x) = A_i(x-h)$ for every $x \in [x_i, x_{i+1}]$.

For example the following formulas give standard triangular membership functions of basic functions:

$$A_0(x) = \begin{cases} \frac{x_1 - x}{h_x} & x_0 \le x \le x_1 \\ 0, & otherwise \end{cases} \qquad A_m(x) = \begin{cases} \frac{x - x_{m-1}}{h_x}, & x_{m-1} \le x \le x_m \\ 0, & otherwise \end{cases} \tag{18}$$

$$for \quad i = 1,2,3,...m-1; \qquad A_i(x) = \begin{cases} \frac{x - x_{i-1}}{h_x}, & x_{i-1} \le x \le x_i \\ \frac{x_{i+1} - x}{h_x}, & x_i \le x \le x_{i+1} \\ 0, & otherwise \end{cases} \tag{19}$$

The F-transform and the inverse F-transform of a one-variable and two-variable functions and their properties were introduced in [26], but only is presented in the following according to the need for two- variable function.

Definition 2.2. [26] Let $A_0(x),...,A_m(x)$ be basic functions which form a fuzzy partition of $[a,b]$ and $B_0(x),...,B_n(x)$ be basic functions which form a fuzzy partition of $[c,d]$. Let $f(x,y)$ be any function from $C([a,b] \times [c,d])$. We say that the $m \times n$-matrix of real numbers $F_{nm}[f] = (F_{ij})$ is the (integral) F-transform of f with respect to $A_1,...,A_m$ and $B_1,...,B_n$ if for each $i = 1,...,m, j = 1,...,n$

$$F_{ij} = \frac{\int_c^d \int_a^b f(x,y) A_i(x) B_j(y) dx dy}{\int_c^d \int_a^b A_i(x) B_j(y) dx dy}, \quad i = 0,1,...,m, \quad j = 0,1,...,n \tag{20}$$

In the discrete case, when an original function $f(x,y)$ is known only at some nodes $(p_k,q_l)\in[a,b]\times[c,d]$, $k=1,...,M, l=1,...N$. We assume additionally that sets $P=\{p_1,...,p_M\}$ and $Q=\{q_1,...,q_N\}$ of these nodes are sufficiently dense with respect to the chosen partitions, i.e.

$$\begin{array}{l}(\forall i)(\exists k)A_i(p_k)>0\\(\forall j)(\exists l)B_j(p_l)>0\end{array}\tag{21}$$

Then (discrete) F-transform of f can be introduced as:

$$F_{ij}=\frac{\sum_{k=1}^{M}\sum_{l=1}^{N}f(p_k,q_l)A_i(p_k)B_j(q_l)}{\sum_{k=1}^{M}\sum_{l=1}^{N}A_i(p_k)B_j(q_l)},\quad i=0,1,...,m,\quad j=0,1,...,n\tag{22}$$

Definition 2.3. [26] Let $A_0,A_1,...,A_m$ and $B_0,B_1,...,B_n$ be basic functions which form fuzzy partitions of $[a,b]$ and $[c,d]$ respectively. Let f be a real function from $C([a,b]\times[c,d])$ and $F_{nm}[f]$ be the F-transform of f with respect to $A_0,A_1,...,A_m$ and $B_0,B_1,...,B_n$. Then the function

$$f_{m,n}^F(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n}F_{ij}A_i(x)B_j(y)\tag{23}$$

is called the inverse F-transform of a real function f on $[a,b]\times[c,d]$.

The below theorem shows that the inverse F-transform f_{mn}^F can approximate the original continuous real function f with an arbitrary precision and also show the convergence of the fuzzy transformation method.

Theorem 2.4. *[26] Let f be any continuous real function on $[a,b]\times[c,d]$. Then for any $\epsilon>0$ there exist m_ϵ, n_ϵ and fuzzy partitions $A_0,A_1,...,A_m$ and $B_0,B_1,...,B_n$ of $[a,b]$ and $[c,d]$ respectively, such that for all $(x,y)\in[a,b]\times[c,d]$*

$$|f_{m_\epsilon n_\epsilon}^F(x,y)-f(x,y)|\le\epsilon\tag{24}$$

The function $f_{m_\epsilon n_\epsilon}^F$ in (24) is the inverse F-transform of f with respect to $A_0,A_1,...,A_m$ and $B_0,B_1,...,B_n$.

3 The Methodology

In this section, we want to provide an approximate solution for the real main equations (1), (2) and (3) with F-transform method. Hence we consider

$$\frac{\partial u(x,t)}{\partial t}=a(x)\frac{\partial u(x,t)}{\partial x}+c(x)\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}}+f(x,t),\quad x\in[a,b],\quad t\in[0,T] \tag{25}$$

By discretizing the interval $[a,b]$ with $x_i=a+i.h_x; i=0,1,...,m$ where $h_x=(b-a)/m$ and $[0,T]$ with $t_k=k.h_t; k=0,1,...,n$ where $h_t=T/n$ and using finite difference method for the ordinary derivative and Eq.(17) for fractional derivative, we will have

$$\frac{u^{k+1}{}_i-u^k{}_i}{h_t}=a_i\frac{u^k{}_{i+1}-u^k{}_i}{h_x}+c_i\frac{(\mathrm{h_x})^{-\alpha}}{\Gamma(3-\alpha)}\sum_{j=0}^{i-1}b_j\left[u^k{}_{i-j+1}-2u^k{}_{i-j}+u^k{}_{i-j-1}\right]+h_t f^k{}_i \tag{26}$$

Where $u_i^k=:u(x_i,t_k)$, $a_i=:a(x_i)$, $c_i=:c(x_i)$ and $f_i^k=:f(x_i,t_k)$. With simple operations, Eq.(26) can be rewritten as follows:

$$\begin{cases} u^{k+1}{}_i=\frac{h_t}{h_x}a_i u^k{}_{i+1}+(1-\frac{h_t}{h_x}a_i)u^k{}_i+c_i\frac{(\mathrm{h_x})^{-\alpha}h_t}{\Gamma(3-\alpha)}\sum_{j=0}^{i-1}b_j\left[u^k{}_{i-j+1}-2u^k{}_{i-j}+u^k{}_{i-j-1}\right]+h_t f^k{}_i, \\ i=0,1,...,m;\quad k=0,1,...n \end{cases} \tag{27}$$

with BCs Eq. (2) will have

$$\begin{matrix} u_0^k=\vartheta_1(\mathbf{t_k}); k=0,1,...,n \\ u_m^k=\vartheta_2(\mathbf{t_k}); k=0,1,...,n \end{matrix} \tag{28}$$

and the IC Eq. (3) will have

$$u_i^0=\vartheta_0(\mathbf{x_i}); i=0,1,...,m \tag{29}$$

Now we calculate $u_i^j:=u_{ij}; i=0,1,...,m, j=0,1,...,n$ with relationships (27), (28) and (29).
To calculate the approximate solution of the original equation, we use the interpolator points $(x_i,t_j,u_{ij}); i=0,1,...,m, j=0,1,...,n$ that apply to the equation and consider the corresponding interpolator function as the approximate solution. There are different methods to calculate the interpolator function, such as Lagrange interpolation, Newton interpolation, spline interpolation, etc. In the following, we want to use the inverse F-transform as an interpolator function.
Suppose x_0, x_1, x_2, ..., x_m be a set of $m+1$ points in $[a,b]$ by h_x-uniform

fuzzy partition, such that $m \geq 3$ and $a = x_0 < x_1 < x_2 < ... < x_m = b$ regarding to definition 2.1 with $h_x = (b-a)/m$ and basic functions $\{A_0, ..., A_m\}$ with triangular membership function and also if t_0, t_1, t_2, ..., t_n be a set of $n+1$ points in $[0, T]$ by h_t-uniform fuzzy partition, such that $n \geq 3$ and $0 = t_0 < t_1 < t_2 < ... < t_n = T$ regarding to definition 2.1 with $h_t = T/n$ and linear basic functions $\{B_0, ..., B_n\}$ with triangular membership function, then regarding to definition (2.3), we consider the interpolation function of analytical solution of the equation (1) with fuzzy transformation method as follows:

$$u(x,t) \approx u^F_{m,n}(x,t) = \textstyle\sum_{i=0}^{m} \sum_{j=0}^{n} A_i(x) B_j(t) u_{ij} \tag{30}$$

Proposition 3.1. *If $u^F_{m,n}(x,t)$ is the function defined by relation (30) then $u^F_{m,n}(x_i,t_j) = u_{ij}; i = 0,1,...,m,\ \ j = 0,1,...,n$ (the property of interpolation is established).*

Proof. With definition (2.1) will have □

$$A_i(x_p) = \begin{cases} 1, & i = p \\ 0 & i \neq p \end{cases} \qquad\qquad B_j(t_q) = \begin{cases} 1, & j = q \\ 0 & j \neq q \end{cases} \tag{31}$$

Then for any p, q

$$u^F_{m,n}(x_p,t_q) = \textstyle\sum_{i=0}^{m} \sum_{j=0}^{n} A_i(x_p) B_j(t_q) u_{ij} = u_{pq} \tag{32}$$

4 Numerical Scheme

In this section, to explain the method presented in this research, we provide two numerical examples with function (30) by using standard triangular membership functions (18), (19).

Example 4.1. Assume following test problem

$$\frac{\partial u(x,t)}{\partial t} = a(x)\frac{\partial u(x,t)}{\partial x} + c(x)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + f(x,t),$$

with BCs

$$\begin{aligned} u(\mathbf{0},t) &= 0, \\ u(\mathbf{1},t) &= \mathbf{e}^{-t}, \end{aligned} \tag{33}$$

and the IC

$$u(x,0) = \mathbf{x}^3, \tag{34}$$

The exact solution is $u(x,t) = \mathbf{x}^3.e^{-t}$,
where $a(\mathbf{x}) = -\mathbf{x}/3, \quad c(\mathbf{x}) = \frac{\Gamma(4-\alpha)}{6} \quad and \quad f(\mathbf{x},\mathbf{t}) = -\mathbf{x}^{(3-\alpha)}.e^{-t}$,

We approximate the solution of this problem in range $[0,1]$ with $\alpha = 1.5$ for various values of Δx and Δt at time $T = 1$. The results of Mean Squared Error (MSE) and Run Time of approach are summarized in Table 1. In Figure 1, we display the results for $\alpha = 1.5$ and $\Delta x = 0.05$ and $\Delta t = 0.0025$ at time $T = 1$.

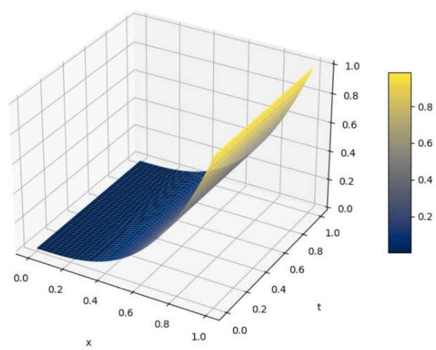

Figure 1: The approximate solution of the problem with $\alpha = 1.5$, $\Delta x = 0.01$ and $\Delta t = 0.0025$

Table 1: Result of proposed method with various Δx and Δt

	Δx	Δt	MSE	Run Time
Case 1	0.04	0.004	0.05	$2s$
Case 2	0.01	0.0025	0.001	$7s$
Case 3	0.005	0.002	0.0007	$16s$

To show effect of fractional order, we calculate example 4.1 with various α and the results has been indicated in Table 2.

Table 2: Effect of fractional order α with $\Delta x = 0.01$ and $\Delta t = 0.0025$ at time $T = 1$ s

	$\alpha_1 = 1.2$	$\alpha_2 = 1.5$	$\alpha_3 = 1.8$
MSE	0.0027	0.001	0.00097

Example 4.2. Assume following test problem

$$\frac{\partial u(x,t)}{\partial t} = a(x)\frac{\partial u(x,t)}{\partial x} + c(x)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + f(x,t),$$

with BCs

$$\begin{aligned} u(\mathbf{0},t) &= 0, \\ u(\mathbf{1},t) &= \mathbf{c}os(-t), \end{aligned} \tag{35}$$

and the IC

$$u(x,0) = \mathbf{x}^2, \tag{36}$$

The exact solution is $u(x,t) = \mathbf{x}^2.cos(-t)$,

where

$a(\mathbf{x}) = -\mathbf{x}/2$, $c(\mathbf{x}) = \frac{\Gamma(3-\alpha)}{2}$, $f(\mathbf{x},t) = -\mathbf{x}^2.sin(-t) - \mathbf{x}^2.cos(-t) - \mathbf{x}^{2-\alpha}.cos(-t)$,

We approximate the solution of this problem in range $[0,1]$ with $\alpha = 1.5$ for various values of Δx and Δt at time $T = 1$. The results are summarized in Table 4 . In Figure 2, we display the results for $\alpha = 1.5$ and $\Delta x = 0.0125$ and $\Delta t = 0.00625$ at time $T = 1$.

Table 3: Result of proposed method with various Δx and Δt

	Δx	Δt	MSE	Run Time
Case 1	0.025	0.0005	0.014	$4s$
Case 2	0.0125	0.00625	0.013	$14s$
Case 3	0.005	0.0005	0.011	$48s$

The results of various α has been shown in Table 4

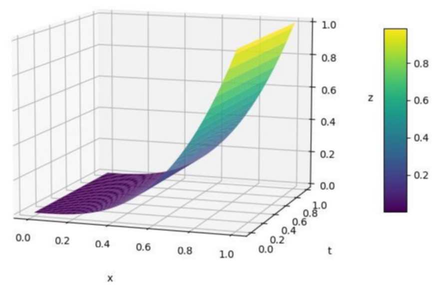

Figure 2: The approximate solution of the problem with $\alpha = 1.5$, $\Delta x = 0.0125$ and $\Delta t = 0.00625$

Table 4: Effect of fractional order α with $\Delta x = 0.0125$ and $\Delta t = 0.00625$ at time $T = 1$ s

	$\alpha_1 = 1.2$	$\alpha_2 = 1.5$	$\alpha_3 = 1.8$
MSE	0.015	0.013	0.001

5 conclusion

A numerical scheme based on finite difference method and inverse F-transfrom method was investigated to solve SFPDEs. We described how we construct shape functions by applying the fuzzy partition and inverse F-transform method. We used the interpolation function with inverse F-transform function because, this method is convergent and to approximate a function in certain intervals, unlike other methods that only use a limited number of points, in this method, all points are used with a degree of accuracy. Obviously, if the number of partition points is increased, the proposed method brings better results and guarantees stability. Finally, the efficiency of the proposed method is shown by two test problems for various values of t. Tables and graphs are used to describe the results, which ensures that results are in an excellent agreement with analytical solution.

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