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Original Research Paper

Fair Fixed Cost Allocation Plan Based on Single-Stage DEA Model in the Presence of Production Trade-offs

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Abstract. In order to apply the DM (DM)'s preferred information in the fixed cost allocation process, the production trade-off method in data envelopment analysis (DEA) can be used. In this paper, we present a fair fixed cost allocation scheme among decision-making units (DMUs) in the presence of production trade-offs from inputs and outputs based on a single-stage DEA model. The model simultaneously considers the importance of inputs and outputs in the fixed cost allocation model. An algorithm is also presented for the fixed cost allocation scheme in the presence of production trade-offs based on the principle of efficiency invariance. In this algorithm, the fixed cost allocation to DMUs is a function of the efficiency scores, the scale of inputs and outputs, and the production trade-offs between their inputs and outputs. We apply the presented algorithm to allocate fixed costs to a set of refineries in Iran which operate under the same management.

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1 Introduction

The radial projection of the inputs and outputs of DMUs often exhibit mixed inefficiency in the standard DEA model. These DMUs are only weakly efficient and not strongly efficient (Thanassoulis et al. [25]). Therefore, these projection DMUs cannot be considered as efficient targets in general. A suitable approach to identify efficient targets is to maximize the component slacks (input and output) as a secondary optimization objective. There are two main methods in the literature that allow us to implement this process: radial improvement factor optimization and component slacks can be performed either in a single optimization step or in separate steps. In the single-stage method, the multiplier DEA model is solved by including a small lower bound $\epsilon \geq 0$ on all weights of input and output, which is equivalent to multiplying the inputs and outputs slacks of by a very small amount of $\epsilon \geq 0$ in the objective function of the envelopment DEA model while simultaneously considering possible mixed inefficiency as a secondary objective (Cooper, Seiford, and Tone [9]). Ali and Seiford [4] shown that the single-stage method may lead to computational difficulties. Then we must select the amount of ? as small as possible to keep the radial efficiency evaluation as the primary objective and the mixed improvement as a secondary objective. Also, using a very small $\epsilon \geq 0$ may lead to unacceptable inaccuracies due to the limited accuracy of the computer solutions. To face these problems, Ali and Seiford [5] developed the two-stage optimization method to obtain the preferred solution method in the DEA models. In this paper, in order to consider the importance of each input and output component, we use DEA models with weight restrictions. These are additional constraints that are often imposed on the input and/or output variable weights as value judgments in the multiplier DEA model that lead to improvements and distinguish between the efficiencies of DMUs. Including weight restrictions in the multiplier DEA model creates additional terms in the envelopment DEA model (Podinovski [19]). Podinovski [19] explained that the trade-offs represent simultaneous changes in inputs and outputs that are possible if executed in each DMU in the technology. Podinovski [20] discussed that the dual relationship between weight restrictions and production trade-offs can be as a basis for constructing weight restrictions and develop the production technology such that all DMUs in this technology be technologically feasible or producible. Podinovski and Bouzdine-Chameeva [21] analyze and review DEA models with weight restrictions and free production. They show that the use of weight restrictions may lead to zero or negative efficiency scores of some DMUs. They stated that such problems arise when weight restrictions allow for free or unlimited output in the production technology. They proposed analytical criteria and computational methods to diagnose the above problems. Atici and Podinovski [3] used production trade-offs method in DEA to assess efficiency of agricultural farms. Podinovski and Bouzdine-Chameeva [22] defined consistent weight restrictions in DEA. They examined the concept of free productions in outputs and showed that in the presence of unlimited and free productions, the envelopment DEA model with production trade-offs can be unbounded. Podinovski et al. [24] developed production trade-offs in DEA model when data are ratio. They proposed these models in the event that both volume and ratio inputs and outputs exist. They showed that, just as in conventional DEA models under constant and variable returns to scale (CRS and VRS), the role of production trade-offs can be included in the model, this role can be shown in the presence of ratio data in the model and its effect on the efficiency scores of the DMUs can be examined. Allocation of fixed costs is one of the important issues in many organizations, including banks, commercial enter prises, and industrial firms. It helps managers take a fair perspective on the organization they manage and prevent the overall waste of resources. Additionally, fixed costs play a key role in decision-making processes related to pricing and determining profitability across different industries. Proper allocation of these costs among various DMUs is essential, such as allocating advertising costs among retailers and distributing health resources and equipment upgrades. In this area, a group of studies have been developed based on the fundamental work of Cook and Kress [7], which introduce two main assumptions: efficiency invariance and Pareto-minimal. The efficiency invariance principle states that after allocating fixed costs among DMUs, the efficiency score of these DMUs should not change compared to the situation before the allocation. In many applications where DEA is used, there is a fixed or common cost imposed on all DMUs. The goal here is to fairly allocate these costs among different units so that each DMUs bears its share of the costs. In Cook and Kress's [7] proposed approach, after fairly allocating shared costs, several computational issues arise that require solving linear programming problems. Beasley [6] introduced an approach called efficiency maximization. The aim of this approach is to improve the average efficiency of DMUs. This approach is particularly useful in environments where balanced and simultaneous allocation among various DMUs is necessary. Beasley [6]'s approach optimizes efficiency score and aid in the optimal allocation of resources and prevent unfair allocation. However, Jahanshahloo et al. [13] provided a simple formula, which resolves these problems easily without the need to solve linear programming problems, thus proposing a straightforward method for cost allocation without computational complexity. This method can be applied to both CCR and BCC DEA models. It should be noted that in the Cook and Kress [7]'s method the principle of Pareto-minimal is not preserved. Subsequently, Cook and Zhu [8] extended the work of Cook and Kress [7] for various models. Lin [15] also suggested modifications based on the approach of Cook and Zhu [8], although the changes were not very significant. Nevertheless, these studies emphasize the importance of allocating fixed costs and addressing concerns in this process, presenting algorithms and theorems that help ensure the correct allocation of costs. Mostafaee [17] introduced a new method for allocating fixed costs to DMUs using DEA, which has some important features as (1) Efficiency preservation: In this method, fixed costs are allocated in a way that the efficiency scores of each DMU remain unchanged before and after the allocation of fixed costs. (2) Minimization of deviation: This method seeks to minimize the deviation between different fixed cost allocations to DMUs. This is crucial because, in many cases, it is necessary to allocate costs in a fair and contextually appropriate manner. (3) Flexibility in managerial perspectives: One notable feature of this method is its high flexibility, allowing it to consider various managerial perspectives when allocating costs. Lin and Chen [16] addressed resource and fixed cost allocation issues using DEA. The researchers proposed a new sharing model in which fixed resources and targets are divided among DMUs. They consider three issues including (1) Efficiency preservation: Similar to previous approaches, the goal of this model is to ensure that the efficiency of DMUs does not change after allocating fixed costs or resources. (2) Reduction of resource wastage: This model also aims to eliminate resource wastage and inefficiencies caused by slack variables. (3) Positive resource and target allocation: In this model, each DMU is allocated both a positive resource and a positive target. An et al. [1] proposed an efficiency-based approach to solve fixed cost allocation issues in two-stage systems, extending this model to general systems. This approach is particularly important in scenarios where decision-making systems involve multiple stages or levels of operation. Utilizing this model in cooperative and non-cooperative systems can help in the optimal allocation of costs while also maintaining system efficiency. Zhu et al. [30] proposed DEA models for measuring fixed cost performance in two-stage systems. In this approach, both input and output factors are carefully considered to ensure the best possible cost allocation. This approach is suitable for complex systems with multi-stage processes where resources must be meticulously allocated. Li et al. [14] extended the traditional fixed cost allocation issue to systems with a two-stage network structure. Using DEA, researchers assess the ratio efficiency of DMUs and make efficient cost allocation possible under a set of common weights. This approach allows DMUs to maximize their efficiency score by choosing different allocations and ratio weights. Due to the diverse allocations existing in the efficient set, allocation programs are optimized with a focus on operational unit sizes, and a minimax model and an algorithm are provided to reduce deviations between efficient allocation and size. Chu et al. [10] expanded the fixed cost allocation method by incorporating the principle of full efficiency offering new approaches for fixed cost allocation in two-stage structures. They propose a range of possible allocations and models to consider competition between two-stage DMUs under a centralized framework. Leader-follower models and the concept of union satisfaction degree have also been utilized to provide stable and acceptable allocations. Dai et al. [12] introduced a two-stage interactive approach for allocating shared revenues or fixed costs. They proposed a method based on the cross-efficiency DEA for evaluating DMUs and incentive allocation. Practical aspects of these methods include improved performance and efficiency in information asymmetry decision-making environments. This approach motivates sub-units to enhance their performance. These studies, by offering diverse methods for cost allocation and emphasizing efficiency and motivation, seek to create innovative strategies in the management of fixed costs and shared revenues in organizations. These approaches have been developed considering complex structures and the need for justice in cost allocation. Pereira et al. [18] used DEA to create composite indices for multidimensional performance that combine DM preferences with weight constraints and artificial targets. They state that understanding the complexity and diversity of complex systems dealing with increasing data volumes is essential for the continuous improvement of public and private institutions. An et al. [2] identified shortcomings in Dai et al. [12]'s approach and proposed two alternative incentive mechanisms for allocating shared revenues or fixed costs. These mechanisms are designed under conditions of informational symmetry and asymmetry and establish criteria for incentive productivity. These mechanisms are tested on real data from a Chinese company based on the efficiency ratio. Chu et al. [11] emphasized fairness in allocation and presented a multi-objective model for cost allocation that considers the needs and preferences of DMUs. This model is particularly important in situations where multiple stake holders with different preferences are involved in decision-making. This approach can create a fair allocation that responds to the diverse needs and desires of DMUs. Zhang et al. [29] introduced aggressive game strategies for cost allocation in a decentralized environment. This approach is particularly useful in systems where DMUs operate independently. Utilizing game theory, this approach seeks agreements among DMUs to allocate costs in a way that ensures both fairness and efficiency. Additionally, this approach demonstrates that after cost allocation, the average efficiencies converge towards the cross-efficiency of the aggressive game. They proposed a fixed cost allocation model based on a DEA aggressive game approach. They developed a computational algorithm based on the DEA aggressive game approach to facilitate consensus and determine the fixed cost allocation plan. Yang et al. [28] introduce a new DEA-based fixed cost allocation method that simultaneously balances individual efficiency assurance goals and collective priority objectives. This approach involves constructing a Priority Value Loss index, which accurately measures the effects of priority considerations. Moreover, our generalized fixed cost allocation strategy ensures the minimization of Priority Value Loss and provides a prioritized evaluation process for selecting the final allocation plan. They examined discrepancies in cost allocation and sought to minimize the deviation between individual efficiency and total preferences. This approach can improve coordination among DMUs and make cost allocation not only mathematically optimal but also minimize disagreements. Xu et al. [27] proposed a fixed cost allocation model based on DEA from inequality aversion perspectives. They developed concept of fairness concern disutility for unique fixed cost allocation. Wang et al. [26] proposed new models for obtaining the equilibrium efficient frontier by proportional frontier shifting in DEA with fixed-sum outputs. We stated that the method presented in this paper uses the strategy of the principle of efficiency invariance. All the previous methods presented for fixed cost allocation used the issue of applying superiority in other ways, but in this paper, we used a suitable method for applying the DM 's superiority information in the fixed cost allocation process, namely the production trade-off method. Using this method, we can provide strongly efficient targets for inefficient units on the new efficiency frontier. These targets are always feasible, and the presentation method is based on an envelopment DEA model, while most methods use multiplier models. So far, no study that uses the production trade-off method for fixed cost allocation has been presented in the DEA literature. This paper aim to obtain a fair fixed cost allocation plan by considering the strategy of efficiency invariance in the presence of production trade-offs. Therefore, we first propose the concept of production trade-offs on inputs and outputs in DEA. Also, we propose a fair fixed cost allocation plan among DMUs in the presence of production trade-offs from inputs and outputs based on a single-stage DEA model. This model applies the importance of inputs and outputs in the fixed cost allocation model simultaneously. We propose an algorithm for donning the fixed cost allocation by considering the input and output values and the efficiency score of DMUs. Finally, we obtain amount of fair fixed cost for DMUs. We can explain that superiority of this paper compared to previous approaches is that by using of proposed algorithm, we obtain a fair fixed cost allocation plane according to preferred acknowledge of DM while the other approaches don't consider the preferred information of DM by considering production trade-offs between inputs and outputs components. The proposed algorithm allocates the fixed cost between efficient and inefficient DMUs simultaneously. The method stated in this paper for applying the DM 's preferred information is the production trade-off method, which is a suitable method according to the literature of DEA. Based on the production trade-off method, we can simultaneously apply the superiority of input and output, which is not possible in other methods of weight restrictions. The proposed algorithm used of reference set of DMUs resulting of single-stage DEA model for determining a fair fixed cost allocation plan among DMUs in the presence of production trade-offs from inputs and outputs. The sections of this paper are as follows: in the second section, we propose the concept of production trade-offs in DEA in order to apply the relative importance of inputs and outputs, in the efficiency evaluation process of DMUs. In the third section, we present a fair fixed cost allocation plan in the presence of production trade-offs based on the single stage DEA model. In the fourth section, we use our approach to a data set of refineries in Iran and provide a cost allocation plan for these refineries, presenting the results of our research.

2 Production Trade-offs in DEA

In this section, we describe how to obtain efficient targets in DEA in the presence of production trade-offs. Consider the set of observation DMU_j, $j = 1, \cdots, n$. The input and output vectors of DMU_j are as $X_j = (x_{1j}, \cdots, x_{mj}) \in R^m_+$ and $Y_j = (y_{1j}, \cdots, y_{sj}) \in R^s_+$ respectively. We consider DMU under evaluation as DMU_o, $o \in \{1, \cdots, n\}$. The weight restrictions on the vectors $v \in R^m_+$ and $u \in R^s_+$ of inputs and outputs are as follows:

$$v^T M_t - u^T N_t \leq 0, \quad t = 1, \cdots, L. \tag{1}$$

This weight restriction is a homogeneous (Podinovski [19]). Components of matrixes M_t and N_t, $t = 1, \cdots, L$ can be negative, positive or equal to zero. To calculate the efficiency score of the DMU_o in the presence of weight restriction, we need to solve model (2) as follows.

$$\begin{aligned}
\min \quad & \sum_{i=1}^{m} v_i x_{io} \\
s.t. \quad & \sum_{r=1}^{s} u_r y_{ro} = 1, \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \qquad j = 1, \cdots, n, \\
& \sum_{r=1}^{s} u_r N_{rt} - \sum_{i=1}^{m} v_i M_{it} \leq 0, \qquad t = 1, \cdots, L, \\
& \quad u_r \geq 0, v_i \geq 0, \ i = 1, \cdots, m, \ r = 1, \cdots, s.
\end{aligned} \tag{2}$$

The dual model (2) in the envelopment form is as follows.

$$\begin{aligned}
\min \quad & \gamma_o \\
s.t. \quad & \sum_{j=1}^{n} \mu_{jo} x_{ij} + \sum_{t=1}^{L} \pi_{to} M_{it} \leq x_{io}, \quad i = 1, \cdots, m, \\
& \sum_{j=1}^{n} \mu_{jo} y_{rj} + \sum_{t=1}^{L} \pi_{to} N_{rt} \geq y_{ro}, \quad r = 1, \cdots, s, \\
& \quad \mu_{jo} \geq 0, j = 1, \cdots, n, \\
& \quad \pi_{to} \geq 0, \ t = 1, \cdots, L.
\end{aligned} \tag{3}$$

Considering the weight restriction (1) in the multiplier model (2) creates an additional term based on matrixes M_t and N_t , $t = 1, \cdots, L$. This term is used in the envelopment model (3) with a multiplier $\pi_t \geq 0$ $t = 1, \cdots, L$. Podinovski [19] defined this term as the production trade-offs on the inputs and outputs. This set includes all DMUs as $(X, Y) \in R_{+}^{m+s}$ such that there exist intensity vectors $\mu \in R_{+}^{n}$, $\pi \in R_{+}^{L}$, and slack vectors $\alpha = (\alpha_1, \cdots, \alpha_m) \in R_{+}^{m}$, $\beta = (\beta_1, \cdots, \beta_s) \in R_{+}^{s}$ such that

$$\begin{aligned}
& \sum_{j=1}^{n} \mu_j X_j + \sum_{t=1}^{L} \pi_t M_t + \alpha = X, \\
& \sum_{j=1}^{n} \mu_j Y_j + \sum_{t=1}^{L} \pi_t N_t + \beta = Y,
\end{aligned} \tag{4}$$

It should be noted that expressions $\sum_{t=1}^{L}\pi_t M_t$ and $\sum_{t=1}^{L}\pi_t N_t$ modify the virtual unit $(\sum_{j=1}^{s}\mu_j X_j, \sum_{j=1}^{s}\mu_j Y_j)$ by applying production trade-offs to some ratios $\pi_{to} \geq 0$, $t = 1, \cdots, L$. Also, the resulting unit changes by increasing its inputs by vector R_{+}^{m} and and decreasing its outputs by a vector R_{+}^{s} that is consistent with the axiom of free disposability in making the production possibility set. In order to obtain the efficient targets corresponding to each of the DMUs in the presence of production trade-offs, Podinovski and Bouzdine-Chameeva [23] presented a single-stage model that simultaneously obtains the radial efficiency of outputs and possible mix inefficiency as follows.

$$
\begin{aligned}
\min \quad & \gamma_o^{TO} + \epsilon(\sum_{i=1}^{m} f_i + \sum_{r=1}^{s} g_r) \\
s.t. \quad & \sum_{j=1}^{n} \mu_{jo} x_{ij} + \sum_{t=1}^{L} \pi_{to} M_{it} + \alpha_i = x_{io} - f_i, \quad i = 1, \cdots, m, \\
& \sum_{j=1}^{n} \mu_{jo} y_{rj} + \sum_{t=1}^{L} \pi_{to} N_{rt} = \gamma_o^{TO} y_{ro} + g_r, \quad r = 1, \cdots, s, \\
& x_{io} - f_i \geq 0, \quad i = 1, \cdots, m, \\
& \mu_{jo} \geq 0, j = 1, \cdots, n, \\
& \pi_{to} \geq 0, \ t = 1, \cdots, L, \\
& f_i \geq 0, \ \alpha_i \geq 0 \ \ i = 1, \cdots, m, \\
& g_r \geq 0, \ r = 1, \cdots, s,
\end{aligned}
\tag{5}
$$

Let $(\bar{\mu}_o, \bar{\pi}_o, \bar{\alpha}, \bar{f}, \bar{g}, \bar{\gamma}_o^{TO})$ to be an optimal solution of model (5). The projection of DMU_o on the efficiency frontier of the production possibility set under CRS technology in the presence of production trade-offs is defined as follows, which is the corresponding efficient target for DMU_o.

$$(\bar{X}, \bar{Y}) = (X_o - \bar{f}, \bar{\gamma}_o^{TO} Y_o + \bar{g}). \tag{6}$$

Theorem 2.1. *The model (5) is always feasible.*

Proof. Put $\mu_{oo} = 1$, $\mu_{jo} = 0$, $j = 1, \cdots, n$, $j \neq o$, $\pi_{to} = 0$, $t = 1, \cdots, L$, $\alpha_i = f_i = 0$, $i = 1, \cdots, m$, $\gamma_o^{TO} = 1$, $g_r = 0$, $r = 1, \cdots, s$ we obtain a feasible solution for model (7) and the proof is completed. □

Theorem 2.2. $(\bar{X}, \bar{Y})$ *is an efficient DMU in production technology in the presence of production trade-offs.*

Proof. Let $(\bar{X}, \bar{Y}) = (X_o - \bar{f}, \bar{\gamma}_o^{TO} Y_o + \bar{g})$ is inefficient DMU in the production technology in the presence of production trade-offs. Then there is a DMU belong to production technology such that
$\hat{X} = \bar{X} - \Delta$ and $\hat{Y} = \bar{X} + q$ where at least one of the vectors $\Delta \in R_+^m$, $q \in R_+^s$, such that is non-zero. Given that a unit $(\bar{X}, \bar{Y})$ is belong to production technology in the presence of production trade-offs. Then, there are vectors $(\hat{\mu}_o, \hat{\pi}_o, \hat{\alpha}, \hat{\beta}, \hat{f}, \hat{g})$ such that $\sum_{j=1}^{n} \hat{\mu}_o x_{ij} + \sum_{t=1}^{L} \hat{\pi}_{to} M_{it} + \hat{\alpha}_i =$
$\hat{x}_i = x_{io} - \bar{f}_i - \Delta_i, \quad i = 1, \cdots, m,$
$\sum_{j=1}^{n} \hat{\mu}_{jo} y_{rj} + \sum_{t=1}^{L} \hat{\pi}_{to} N_{rt} - \hat{\beta}_r = \hat{y}_r = \hat{\gamma}_o^{TO} y_{ro} + \bar{g}_r + q_r, \quad r = 1, \cdots, s,$
Put $\check{f} = \bar{f} + \Delta$, $\check{g} = \bar{g} + \hat{\beta} + q$. Then $(\hat{\mu}_o, \hat{\pi}_o, \hat{\alpha}, \check{f}, \check{g}, \bar{\gamma}_o^{TO} Y_o)$ is an optimal solution of model (5) and we will have
$\sum_{i=1}^{m} \check{f}_i + \sum_{r=1}^{s} \check{g}_r > \sum_{i=1}^{m} \bar{f}_i + \sum_{r=1}^{s} \bar{g}_r.$
Then the solution $(\bar{\mu}_o, \bar{\pi}_o, \bar{\alpha}, \bar{\beta}, \bar{f}, \bar{g})$ is not an optimal solution of model (5) and we will have a contradiction. Then $(\bar{X}, \bar{Y})$ is an efficient DMU in production technology in the presence of production trade-offs. The proof is completed. □

3 Fixed Cost Allocation in the Presence of Production Trade-offs

In this section, we present a fixed cost allocation scheme among DMUs based on DEA in the presence of production trade-offs of inputs and outputs. For this purpose, we use the strategy of not changing efficiency score after the fixed cost allocation process. To obtain a fair cost allocation scheme, we use the simultaneous effect of inputs, the scale of inputs and outputs, and the efficiency scores of DMUs. Suppose that we want to allocate a total cost R among DMUs. In this paper, we consider the cost allocated to each DMU as a new input for the DMUs. We denote this cost by R_j . A single-stage model for calculating the efficiency score and efficient targets in the presence of production trade-offs by considering cost as a new input is presented below.

$$
\begin{aligned}
\min\ & \gamma_o^{CA} + \epsilon(\sum_{i=1}^{m} f_i + \sum_{r=1}^{s} g_r) \\
s.t.\ & \sum_{j=1}^{n} \mu_{jo} x_{ij} + \sum_{t=1}^{L} \pi_{to} M_{it} + \alpha_i = x_{io} - f_i, \quad i = 1, \cdots, m, \\
& \sum_{j=1}^{n} \mu_{jo} R_j + \sum_{t=1}^{L} \pi_{to} M_{m+1t} + \alpha_{m+1} = R_o - f_{m+1}, \\
& \sum_{j=1}^{n} \mu_{jo} y_{rj} + \sum_{t=1}^{L} \pi_{to} N_{rt} = \gamma_o^{CA} y_{ro} + g_r, \quad r = 1, \cdots, s, \\
& \sum_{j=1}^{n} R_j = R, \quad R_j \geq 0, \\
& R_o - f_{m+1} \geq 0, \\
& \mu_{jo} \geq 0, j = 1, \cdots, n, \\
& \pi_{tO} \geq 0,\ t = 1, \cdots, L, \\
& f_i \geq 0,\ \alpha_i \geq 0 \ \ i = 1, \cdots, m, \\
& g_r \geq 0,\ r = 1, \cdots, s, \\
& \gamma_o^{CA}\ free\ sign, \quad \epsilon\ is\ sufficiently\ small.
\end{aligned} \tag{7}
$$

Theorem 3.1. *The model (7)is always feasible.*

Proof. Put $\mu_{oo} = 1$, $\mu_{jo} = 0$, $j = 1, \cdots, n$, $j \neq o$, $\pi_{to} = 0$, $t = 1, \cdots, L$, $\alpha_i = f_i = 0$, $i = 1, \cdots, m$, $\gamma_o^{CA} = 1$, $g_r = 0$, $r = 1, \cdots, s$, $R_o = R$, $R_j = 0$, $j = 1, \cdots, n$, $j \neq o$, in this case, we obtain a feasible solution for model (7) and the proof is completed. □

Model (7) is a nonlinear model that must be solved with nonlinear software. For obtaining the optimal fixed cost allocation corresponding to each of DMUs, we solve model (7), that is a difficult work, by attention to this fact model (7) is nonlinear. As results, we can solve model (5), for obtaining a fair fixed cost allocation plan in the presence of production trade-offs of inputs and outputs. In the following, we propose this process.

Now we present an algorithm to obtain a fair fixed cost allocation plan in the presence of production trade-offs of inputs and outputs. Suppose we want the efficiency scores of the DMUs not to change after the fixed cost is assigned to them. Suppose $(\gamma_o^{CA^*}, \mu_{jo}^*, j = 1, \cdots, n)$ is an optimal solution of model (5) in the evaluation of DMU_o. We denote the set of efficient and inefficient DMUs obtained from model (5) corresponding to DMU_o by $ET = \{DMU_j | \mu_{jo}^* > 0\}$ and $IET = \{DMU_1, \cdots, DMU_n\} - ET$ respectively. The fixed cost allocation plan must be such that the fixed cost is assigned simultaneously to the efficient and inefficient units. In order for the efficiency score in the desired allocation plan not to change. The equation (8) is valid only for the efficient DMUs in the reference set corresponding to DMU_o based on the model (5).

$$\sum_{j=1}^{n} \mu_{jo}^* R_j + \sum_{t=1}^{L} \pi_{to}^* M_{m+1t} \leq R_o, \tag{8}$$

As can be seen, we let $(\gamma_o^{CA^*}, \mu_{jo}^*, j = 1, \cdots, n)$ be an optimal solution of model (5) in the evaluation of DMU_o. That is μ_{jo}^* for indices that do not belong to the reference set corresponding to DMU_o must be zero. That is $\gamma_o^{CA^*} = \gamma_o^{TO^*} > 1$ is established for inefficient DMUs.
Now, if we want the amount of fixed cost allocated to each DMUs to be dependent on the scale of the input and output components and the efficiency score of DMU_j based on model (5). We define the proportion δ_j of total fixed cost R corresponding to DMU_j as follows.

$$\delta_j = \frac{\dfrac{(\sum_{i=1}^{m} x_{ij} * \sum_{r=1}^{s} y_{rj})}{\gamma_o^{TO^*}}}{\sum_{j=1}^{n} (\dfrac{(\sum_{i=1}^{m} x_{ij} * \sum_{r=1}^{s} y_{rj})}{\gamma_o^{TO^*}})}, \quad j = 1, \cdots, n. \tag{9}$$

According to equation (9), the allocated cost ratio to DMU_j depends on the scale of inputs and outputs and the efficiency score of the DMU_j. Namely DMUs with a larger input-output scale should bear a larger proportion of the fixed cost. Also $\frac{1}{\gamma_o^{TO^*}}$ indicates the efficiency score corresponding to DMU_o obtained from model (5), which is multiplied by the cost ratio, therefore, the larger the corresponding efficiency score, the greater the cost allocated to this DMU. Then we let

$$\hat{R}_j = \delta_j R = \frac{\frac{(\sum_{i=1}^{m} x_{ij} * \sum_{r=1}^{s} y_{rj})}{\gamma_o^{TO*}}}{\sum_{j=1}^{n}(\frac{(\sum_{i=1}^{m} x_{ij} * \sum_{r=1}^{s} y_{rj})}{\gamma_o^{TO*}})} R, \quad j = 1, \cdots, n. \tag{10}$$

In order for the efficiency score of the DMUs not to change after the allocation of fixed cost, we define another principle called minimum deviation, which means that without violating the invariance of efficiency score, the distance between the allocated cost and the corresponding costs must be minimized. So, we define.

$$\tau(r) = \sqrt{\sum_{j=1}^{n}(R_j - \hat{R}_j)^2} \tag{11}$$

Therefore, to obtain the allocated cost to each of the DMUs, we present the following model.

$$\begin{aligned} \min \quad & \sum_{j=1}^{n}(R_j - \hat{R}_j)^2 \\ s.t. \quad & \sum_{j \in ET} \mu_{jo}^* R_j \leq R_o, \quad o \in IET, \\ & \sum_{j=1}^{n} R_j = R, \quad R_j \geq 0, \end{aligned} \tag{12}$$

In model (12), we modify the set IET based on model (5), and μ_{jo}^* are obtained from solving model (5) corresponding to DMU_o. It is possible for each DMU to pay its corresponding relative cost, that is $R_j = \hat{R}_j$ without the defect of the principle of efficiency invariance. The presented model can be presented under VRS technology by adding a constraint $\sum_{j=1}^{n} \mu_{jo} = 1$. Therefore, we present the following algorithm for obtaining the fair fixed cost allocation plan as follows.

Table 1: An algorithm for obtaining the fair fixed cost allocation plan.

Step 1: Define the production trade-off matrices M_t and N_t, $j = 1, \cdots, n$ and solve model (5) to obtain the sets ET and IET.
Step 2: Obtain the values of $\hat{R}_j$ from equation (10).
Step 3: Solve model (12) to obtain the fair fixed cost allocation plan.

Table 2: Data set and the results of Cook and Kress [7].

DMUs	Input 1	Input 2	Input 3	Output 1	Output 2
1	350	39	9	67	751
2	298	26	8	73	611
3	422	31	7	75	584
4	281	16	9	70	665
5	301	16	6	75	445
6	360	29	17	83	1070
7	540	18	10	72	457
8	276	33	5	78	590
9	323	25	5	75	1074
10	444	64	6	74	1072
11	323	25	5	25	350
12	444	64	6	104	1199

4 Numerical Examples

In this section, we use a numerical example to illustrate the models presented in this paper for the fixed cost allocation scheme. This example has been used of Cook and Kress [7] paper. Data set includes 12 DMUs that each has three inputs and two outputs. These data are in the Table 2. We want to allocate a fixed cost of 100 units among the DMUs. At first, to incorporate the DM 's preferred information in the fixed cost allocation plan, we apply the method of production trade-offs. We consider the production trade-offs (1). For this purpose, we define the matrices M_1 and N_1 for inputs and outputs as follows: Production trade-offs (1): $M_1 = (2, 0.75, -4)$, $N_1 = (-1, 3)$.
Then, the equivalent weight restriction of production trade-off (1) is as follows:
$3u_2 - u_1 - 2v_1 - 0.75v_2 + 4v_3 \leq 0.$

Table 3: The results of model (5) with trade-offs 1.

DMUs	The efficiency score of model (5)	Set ET
1	0.7825	DMU6, MU8,
2	0.9453	DMU9, DMU12
3	0.8921	DMU4, MU5,
4	1	DMU6, DMU8
5	1	DMU5, MU6,
6	1	DMU12
7	0.9445	DMU4
8	1	DMU5
9	1	DMU6
10	0.8882	DMU5, DMU6
11	0.3325	DMU8
12	1	DMU9
DMUs	The multipliers of the model (5)	π_o^*
1	$\mu_6^* = 0.2219$, $\mu_8^* = 0.3048$, $\mu_9^* = 0.1996$, $\mu_{12}^* = 0.2737$	0
2	$\mu_4^* = 0.111$, $\mu_5^* = 0.2588$, $\mu_6^* = 0.1783$, $\mu_8^* = 0.4519$	0
3	$\mu_5^* = 0.6212$, $\mu_6^* = 0.0909$, $\mu_{12}^* = 0.2879$	0
4	$\mu_4^* = 1$	0
5	$\mu_5^* = 1$	0
6	$\mu_6^* = 1$	0
7	$\mu_5^* = 0.8462$, $\mu_6^* = 0.1538$	0
8	$\mu_8^* = 1$	0
9	$\mu_9^* = 1$	0
10	$\mu_9^* = 0.2554$, $\mu_{12}^* = 0.7446$	13.2803
11	$\mu_5^* = 0.0331$, $\mu_8^* = 0.0027$, $\mu_9^* = 0.9573$, $\mu_{12}^* = 0.0069$	0.01
12	$\mu_{12}^* = 1$	0

This production trade-off presents the relationship between amount of outputs produced from inputs and the corresponding input-output weights. Column second of Table (3) propose the results of model (5) in evaluating the efficiency of the DMUs in the presence of production trade-off (1). We select $\epsilon = 0.0001$. The DMUs 4, 5, 6, 8, 9, and 12 are efficient, while the others DMUs are inefficient. Column third and fourth of Table 3 shows the reference set and the corresponding multipliers for the DMUs in the reference set for each DMU of model (5). Additionally, the last column of the table shows the multipliers corresponding to only the considered production trade-offs. Now, to propose the efficient targets

Table 4: Targets of DMUs of model (5) with trade-offs 1.

DMUs	Input 1	Input 2	Input 3	Output 1	Output 2
1	350	39	7.9368	85.6267	959.7859
2	298	26	7.8422	77.2268	646.3774
3	347.5303	31	7	84.0758	718.8788
4	281	16	9	70	665
5	301	16	6	75	445
6	360	29	17	83	1070
7	310.0769	18	7.6923	76.2308	541.1538
8	276	33	5	78	590
9	323	25	5	75	1074
10	439.6583	64	0	83.3133	1206.9171
11	323	25	5	75.198	1052.7724
12	444	64	6	104	1199

for each DMU in the presence of production trade-off (1) of model (5) by relation (6). These targets are efficient in the presence of the production trade-offs (1) according to Theorem 2. Table 4 shows efficient targets of model (5). In the future, to obtain the fair fixed cost allocation plan by production trade-off (1), we first use equation (9) to adjust the δ_j values corresponding to each DMU according to their efficiency scores from model (5) and their input-output scales. These values are presented in the third column of Table (5). Also, the fourth column of Table 5 shows the cost allocated to the units based on relation (10). This cost is determined according to the efficiency scores of the DMUs in the presence of production trade-offs and the input-output measures. Now, to obtain the fixed cost allocation plan in the presence of production trade-offs, we solve model (12). The results obtained from model (12) are in the last column of Table 5. This cost varies according to the efficiency scores, input-output measures and the production trade-off (1).

In order to done a sensitivity analysis of the results of models (5) and (12) relative to changes in the production trade-off matrix, we select these matrices differently. In the second choice, we select the production trade-off matrices as follows:

Production trade-offs (2): $M_1 = (3, -4, 0.5)$, $N_1 = (5, -1)$.

Then, the equivalent weight restriction of production trade-off (2) is as

Table 5: The results of fixed cost allocation plan with trade-offs 1.

DMUs	Scale of data	δ_j	$\hat{R}_j$	Fixed cost allocation
1	416056.23	0.092	9.2015	7.5454
2	240228.4989	0.0531	5.3129	6.8979
3	339804.9546	0.0752	7.5151	10.8782
4	224910	0.0497	4.9741	0
5	167960	0.0371	3.7146	12.6117
6	468118	0.1035	10.3529	0
7	318128.1101	0.0704	7.0357	10.672
8	209752	0.0464	4.6389	8.0416
9	405597	0.0897	8.9702	11.0257
10	663188.4711	0.1467	14.6671	10.6881
11	398120.3008	0.088	8.8048	11.067
12	669742	0.1481	14.812	10.5723
Sum	4521605.566	1	100	100

follows: $-1u_2 + 5u_1 - 3v_1 + 4v_2 - 0.5v_3 \leq 0$. The results of the presented algorithm are given in Tables 6 to 8.

As can be seen, the cost allocated to the DMUs changes by changing the production trade-off matrix, and this is one of the strengths of the algorithm presented in this paper. By using the production trade-off matrix in the model (5), we can apply the DM 's opinion in the fixed cost allocation process.

5 An Application of Proposed Approach in the Iranian Oil Refineries Companies

Now, we use the proposed approach in Iranian oil refineries. In an oil well, drilling is done in the ground to explore and extract crude oil. An oil field is a geographical area where several oil wells can be drilled on oil reservoirs or fields and crude oil can be extracted from them. Oil refining companies are important in the economy of all countries. Given the importance of these companies, we conducted our case study for these companies. Given the importance of oil refining companies in Iran, we chose a case study for these companies. In many cases, company management must allocate a fixed cost to the companies under

Table 6: The results of model (5) with trade-offs 2.

DMUs	The efficiency score of model (5)	Set ET
1	0.6909	DMU9, DMU12
2	0.8231	DMU8, DMU9
3	0.7219	DMU9, DMU12
4	1	DMU4
5	1	DMU5
6	0.9677	DMU9, DMU12
7	0.5205	DMU9, DMU12
8	1	DMU8
9	1	DMU9
10	0.8941	DMU12
11	0.3333	DMU9
12	1	DMU12
DMUs	The multipliers of the model (5)	π_o^*
1	$\mu_9^* = 0.8667$, $\mu_{12}^* = 0.1333$	3.6227
2	$\mu_8^* = 0.6806$, $\mu_9^* = 0.3194$	2.3287
3	$\mu_9^* = 0.5319$, $\mu_{12}^* = 0.4681$	3.0638
4	$\mu_4^* = 1$	0
5	$\mu_5^* = 1$	0
6	$\mu_9^* = 0.7338$, $\mu_{12}^* = 0.2662$	1.5957
7	$\mu_9^* = 0.2979$, $\mu_{12}^* = 0.7021$	8.5957
8	$\mu_8^* = 1$	0
9	$\mu_9^* = 1$	0
10	$\mu_{12}^* = 1$	0
11	$\mu_9^* = 1$	0
12	$\mu_{12}^* = 1$	0

their management, and this allocation should be in a way that does not worsen the performance of the companies. Given the importance of these companies in areas such as energy and the stock market, we chose them for our case study. After the crude oil leaves the well, it requires a series of refining, storage and transportation of oil. Crude oil obtained from the well contains impurities such as salt, water and dissolved gases. Special equipment is provided to separate each of these impurities. After refining the crude oil extracted from the well, it must first be stored. In the next stage, the crude oil is sent to ports for export or to refineries for refining and processing in various ways. An oil refinery is an industrial unit

Table 7: Targets of DMUs of model (5) with trade-offs 2.

DMUs	Input 1	Input 2	Input 3	Output 1	Output 2
1	350	15.7087	6.9447	96.9798	1087.0425
2	298	21.1297	6.1643	88.6852	742.2826
3	388.8298	31	7	103.8936	1129.4468
4	281	16	9	70	665
5	301	16	6	75	445
6	360	29	6.0641	90.6988	1105.6822
7	433.7447	18	10	138.3404	1153.1702
8	276	33	5	78	590
9	323	25	5	75	1074
10	444	64	6	104	1199
11	323	25	5	75	1074
12	444	64	6	104	1199

Table 8: The results of fixed cost allocation plan with trade-offs 2.

DMUs	Scale of data	δ_j	$\hat{R}_j$	Fixed cost allocation
1	471217.2529	0.095	9.4965	11.7372
2	275893.5731	0.0556	5.5601	6.8006
3	419919.6565	0.0846	8.4627	10.7275
4	224910	0.0453	4.5327	0
5	167960	0.0338	3.3849	2.5558
6	483742.8955	0.0975	9.749	11.3364
7	577275.6964	0.1163	11.634	10.0218
8	209752	0.0423	4.2272	4.2953
9	405597	0.0817	8.1741	12.1392
10	658812.2134	0.1328	13.2772	9.1234
11	397164.7165	0.08	8.0041	12.1392
12	669742	0.135	13.4975	9.1234
Sum	4961987.004	1	100	100

where crude oil is converted into more useful substances such as liquefied petroleum gas, kerosene, gasoline, diesel, fuel oil, bitumen and other petroleum products. In this paper, we will evaluate the oil refineries under the management of the National Petroleum Products Company. One of the goals of management is to allocate a fixed cost to the units under its management. It should allocate this cost to the oil refineries in such a way that the performance of these refineries does not change and these companies continue to have their previous level of efficiency. In this regard, we will use the approach presented in this paper. Table (9) presents the input, output data. Input data are including: The cost of building an oil rig ($I1$), The cost of building an oil well ($I2$), Cost of transporting petroleum products ($I3$). Also output data are including: The amount of fuel oil exports ($O1$), Sales price of petroleum products ($O2$), Total production of petroleum products ($O3$).

Suppose we want to allocate a cost of 6,800 Dollar between these companies. In order to apply superior management information to the fixed cost allocation process, we use the production trade-offs between inputs or outputs in the model as follows.
We select the production trade-offs (3). In this way, we define the matrices M_1, M_2, N_1 and N_2 for inputs and outputs respectively as follows:
Production trade-offs (3): $M_1 = (1, 0.5, 2)$, $N_1 = (5, 1, -2)$, $M_2 = (3, 1, -1)$, $N_2 = (1, -2, 0.75)$. Then, the equivalent weight restrictions of production trade-off (2) is as follows:
$-2u_3 + 1u_2 + 5u_1 - 1v_1 - 0.5v_2 - 2v_3 \leq 0,$
$0.75u_3 - 2u_2 + 1u_1 - 3v_1 - 1v_2 + 1v_3 \leq 0,$
Column second of Tables 10 and 11 propose the results of model (5) in evaluating the efficiency scores of the companies in the presence of production trade-off (3). We select $\epsilon = 0.000001$. The companies 1, 2, 3, 19, and 20 are efficient, while the others DMUs are inefficient. Column third and fourth of Table 11 shows the set ET and the corresponding multipliers for the DMUs in the reference set for each company of model (5). Also, the two-last column of the table shows the multipliers corresponding to only the considered production trade-offs.

Table 12 proposes the efficient targets for companies in the presence of production trade-off (3) of model (5) by relation (6).

Now, we obtain the fair fixed cost allocation plan by production trade-off (3) for companies, we apply relation (9) to obtain the δ_j values corresponding to each company according to their efficiency scores from model (5) and their input-output scales. These values are presented in the third column of Table 13. Also, the fourth column of Tables 13 and 14 shows the cost allocated to the companies based on relation (10). This cost is determined according to the efficiency scores of the DMUs in

Table 9: The data set of Iranian oil refineries.

Companies	I1	I2	I3
C01	14432	299731	557
C02	16381	288081	587
C03	21163	241465	582
C04	26747	261624	574
C05	22296	271722	566
C06	22640	287666	567
C07	29427	331914	575
C08	16141	345004	583
C09	21921	326139	602
C10	23207	346213	600
C11	18063	391630	605.2
C12	25318	334375	612.2
C13	15570	356752	607.1
C14	13096	454372	627.7
C15	12318	417493	645.6
C16	6250	339639	661.8
C17	11755	382784	637.4
C18	9430	356290	613.2
C19	6170	293606	610.4
C20	2704	254269	614.2
Companies	O1	O2	O3
C01	15805	12	18035
C02	18188	17.2	17838
C03	16056	26.8	19330
C04	14293	22.9	19176
C05	15410	23.5	17103
C06	14925	26.9	17719
C07	13641	34.6	20717
C08	13457.6	50.7	21845
C09	14183.8	61.1	23308
C10	10217.2	69.3	24154
C11	9132.3	94.7	21453
C12	8213.6	61.3	25960
C13	10700.3	78.2	24968
C14	10922.3	108.3	26188
C15	6105.2	109.7	28005
C16	8982.7	107.2	28644
C17	6740.1	97.3	30578
C18	10063.4	51.4	35401
C19	16747.4	41.6	38740
C20	16775.4	120.4	33502

Table 10: The results of model (5) with trade-offs 3.

DMUs	The efficiency score	Set ET	π^*_{1o}
C01	1	DMU1	3184.6282
C02	1	DMU2	3814.7001
C03	1	DMU3	0
C04	0.8349	DMU2, DMU20	2388.7304
C05	0.9284	DMU1, DMU20	3302.8108
C06	0.9181	DMU1, DMU20	2994.232
C07	0.815	DMU1, DMU20	1723.6111
C08	0.8335	DMU1, DMU20	1334.961
C09	0.822	DMU2, DMU20	1243.4378
C10	0.7492	DMU1, DMU20	0
C11	0.9248	DMU20	0
C12	0.7243	DMU19, DMU20	0
C13	0.7435	DMU19, DMU20	0
C14	0.8517	DMU20	27
C15	0.8301	DMU19, DMU20	64.5169
C16	0.8306	DMU19, DMU20	98.7426
C17	0.8761	DMU19, DMU20	49.5195
C18	0.9336	DMU19, DMU20	4.5123
C19	1	DMU19	0
C20	1	DMU20	0

Table 11: The results of model (5) with trade-offs 3.

DMUs	The multipliers	π^*_{2o}
C1	$\mu^*_1 = 1$	1649.5141
C2	$\mu^*_2 = 1$	1934.55
C3	$\mu^*_3 = 1$	0
C4	$\mu^*_2 = 0.2165$, $\mu^*_{20} = 0.3048$	1234.5652
C5	$\mu^*_1 = 0.2188$, $\mu^*_{20} = 0.7812$	1699.6054
C6	$\mu^*_1 = 0.5502$, $\mu^*_{20} = 0.4498$	1544.316
C7	$\mu^*_1 = 0.2979$, $\mu^*_{20} = 0.0755$	901.0055
C8	$\mu^*_1 = 0.4717$, $\mu^*_{20} = 0.5283$	698.6805
C9	$\mu^*_2 = 0.3286$, $\mu^*_{20} = 0.6714$	633.9189
C10	$\mu^*_1 = 0.0822$, $\mu^*_{20} = 0.9178$	13.7745
C11	$\mu^*_{20} = 1$	9
C12	$\mu^*_{19} = 0.4462$, $\mu^*_{20} = 0.5538$	0.2911
C13	$\mu^*_{19} = 0.0143$, $\mu^*_{20} = 0.9857$	6.752
C14	$\mu^*_{20} = 1$	0
C15	$\mu^*_{19} = 0.0512$, $\mu^*_{20} = 0.9488$	0
C16	$\mu^*_{19} = 0.1969$, $\mu^*_{20} = 0.8031$	0
C17	$\mu^*_{19} = 0.2722$, $\mu^*_{20} = 0.7278$	0
C18	$\mu^*_{19} = 0.8433$, $\mu^*_{20} = 0.1567$	0
C19	$\mu^*_{19} = 1$	0
C20	$\mu^*_{20} = 1$	0

Table 12: Targets of DMUs of model (5) with trade-offs 3.

Companies	I1	I2	I3
C01	14432	299731	557
C02	16381	288081	587
C03	21163	241465	582
C04	5772.1713	261624	574
C05	5377.2036	264252.0188	566
C06	9203.7991	279297.4016	567
C07	3694.0479	257736.0255	575
C08	8248.5001	275716.6472	583
C09	7221.0769	265387.3048	602
C10	3696.4827	258015.2649	600
C11	2731	254278	605.2
C12	4251.5264	271822.4285	612.2
C13	2774.7847	254839.5198	607.1
C14	2710.75	254272.375	627.7
C15	2897.3754	256292.302	645.6
C16	3410.5731	262025.9022	661.8
C17	3659.7036	264984.1914	637.4
C18	5627.8634	287441.1196	613.2
C19	6170	293606	610.4
C20	2704	254269	614.2
Companies	O1	O2	O3
C01	15805	12	18035
C02	18188	17.2	17838
C03	16056	26.8	19330
C04	17120.1696	27.4296	30136.7379
C05	16598.7555	25.3128	30144.499
C06	16257.2262	29.3011	25004.0304
C07	16737.0216	42.453	32360.4835
C08	16321.9036	60.8301	26209.7207
C09	17255.9694	74.3341	28356.4443
C10	16705.136	92.4932	32237.8078
C11	16784.4	102.4	33508.75
C12	16763.2101	84.6288	35839.551
C13	16782.0445	105.1801	33582.3148
C14	16809.15	127.15	33488.5
C15	16852.9522	132.1601	33738.7705
C16	16890.7577	129.0596	34484.9288
C17	16828.3635	111.0647	34903.7623
C18	16757.2996	55.0528	37916.8247
C19	16747.4	41.6	38740
C20	16775.4	120.4	33502

Table 13: The results of fixed cost allocation plan with trade-offs 3.

DMUs	Scale of data	δ_j	$\hat{R}_j$
C01	10653901440	0.0362	246.069
C02	10994942117	0.0373	253.9459
C03	9321003088	0.0317	215.2836
C04	11590989395	0.0394	267.7125
C05	10323925373	0.0351	238.4477
C06	11062521180	0.0376	255.5067
C07	15272677573	0.0519	352.7471
C08	15342865630	0.0521	354.3682
C09	15928551362	0.0541	367.8955
C10	17009708769	0.0578	392.8666
C11	13611536306	0.0462	314.3803
C12	17030253336	0.0578	393.3411
C13	17929939574	0.0609	414.1208
C14	20455402865	0.0695	472.4504
C15	17745069035	0.0603	409.8509
C16	15743695199	0.0535	363.626
C17	16876707084	0.0573	389.7947
C18	17859842186	0.0607	412.5018
C19	16680156406	0.0567	385.255
C20	12981828188	0.0441	299.8362
Sum	294415516105.57	1	6800

the presence of production trade-offs and the input-output measures. In order to obtain the fixed cost allocation plan in the presence of production trade-offs, we solve model (12). The results obtained from model (12) are in the last column of Table 14. This cost relates to the efficiency scores, input-output measures and the production trade-off (3).

As can be seen, based on the proposed approach in this paper, we can obtain a fair allocation plan for the companies, and the cost of 6,800 Dollar is divided between each of the companies. Given that the model (12) has multiple optimal solutions, we presented two solutions among the solutions in the two-last column of the Table 14.

Table 14: The results of fixed cost allocation plan with trade-offs 3.

DMUs	Fixed cost allocation:solution 1	Fixed cost allocation: solution 2
C01	0	227.321
C02	0	283.8649
C03	2266.6667	235.4484
C04	2266.6667	289.5471
C05	2266.6667	211.894
C06	0	265.5992
C07	0	301.4592
C08	0	357.4193
C09	0	378.5481
C10	0	402.5446
C11	0	289.6923
C12	0	389.5417
C13	0	418.8532
C14	0	489.5552
C15	0	412.7554
C16	0	321.7693
C17	0	432.8742
C18	0	410.4873
C19	0	399.4523
C20	0	281.3733
Sum	6800	6800

6 Conclusion

In this paper, we presented a fair fixed cost allocation scheme considering production trade-offs among inputs and outputs. We employed the principle of efficiency invariance of DMUs for cost allocation. Based on this principle, the efficiency score of DMUs remains unchanged after the allocation of fixed costs. To account for the relative importance of inputs and outputs, we apply the production trade-offs in the cost allocation model. We also considered two goals in the cost allocation plane of units: the first goal applying the efficiency score of the DMUs, and the second goal applying the scale of the inputs and outputs. In the practical example section, we demonstrated that by selecting different production trade-off matrices, we can incorporate the relative importance of inputs and outputs in the cost allocation process, resulting in divergent outcomes. We propose an applied aspect of the approach presented in this paper to the evaluation of oil refining companies in Iran. We showed that based on the presented models, we can provide a fair allocation plan for these companies. The importance of each input and output component in this assessment was considered in the fixed cost allocation plan using the production trade-offs method. Another advantage of the cost allocation plan presented in this paper is that fixed costs are allocated among all DMUs, including efficient and inefficient DMUs. In this paper, a fixed cost allocation scheme was presented for applying the DM's preferred information in the fixed cost allocation process. The cost allocated to the DMUs changes by altering the production trade-off matrix, and this is one of the strengths of the algorithm presented in this paper. By using the production trade-off matrix, we can apply the DM 's opinion in the fixed cost allocation process. We used of reference set of inefficient DMUs based on a single-stage DEA model for obtaining fair fixed cost allocation plan. For this purpose, we used the production trade-off method in DEA. Some advantages of this method are as (i) Applying DM's preferred information through all input and output components. (ii) The high flexibility of the proposed model in applying the DM's opinion, so that we can change this information. (iii) The allocated cost depends on the input and output values and the efficiency score of the units. (iv) The feasibility of the presented models and their easy solution. As future work, the proposed models can be developed for network DEA structure. Also, we can utilize other strategies, such as making all the DMUs efficient after the allocation process.

References

[1] Q. An, P. Wang, E. Ali, J. Hu, Fixed cost allocation based on the principle of efficiency invariance in two-stage systems. *European Journal of Operational Research*, 283(2), (2020), 662-675.

[2] Q. An, X. Tao, B. Xiong, X. Chen, Frontier-based incentive mechanisms for allocating common revenues or fixed costs. *European Journal of Operational Research*, 302(1), (2022), 294-308.

[3] K. B. Atici, V. V. Podinovski, Using data envelopment analysis for the assessment of technical efficiency of units with different specializations: An application to agriculture. *Omega*, 54, (2015), 72-83.

[4] A. I. Ali, L. M. Seiford, Computational accuracy and infinitesimals in data envelopment analysis. *Information Systems and Operational Research*, 31, (1993), 290-297.

[5] A. I. Ali, L. M. Seiford, The mathematical programming approach to efficiency analysis. In H. O. Fried, C. A. K. Lovell, S. S. Schmidt (Eds.), The measurement of productive efficiency: Techniques and applications (pp.120159). NewYork: Oxford University Press. (1993).

[6] J. Beasley, Allocating fixed costs and resources via data envelopment analysis. *European Journal of Operational Research*, 147(1), (2003), 198-216.

[7] W.D. Cook, M. Kress, Characterizing an equitable allocation of shared costs: A DEA approach. *European Journal of Operational Research*, 119(3), (1999), 652-661.

[8] W.D. Cook, J. Zhu, Allocation of shared costs among decision making units: A DEA approach. *Computers and Operations Research*, 32(8), (2005), 2171-2178.

[9] W. W. Cooper, L. M. Seiford, K. Tone, Data envelopment analysis. A comprehensive text with models, applications, references and DEA-Solvers software (2nded.). *New York: Springer Science Business Media*, (2007).

[10] J. Chu, J. Wu, C. Chu, T. Zhang, DEA-based fixed cost allocation in two-stage systems: Leader-follower and satisfaction degree bargaining game approaches. *Omega*, 94, (2020), 102054.

[11] J. Chu, W. Su, F. Li, Z. Yuan, Individual rationality and overall fairness in fixed cost allocation: An approach under DEA cross-efficiency evaluation mechanism. *Journal of the Operational Research Society*, 74(3), (2022), 992-1007.

[12] Q. Dai, Y. Li, X. Lei, D. Wu, A DEA-based incentive approach for allocating common revenues or fixed costs. *European Journal of Operational Research*, 292, (2021), 675-686.

[13] G.R. Jahanshahloo, J. Sadeghi, M. Khodabakhshi, Proposing a method for fixed cost allocation using DEA based on the efficiency invariance and common set of weights principles. *Mathematical Methods of Operations Research*, 85(2), (2017), 223-240.

[14] F. Li, Q. Zhu, Z. Chen, Allocating a fixed cost across the decision making units with two-stage network structures. *Omega*, 83, (2019), 139-154.

[15] R. Lin, Allocating fixed costs or resources and setting targets via data envelopment analysis. *Applied Mathematics and Computation*, 217(13), (2011), 6349-6358.

[16] R. Lin, Z. Chen, Fixed input allocation methods based on super CCR efficiency invariance and practical feasibility. *Applied Mathematical Modelling*, 40 (9-10), (2016), 5377-5392.

[17] A. Mostafaee, An equitable method for allocating fixed costs by using data envelopment analysis. *Journal of the Operational Research Society*, 64(3), (2013), 326-335.

[18] M. A. Pereira, A. S. Camanho, J. R. Figueira, R. C. Marques, Incorporating preference information in a range directional composite indicator: the case of Portuguese public hospitals. *European Journal of Operational Research*, 294(2), (2021), 633-650.

[19] V. V. Podinovski. Production trade-offs and weight restrictions in data envelopment analysis. *Journal of the Operational Research Society*, 55(12), (2004), 1311-1322.

[20] V. V. Podinovski, Computation of efficient targets in DEA models with production trade-offs and weight restrictions. *European Journal of Operational Research*, (2007), 181,586-591.

[21] V. V. Podinovski, T. Bouzdine-Chameeva, Weight restrictions and free production in data envelopment analysis. *Operations Research*, 61 (2), (2013), 426-437.

[22] V. V. Podinovski, T. Bouzdine-Chameeva, Consistent weight restrictions in data envelopment analysis. *European Journal of Operational Research*, 244 (1), (2015), 201-209.

[23] V. V. Podinovski, T. Bouzdine-Chameeva, On single-stage DEA models with weight restrictions. *European Journal of Operational Research*, (2016), 248, 1044-1050.

[24] V. V. Podinovski, J. Wu, N. Argyris, Production trade-offs in models of data envelopment analysis with ratio inputs and outputs: An application to schools in England. *European Journal of Operational Research*, 313, (2024), 359-372.

[25] E. Thanassoulis, M. C. S. Portela, O. Despi'c, Data envelopment analysis: The mathematical programming approach to efficiency analysis. In H.O. Fried, C.A.K. Lovell, S.S. Schmidt (Eds.), The measurement of productive efficiency and productivity growth (pp. 251-420). *New York: Oxford University Press*, (2008).

[26] W. Wang, F. Xu, J. Chu, Y. Dong, Z. Yuan, Determining the equilibrium efficient frontier by proportional frontier shifting for data envelopment analysis with fixed-sum outputs,*Omega*, 130, (2025), 103174.

[27] G. Xu, J. Wu, Q, Zhu, Y, Pan, Fixed cost allocation based on data envelopment analysis from inequality aversion perspectives. *European Journal of Operational Research*, 313 (1), (2024), 281-295.

[28] J. Yang, D. Li, Y. Li, A generalized data envelopment analysis approach for allocating fixed cost with preference information. *Omega*, 122, (2024), 102948.

[29] D. Zhang, H. Wu, F. Li, Y. Song, Fixed cost allocation based on a data envelopment analysis aggressive game approach. *Computers and Industrial Engineering*, 193, (2024), 110316.

[30] W. Zhu, Q. Zhang, H. Wang, Fixed costs and shared resources allocation in two-stage network DEA. *Annals of Operations Research*, 278, (2019), 177-194.

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