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On Wardowski Type Multi-valued Contractions on Complex Valued Metric Spaces and Related Fixed Point Results

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Abstract. Wardowski type admissible multi-valued contractions on complex valued metric spaces are introduced and some fixed point results in this direction are proved. The new results is illustrated with an example.

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1 Introduction

The most well-known fixed point theorem in metric spaces, the Banach contraction principle [8], has numerous applications in a wide range of applied scientific domains and has been expanded upon in a large body of literature from a variety of perspectives (see, for instance, [1, 18, 19, 21, 13, 9, 14, 17, 3, 4, 5, 10, 11, 15, 12, 20]). The concept of α-ψ-contractive type mappings was first presented by Samet et al. [20] in 2012. The principle of Banach's contraction was expanded to multi-valued mappings by Nadler [16] in 1969. The F-contraction is a new generalization of the Banach contraction introduced by Wardowski [23] in 2012. Subsequently, Wardowski and Van Dung [24] have provided an expansion of Wardowski's finding. In 2013, Mohammadi et al. [15] introduced α-admissible multi-valued mappings and demonstrated that α-admissible α-ψ-contractive multi-valued mappings have a fixed point. In 2011, Azam et al. introduced the complex valued metric spaces and provided some sufficient conditions for the existence of common fixed points of a pair of mappings satisfying contractive type conditions [7]. In [2], Ahmad et al. investigated the existence of common fixed point for multivalued mappings satisfying contractive type contractions in complex valued metric spaces. The aim of this article is to introduce Wardowski type multi-valued contractions on complex valued metric spaces and prove the existence of at least one fixed point for these type of contractions. Our main result generalize and improve some results in complex valued metric spaces. In the sequel, the new results will be illustrated with an example.

2 Preliminaries

Let us firstly introduce some notations and definitions which will be need in the sequel in this research.

Definition 2.1. ([7]) Let $\mathbb{C}$ be the set of all complex numbers and $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathbb{C}$. The partial order on $\mathbb{C}$ is defined as: $\mathfrak{z}_1 \preceq \mathfrak{z}_2$ if and only if $\mathcal{R}_e(\mathfrak{z}_1) \leq \mathcal{R}_e(\mathfrak{z}_2)$ and $\mathcal{I}_m(\mathfrak{z}_1) \leq \mathcal{I}_m(\mathfrak{z}_2)$. This implies that $\mathfrak{z}_1 \preceq \mathfrak{z}_2$ if and only if one of the following conditions holds:

(i) $\mathcal{R}_e(\mathfrak{z}_1) = \mathcal{R}_e(\mathfrak{z}_2), \mathcal{I}_m(\mathfrak{z}_1) < \mathcal{I}_m(\mathfrak{z}_2)$,

(ii) $\mathcal{R}_e(\mathfrak{z}_1) < \mathcal{R}_e(\mathfrak{z}_2), \mathcal{I}_m(\mathfrak{z}_1) = \mathcal{I}_m(\mathfrak{z}_2)$,

(iii) $\mathcal{R}_e(\mathfrak{z}_1) < \mathcal{R}_e(\mathfrak{z}_2), \mathcal{I}_m(\mathfrak{z}_1) < \mathcal{I}_m(\mathfrak{z}_2)$,

(iv) $\mathcal{R}_e(\mathfrak{z}_1) = \mathcal{R}_e(\mathfrak{z}_2), \mathcal{I}_m(\mathfrak{z}_1) = \mathcal{I}_m(\mathfrak{z}_2)$.

In particular, if one of (i), (ii) and (iii) is satisfied, then we will write $\mathfrak{z}_1 \precnsim \mathfrak{z}_2$ and we will write $\mathfrak{z}_1 \prec \mathfrak{z}_2$ if and only if (iii) is satisfied.

Definition 2.2. ([7]) Let $\mathcal{X}$ is a nonempty set. If a function $\boldsymbol{\xi} : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ satisfies the following conditions for all $\rho, \sigma, \delta \in \mathcal{X}$:

(i) $0 \preceq \boldsymbol{\xi}(\rho, \sigma)$ and $\boldsymbol{\xi}(\rho, \sigma) = 0$ if and only if $\rho = \sigma$,

(ii) $\boldsymbol{\xi}(\rho, \sigma) = \boldsymbol{\xi}(\sigma, \rho)$,

(iii) $\boldsymbol{\xi}(\rho, \sigma) \preceq \boldsymbol{\xi}(\rho, \delta) + \boldsymbol{\xi}(\delta, \sigma)$,

then $\boldsymbol{\xi}$ is known as a complex valued (C.V.) metric on $\mathcal{X}$, and the pair $(\mathcal{X}, \boldsymbol{\xi})$ is said to be a complex valued metric space.

Example 2.3. Let $\mathcal{X}_1 = \{\mathfrak{z} \in \mathbb{C} : \mathcal{R}_e(\mathfrak{z}) \geq 0 \text{ and } \mathcal{I}_m(\mathfrak{z}) = 0\}$ and $\mathcal{X}_2 = \{\mathfrak{z} \in \mathbb{C} : \mathcal{R}_e(\mathfrak{z}) = 0 \text{ and } \mathcal{I}_m(\mathfrak{z}) \geq 0\}$. Set $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$, Define a function $\boldsymbol{\xi} : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ as follows:

$$\boldsymbol{\xi}(\mathfrak{z}_1, \mathfrak{z}_2) = \boldsymbol{\xi}(\mathfrak{z}_2, \mathfrak{z}_1) = \begin{cases} \frac{1}{2}|\lambda_1 - \lambda_2| + \frac{1}{2}|\lambda_1 - \lambda_2| i, \mathfrak{z}_1, \mathfrak{z}_2 \in \mathcal{X}_1, \\ \frac{1}{2}|\mu_1 - \mu_2| + \frac{1}{2}|\mu_1 - \mu_2| i, \mathfrak{z}_1, \mathfrak{z}_2 \in \mathcal{X}_2, \\ \frac{1}{2}(\lambda_1 + \mu_2) + \frac{1}{2}(\lambda_1 + \mu_2) i, \mathfrak{z}_1 \in \mathcal{X}_1, \mathfrak{z}_2 \in \mathcal{X}_2, \end{cases}$$

where $\mathfrak{z}_1 = \lambda_1 + i\mu_1$ and $\mathfrak{z}_2 = \lambda_2 + i\mu_2$. Then, $(\mathcal{X}, \boldsymbol{\xi})$ is a C.V. metric space.

Example 2.4. Let $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$, where $\mathcal{X}_1 = \{\mathfrak{z} \in \mathbb{C} : \mathcal{R}_e(\mathfrak{z}) \geq 0 \text{ and } \mathcal{I}_m(\mathfrak{z}) = 0\}$ and $\mathcal{X}_2 = \{\mathfrak{z} \in \mathbb{C} : \mathcal{R}_e(\mathfrak{z}) = 0 \text{ and } \mathcal{I}_m(\mathfrak{z}) \geq 0\}$. Define $\boldsymbol{\xi} : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ as follows:

$$\boldsymbol{\xi}(\mathfrak{z}_1, \mathfrak{z}_2) = \boldsymbol{\xi}(\mathfrak{z}_2, \mathfrak{z}_1) = \begin{cases} |\lambda_1 - \lambda_2|, \mathfrak{z}_1, \mathfrak{z}_2 \in \mathcal{X}_1, \\ |\mu_1 - \mu_2| i, \mathfrak{z}_1, \mathfrak{z}_2 \in \mathcal{X}_2, \\ \lambda_1 + \mu_2 i, \mathfrak{z}_1 \in \mathcal{X}_1, \mathfrak{z}_2 \in \mathcal{X}_2, \end{cases}$$

where $\mathfrak{z}_1 = \lambda_1 + \mu_1 i$ and $\mathfrak{z}_2 = \lambda_2 + \mu_2 i$. Then, $(\mathcal{X}, \boldsymbol{\xi})$ is a C.V. metric space.

Example 2.5. Let $\mathcal{X} = \mathbb{C}$. Define $\boldsymbol{\xi} : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ as follows:

$$\boldsymbol{\xi}(\mathfrak{z}_1, \mathfrak{z}_2) = a|\lambda_1 - \lambda_2| + b|\mu_1 - \mu_2|i$$

where $\mathfrak{z}_1 = \lambda_1 + \mu_1 i$ and $\mathfrak{z}_2 = \lambda_2 + \mu_2 i$ and a, b are two positive real constants. Then, $(\mathcal{X}, \boldsymbol{\xi})$ is a C.V. metric space.

Example 2.6. ([22]) Let $\mathcal{X} = [0, 1]$. Define $\boldsymbol{\xi} : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ by

$$\boldsymbol{\xi}(\rho, \sigma) = \begin{cases} 0, \rho = \sigma, \\ \frac{i}{2}, \rho \neq \sigma, \end{cases}$$

for all $\rho, \sigma \in \mathcal{X}$. Then $\boldsymbol{\xi}$ is a C.V. metric on $\mathcal{X}$ and hence $(\mathcal{X}, \boldsymbol{\xi})$ is a C.V. metric space.

Definition 2.7. ([7]) Let $(\mathcal{X}, \boldsymbol{\xi})$ be a C.V. metric space.

(i) It is said that a point $\rho \in \mathcal{X}$ is an interior point of a set $U \subseteq \mathcal{X}$, whenever there exists $0 \prec c \in \mathbb{C}$ such that

$$B(\rho, c) = \{\sigma \in \mathcal{X} : \boldsymbol{\xi}(\rho, \sigma) \prec c\} \subseteq U,$$

where $B(\rho, c)$ is an open ball centered at ρ, with radius c.

(ii) It is said that a point $\rho \in \mathcal{X}$ is a limit point of a set $U \subseteq \mathcal{X}$ whenever for every $0 \prec c \in \mathbb{C}$, we have

$$B(\rho, c) \cap (U \setminus \{\rho\}) \neq \emptyset.$$

Let $\{\rho_n\}$ be a sequence in $\mathcal{X}$ and $\rho \in \mathcal{X}$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $\boldsymbol{\xi}(\rho_n, \rho) \prec c$, then $\{\rho_n\}$ is said to be convergent, $\{\rho_n\}$ converges to ρ, and ρ is the limit point of $\{\rho_n\}$. We denote this by $\lim_{n \to \infty} \rho_n = \rho$, or $\rho_n \to \rho$ as $n \to \infty$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$ and $m \in \mathbb{N}$, $\boldsymbol{\xi}(\rho_n, \rho_{n+m}) \prec c$, then $\{\rho_n\}$ is said to be a Cauchy sequence in $(\mathcal{X}, \boldsymbol{\xi})$. If every Cauchy sequence is convergent in $(\mathcal{X}, \boldsymbol{\xi})$, then $(\mathcal{X}, \boldsymbol{\xi})$ is called a complete C.V. metric space.

A set $U \subseteq \mathcal{X}$ is called open whenever each element of U is an interior point of U. Moreover, a subset $C \subseteq \mathcal{X}$ is called closed whenever each limit point of C belongs to C (see [2]). We denote by $\mathcal{N}(\mathcal{X})$, $\mathcal{C}(\mathcal{X})$ and $\mathcal{CB}(\mathcal{X})$ the set of all nonempty, nonempty closed and nonempty closed bounded subsets of a complex valued metric space $\mathcal{X}$, respectively.

Lemma 2.8. *([7]) Let $(\mathcal{X}, \boldsymbol{\xi})$ be a C.V. metric space and $\{\rho_n\}$ be a sequence in $\mathcal{X}$. Then, $\{\rho_n\}$ is a convergent sequence to a point ρ if and only if $|\boldsymbol{\xi}(\rho_n, \rho)| \to 0$ as $n \to \infty$.*

Lemma 2.9. *([7]) Let $(\mathcal{X}, \boldsymbol{\xi})$ be a C.V. metric space and $\{\rho_n\}$ be a sequence in $\mathcal{X}$. Then, $\{\rho_n\}$ is a Cauchy sequence if and only if $|\boldsymbol{\xi}(\rho_n, \rho_m)| \to 0$ as $m, n \to \infty$.*

Definition 2.10. ([6]) Let $(\mathcal{X}, \boldsymbol{\xi})$ be a C.V. metric space. For any $\mathfrak{z} \in \mathbb{C}$, let $\mathcal{S}(\mathfrak{z}) = \{\varpi \in \mathbb{C} : \mathfrak{z} \preceq \varpi\}$. For $l \in \mathcal{X}$ and $\mathrm{P} \in \mathcal{C}(\mathcal{X})$ set

$$\mathrm{S}(l, \mathrm{P}) = \bigcup_{a \in \mathrm{P}} \mathcal{S}(\boldsymbol{\xi}(l, a)) \tag{1}$$

and for $\mathrm{P}, \mathrm{Q} \in \mathcal{C}(\mathcal{X})$, define

$$\mathfrak{G}(\mathrm{P}, \mathrm{Q}) = (\bigcap_{a \in \mathrm{P}} \mathrm{S}(a, \mathrm{Q})) \cap (\bigcap_{b \in \mathrm{Q}} \mathrm{S}(b, \mathrm{P})). \tag{2}$$

Remark 2.11. ([2]) Let $(\mathcal{X}, \boldsymbol{\xi})$ be a C.V. metric space. If we replace $\mathbb{C}$ by $\mathbb{R}$, then $(\mathcal{X}, \boldsymbol{\xi})$ is a metric space. Moreover, for $\mathrm{P}, \mathrm{Q} \in \mathcal{C}(\mathcal{X})$, $\mathcal{H}(\mathrm{P}, \mathrm{Q}) = \inf\{\varpi : \varpi \in \mathfrak{G}(\mathrm{P}, \mathrm{Q})\}$ is the generalized Hausdorff distance induced by $\boldsymbol{\xi}$.

Given a function $\hbar : \mathcal{X} \times \mathcal{X} \to [0, \infty)$, a self-mapping Γ on $\mathcal{X}$ is said to be $\hbar$-admissible if for any $\rho, \sigma \in \mathcal{X}$, $\hbar(\rho, \sigma) \geq 1$ implies $\hbar(\Gamma\rho, \Gamma\sigma) \geq 1$.

Definition 2.12. Let $(\mathcal{X}, \boldsymbol{\xi})$ be a C.V. metric space and $\hbar : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is a function. It is said that $\mathcal{X}$ is $\hbar-$regular if for each sequence $\{\rho_n\}$ in $\mathcal{X}$ with $\hbar(\rho_n, \rho_{n+1}) \geq 1$ for all n and $\rho_n \to \rho$, then we have $\hbar(\rho_n, \rho) \geq 1$ for all n.

Let $\mathcal{X}$ be a nonempty set and $\digamma : \mathcal{X} \to \mathcal{N}(\mathcal{X})$ be a multivalued mapping. An element ρ in $\mathcal{X}$ is called a fixed point of $\digamma$ if $\rho \in \digamma\rho$.

Definition 2.13. ([15]) Let $\mathcal{X}$ be a nonempty set and $\mathcal{F} : \mathcal{X} \to \mathcal{N}(\mathcal{X})$ be a multivalued mapping. Let $\hbar : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a function. We say that $\mathcal{F}$ is $\hbar$-admissible whenever for each $\rho \in \mathcal{X}$ and $\sigma \in \mathcal{F}\rho$ with $\hbar(\rho, \sigma) \geq 1$, we have $\hbar(\sigma, \delta) \geq 1$ for all $\delta \in \mathcal{F}\sigma$.

3 Main Results

Let us firstly define some notions in C.V. metric spaces.

Definition 3.1. Let $(\mathcal{X}, \boldsymbol{\xi})$ be a C.V. metric space and $\hbar : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a function. Then $(\mathcal{X}, \boldsymbol{\xi})$ is called $\hbar-$complete if for every Cauchy sequence $\{\rho_n\}$ in Ω with $\hbar(\rho_n, \rho_{n+1}) \geq 1$ for all n, then ρ_n converges to some point $\rho \in \mathcal{X}$.

Throughout this paper, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}^+$ stand for the set of all natural numbers, real numbers and positive real numbers, respectively. We denote by $\mathbb{C}^{\succeq 0}$, $\mathbb{C}^{\succnsim 0}$ and $\mathbb{C}^{\succ 0}$ the set of all complex numbers c with $0 \preceq c$, complex numbers c with $0 \precnsim c$ and complex numbers c with $0 \prec c$, respectively. Ξ represents the collection of all functions $\mathfrak{F} : \mathbb{C}^{\succnsim 0} \to \mathbb{R}$ satisfying the following conditions:

($\mathfrak{F}1$) For each sequence $\{\mathfrak{z}_n\}$ in $\mathbb{C}^{\succnsim 0}$, $\lim\limits_{n\to\infty} \mathfrak{z}_n = 0$ if and only if $\lim\limits_{n\to\infty} \mathfrak{F}(\mathfrak{z}_n) = -\infty$.

($\mathfrak{F}2$) There exists $\alpha \in (0, 1)$ such that for any sequence $\{\mathfrak{z}_n\}$ in $\mathbb{C}^{\succnsim 0}$ with $\lim\limits_{n\to\infty} \mathfrak{z}_n = 0$, we have $\lim\limits_{n\to\infty} |\mathfrak{z}_n|^\alpha \mathfrak{F}(\mathfrak{z}_n) = 0$.

Example 3.2. The functions $\mathfrak{F} : \mathbb{C}^{\succnsim 0} \to \mathbb{R}$ defined by

(1) $\mathfrak{F}_1(\mathfrak{z}) = \ln |\mathfrak{z}|$,

(2) $\mathfrak{F}_2(\mathfrak{z}) = \ln |\mathfrak{z}| + |\mathfrak{z}|$,

(3) $\mathfrak{F}_3(\mathfrak{z}) = \frac{-1}{|\mathfrak{z}|^\alpha}, 0 < \alpha < 1$,

(4) $\mathfrak{F}_4(\mathfrak{z}) = \ln(\frac{\mathcal{R}_e\,(\mathfrak{z})}{\cos\theta} + \frac{\mathcal{I}_m\,(\mathfrak{z})}{\sin\theta}) + \frac{\mathcal{R}_e\,(\mathfrak{z})}{\cos\theta} + \frac{\mathcal{I}_m\,(\mathfrak{z})}{\sin\theta}$, $\theta \in (0, \frac{\pi}{2})$

belong to Ξ.

For any function $\mathfrak{F} : \mathbb{C}^{\succnsim 0} \to \mathbb{R}$ and any nonempty set $K \subseteq \mathbb{C}^{\succeq 0}$, take

$$\mathbb{U}_{\mathfrak{F}}(K) = \bigcup \{[\mathfrak{F}(k), \infty) : {\scriptstyle k \in K}, k \neq 0\}.$$

Particularly, for $K_0 = \{0\}$, take $\mathbb{U}_{\mathfrak{F}}(K_0) = \emptyset$.

Lemma 3.3. *Let $(\mathcal{X}, \boldsymbol{\xi})$ be a C.V. metric space and $\mathfrak{F} : \mathbb{C}^{\succnsim 0} \to \mathbb{R}$ be a function. Let $\mathrm{A}, \mathrm{B} \in \mathcal{N}(\mathcal{X})$ and $c \in \mathbb{U}_{\mathfrak{F}}(\mathfrak{S}(\mathrm{A}, \mathrm{B}))$. Then, for any $\alpha \in \mathrm{A}$, there exists $\beta \in \mathrm{B}$ so that*

$$\mathfrak{F}(\boldsymbol{\xi}(\alpha, \beta)) \leq c.$$

Proof. Since $c \in \mathbb{U}_{\mathfrak{F}}(\mathfrak{S}(\mathrm{A}, \mathrm{B}))$, there exists $d \in \mathbb{S}(\mathrm{A}, \mathrm{B})$ with $d \neq 0$ so that $c \in [\mathfrak{F}(d), \infty)$, that is,

$$\mathfrak{F}(d) \leq c.$$

Now since,

$$d \in \mathfrak{S}(\mathrm{A}, \mathrm{B}) = (\bigcap_{a \in \mathrm{A}} \mathrm{S}(a, \mathrm{B})) \cap (\bigcap_{b \in \mathrm{B}} \mathrm{S}(b, \mathrm{A})) \tag{3}$$

and $\alpha \in \mathrm{A}$, thus $d \in \mathrm{S}(\alpha, \mathrm{B})$. By (1), there exists $\beta \in \mathrm{B}$ such that $d \in \mathcal{S}(\boldsymbol{\xi}(\alpha, \beta))$ and so $\boldsymbol{\xi}(\alpha, \beta) \preceq d$. Now, since $\mathfrak{F}$ is nondecreasing, we have

$$\mathfrak{F}(\boldsymbol{\xi}(\alpha, \beta)) \leq \mathfrak{F}(d) \leq c.$$

□

Theorem 3.4. *Let $(\mathcal{X}, \boldsymbol{\xi})$ be a C.V. metric space and there exists a function $\hbar : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ so that $(\mathcal{X}, \boldsymbol{\xi})$ be $\hbar$-complete and let $\digamma : \mathcal{X} \to \mathcal{C}(\mathcal{X})$ be a multivalued mapping such that there exist $\mathfrak{F} \in \Xi$ and $\tau > 0$ so that*

$$\mathfrak{F}(\boldsymbol{\xi}(\rho, \sigma)) - \tau \in \mathbb{U}_{\mathfrak{F}}(\mathfrak{S}(\digamma\rho, \digamma\sigma)) \tag{4}$$

for all $\rho, \sigma \in \mathcal{X}$ with $\digamma\rho \neq \digamma\sigma$ and $\hbar(\rho, \sigma) \geq 1$. Moreover, let

i) there exist $\rho_0 \in \mathcal{X}$ and $\rho_1 \in \digamma\rho_0$ such that $\hbar(\rho_0, \rho_1) \geq 1$,

ii) $\digamma$ is $\hbar$-admissible,

iii) $\mathcal{X}$ is $\hbar$-regular.

Then, $\digamma$ has a fixed point.

Proof. By hypothesis (i), there exist $\rho_0 \in \mathcal{X}$ and $\rho_1 \in \digamma\rho_0$ such that $\hbar(\rho_0, \rho_1) \geq 1$. If $\digamma\rho_0 = \digamma\rho_1$, then ρ_1 is a fixed point of $\digamma$ and we have nothing to prove. Let $\digamma\rho_0 \neq \digamma\rho_1$. From (16),

$$\mathfrak{F}(\boldsymbol{\xi}(\rho_0, \rho_1)) - \tau \in \mathbb{U}_{\mathfrak{F}}(\mathfrak{S}(\digamma\rho_0, \digamma\rho_1)). \tag{5}$$

As $\rho_1 \in \digamma\rho_0$, from lemma 3.3, there exists $\rho_2 \in \digamma\rho_1$ so that

$$\mathfrak{F}(\boldsymbol{\xi}(\rho_1, \rho_2)) \leq \mathfrak{F}(\boldsymbol{\xi}(\rho_0, \rho_1)) - \tau.$$

Since $\digamma$ is $\hbar$-admissible, so $\hbar(\rho_1, \rho_2) \geq 1$. Continuing this process we obtain a sequence $\{\rho_n\}$ in $\mathcal{X}$ so that $\rho_{n+1} \in \digamma\rho_n$, $\digamma\rho_n \neq \digamma\rho_{n+1}$, $\hbar(\rho_n, \rho_{n+1}) \geq 1$ and

$$\mathfrak{F}(\boldsymbol{\xi}(\rho_n, \rho_{n+1})) \leq \mathfrak{F}(\boldsymbol{\xi}(\rho_{n-1}, \rho_n)) - \tau \tag{6}$$

for all $n \in \mathbb{N}$. Therefore

$$\mathfrak{F}(\boldsymbol{\xi}(\rho_n, \rho_{n+1})) \leq \mathfrak{F}(\boldsymbol{\xi}(\rho_{n-1}, \rho_n)) - \tau \leq \cdots \leq \mathfrak{F}(\boldsymbol{\xi}(\rho_0, \rho_1)) - n\tau. \tag{7}$$

Taking limit on both sides of the above inequality as $n \to \infty$, we find $\lim\limits_{n\to\infty} \mathfrak{F}(\boldsymbol{\xi}(\rho_n, \rho_{n+1})) = -\infty$. From ($\mathfrak{F}$1), we get $\lim\limits_{n\to\infty} \boldsymbol{\xi}(\rho_n, \rho_{n+1}) = 0$. Now, we shall show that $\{\rho_n\}$ is a Cauchy sequence. From (7), we have

$$n\tau \leq \mathfrak{F}(\boldsymbol{\xi}(\rho_0, \rho_1)) - \mathfrak{F}(\boldsymbol{\xi}(\rho_n, \rho_{n+1})). \tag{8}$$

Multiplying $|\boldsymbol{\xi}(\rho_n, \rho_{n+1})|^{\alpha}$ in both sides of (8) and taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} |\boldsymbol{\xi}(\rho_n, \rho_{n+1})|^{\alpha} n\tau = 0.$$

Thus

$$\lim_{n\to\infty} |\boldsymbol{\xi}(\rho_n, \rho_{n+1})|^{\alpha} n = 0.$$

Therefore, there exists $N \in \mathbb{N}$ such that

$$|\boldsymbol{\xi}(\rho_n, \rho_{n+1})| \leq \frac{1}{n^{\frac{1}{\alpha}}}.$$

From triangular inequality, for any $n, m \in \mathbb{N}$, we have

$$|\boldsymbol{\xi}(\rho_p, \rho_{p+q})| \leq \sum_{i=p}^{p+q-1} \frac{1}{n^{\frac{1}{\alpha}}}.$$

Since the series $\sum_{i=1}^{\infty} \frac{1}{n^{\frac{1}{\alpha}}}$ is convergent, as $p \to \infty$ the right side of the above inequality tends to 0 and so $\lim\limits_{p\to\infty} |\boldsymbol{\xi}(\rho_p, \rho_{p+q})| = 0$. Thus, $\{\rho_n\}$ is a Cauchy sequence in the $\hbar$-complete C.V. metric space $(\mathcal{X}, \boldsymbol{\xi})$. Hence there is $\rho^* \in \mathcal{X}$ so that

$$\lim_{n\longrightarrow\infty} \boldsymbol{\xi}(\rho_n, \rho^*) = 0.$$

Now, since $\mathcal{X}$ is $\hbar$-regular, so $\hbar(\rho_n, \rho^*) \geq 1$ for all $n \in \mathbb{N}$. We consider two cases:

(case1): There exists $N \in \mathbb{N}$ so that $\digamma\rho_n \neq \digamma\rho^*$ for each $n \geq N$. Obviously, $\rho_n \neq \rho^*$ for each $n \geq N$. From (16),

$$\mathfrak{F}(\boldsymbol{\xi}(\rho_n, \rho^*)) - \tau \in \mathbb{U}_{\mathfrak{F}}(\mathfrak{S}(\digamma\rho_n, \digamma\rho^*)).$$

As $\rho_{n+1} \in \digamma\rho_n$, from lemma (3.3), there exists $\upsilon_n \in \digamma\rho^*$ such that

$$\mathfrak{F}(\boldsymbol{\xi}(\rho_{n+1}, \upsilon_n)) \leq \mathfrak{F}(\boldsymbol{\xi}(\rho_n, \rho^*)) - \tau.$$

Taking the limit as $n \to \infty$ in the above inequality, we have $\lim\limits_{n\to\infty} \mathfrak{F}(\boldsymbol{\xi}(\rho_{n+1}, \upsilon_n)) = -\infty$ and so $\lim\limits_{n\to\infty} \boldsymbol{\xi}(\rho_{n+1}, \upsilon_n) = 0$. By triangular inequality,

$$\boldsymbol{\xi}(\rho^*, \upsilon_n) \preceq \boldsymbol{\xi}(\rho^*, \rho_{n+1}) + \boldsymbol{\xi}(\rho_{n+1}, \upsilon_n).$$

So

$$|\boldsymbol{\xi}(\rho^*, \upsilon_n)| \leq |\boldsymbol{\xi}(\rho^*, \rho_{n+1})| + |\boldsymbol{\xi}(\rho_{n+1}, \upsilon_n)|.$$

Taking the limit as $n \to \infty$ in the above inequality, we get $\lim\limits_{n\to\infty} |\boldsymbol{\xi}(\rho^*, \upsilon_n)| = 0$ and so $\lim\limits_{n\to\infty} \boldsymbol{\xi}(\rho^*, \upsilon_n) = 0$. Thus ρ^* is a limit point of $\digamma\rho^*$. Since $\digamma\rho^*$ is closed, so $\rho^* \in \digamma\rho^*$.

case(2): There exists a subsequence $\{\rho_{n_k}\}$ of $\{\rho_n\}$ so that $\digamma\rho_{n_k} = \digamma\rho^*$ for each $k \geq 1$. In this case,

$$\rho_{n_k+1} \in \digamma\rho_{n_k} = \digamma\rho^*$$

for each $k \geq 1$. Since $\rho_{n_k+1} \to \rho^*$ as $k \to \infty$, thus ρ^* is a limit point of $\digamma\rho^*$. Since $\digamma\rho^*$ is closed, so $\rho^* \in \digamma\rho^*$.

The proof is completed. □

Given a metric space $(\mathcal{X}, d)$, a multivalued mapping $\digamma : \mathcal{X} \to \mathcal{CB}(\mathcal{X})$ is said to satisfies approximate valued property, whenever for any $x \in \mathcal{X}$ there exists $y \in \digamma x$ such that $d(x, y) = d(x, \digamma x)$.

Remark 3.5. If we replace $\mathbb{C}$ with $\mathbb{R}$ (that is $\boldsymbol{\xi}$ be a real valued usual metric on $\mathcal{X}$ and $\mathfrak{F} : \mathbb{R}^{>0} \to \mathbb{R}$) in Theorem 3.4, this theorem reduces to the following Wardowski type multi-valued contraction result in metric spaces:

Corollary 3.6. *Let $(\mathcal{X}, \boldsymbol{\xi})$ be a metric space and there exists a function $\hbar : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ so that $(\mathcal{X}, \boldsymbol{\xi})$ be $\hbar$-complete and let $\digamma : \mathcal{X} \to \mathcal{CB}(\mathcal{X})$ be a multivalued mapping satisfying approximate valued property such that there exist $\mathfrak{F} \in \Xi$ and $\tau > 0$ so that*

$$\tau + \mathfrak{F}(H_{\boldsymbol{\xi}}(\digamma\rho, \digamma\sigma)) \leq \mathfrak{F}(\boldsymbol{\xi}(\rho, \sigma)) \tag{9}$$

for all $\rho, \sigma \in \mathcal{X}$ with $\digamma\rho \neq \digamma\sigma$ and $\hbar(\rho, \sigma) \geq 1$. Moreover, let

i) there exist $\rho_0 \in \mathcal{X}$ and $\rho_1 \in \digamma\rho_0$ such that $\hbar(\rho_0, \rho_1) \geq 1$,

ii) $\digamma$ is $\hbar$-admissible,

iii) $\mathcal{X}$ is $\hbar$-regular.

Then, $\digamma$ has at least a fixed point.

Taking $\mathfrak{F}(c) = \ln|c|$ in Theorem 3.4, we get the following contractive type result in C.V. metric spaces:

Corollary 3.7. *Let $(\mathcal{X}, \boldsymbol{\xi})$ be a C.V. metric space and there exists a function $\hbar : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ so that $(\mathcal{X}, \boldsymbol{\xi})$ be $\hbar$-complete and let $\digamma : \mathcal{X} \to \mathcal{C}(\mathcal{X})$ be a multivalued mapping such that there exists $\lambda \in \mathbb{C}$ with $0 < |\lambda| < 1$ so that*

$$\lambda\boldsymbol{\xi}(\rho, \sigma) \in \mathfrak{S}(\digamma\rho, \digamma\sigma) \tag{10}$$

for all $\rho, \sigma \in \mathcal{X}$ with $\digamma\rho \neq \digamma\sigma$ and $\hbar(\rho, \sigma) \geq 1$. Moreover, let *i) there exist* $\rho_0 \in \mathcal{X}$ *and* $\rho_1 \in \digamma\rho_0$ *such that* $\hbar(\rho_0, \rho_1) \geq 1$,

ii) $\digamma$ *is* $\hbar$*-admissible,*

iii) $\mathcal{X}$ *is* $\hbar$*-regular.*

Then, $\digamma$ *has at least a fixed point.*

If we take $\mathfrak{F}(c) = \ln(|c|) + |c|$ in Theorem 3.4, we get the following result:

Corollary 3.8. *Let* $(\mathcal{X}, \boldsymbol{\xi})$ *be a C.V. metric space and there exists a function* $\hbar : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ *so that* $(\mathcal{X}, \boldsymbol{\xi})$ *be* $\hbar$*-complete and* $\digamma : \mathcal{X} \to \mathcal{CB}(\mathcal{X})$ *be a multivalued mapping such that there exists* $\tau > 0$ *and* $c \in \mathfrak{S}(\digamma\rho, \digamma\sigma)$ *with* $c \neq 0$ *satisfying*

$$\frac{|c|}{|\boldsymbol{\xi}(\rho, \sigma)|} e^{|c| - |\boldsymbol{\xi}(\rho,\sigma)|} \leq e^{-\tau} \tag{11}$$

for all $\rho, \sigma \in \mathcal{X}$ *with* $\digamma\rho \neq \digamma\sigma$ *and* $\hbar(\rho, \sigma) \geq 1$*. Moreover, let*

i) there exist $\rho_0 \in \mathcal{X}$ *and* $\rho_1 \in \digamma\rho_0$ *such that* $\hbar(\rho_0, \rho_1) \geq 1$,

ii) $\digamma$ *is* $\hbar$*-admissible,*

iii) $\mathcal{X}$ *is* $\hbar$*-regular.*

Then, $\digamma$ *has at least a fixed point* ρ^**.*

Choosing $\mathfrak{F}(c) = \ln(\frac{\mathcal{R}_e c}{\cos\theta} + \frac{\mathcal{I}_m c}{\sin\theta})$ $(\theta \in (0, \frac{\pi}{2}))$ in Theorem 3.4, we earn the following result:

Corollary 3.9. *Let* $(\mathcal{X}, \boldsymbol{\xi})$ *be a C.V. metric space. Suppose there exists a function* $\hbar : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ *so that* $(\mathcal{X}, \boldsymbol{\xi})$ *be* $\hbar$*-complete and* $\digamma : \mathcal{X} \to \mathcal{CB}(\mathcal{X})$ *be a multivalued mapping such that there exists* $\tau > 0$ *and* $c \in \mathfrak{S}(\digamma\rho, \digamma\sigma)$ *with* $c \neq 0$ *satisfying*

$$\frac{\mathcal{R}_e c}{\cos\theta} + \frac{\mathcal{I}_m c}{\sin\theta} \leq k(\frac{\mathcal{R}_e \boldsymbol{\xi}(\rho, \sigma)}{\cos\theta} + \frac{\mathcal{I}_m \boldsymbol{\xi}(\rho, \sigma)}{\sin\theta}) \tag{12}$$

for all $\rho, \sigma \in \mathcal{X}$ *with* $\digamma\rho \neq \digamma\sigma$ *and* $\hbar(\rho, \sigma) \geq 1$ *where* $\theta \in (0, \frac{\pi}{2})$*. Moreover, let* *i) there exist* $\rho_0 \in \mathcal{X}$ *and* $\rho_1 \in \digamma \rho_0$ *such that* $\hbar(\rho_0, \rho_1) \geq 1$*,*

ii) $\digamma$ *is* $\hbar$*-admissible,*

iii) $\mathcal{X}$ *is* $\hbar$*-regular.*

Then, $\digamma$ *has at least one fixed point* ρ^**.*

Designing $\mathfrak{F}(c) = \ln(\frac{\mathcal{R}_e c}{\cos\theta} + \frac{\mathcal{I}_m c}{\sin\theta}) + \frac{\mathcal{R}_e c}{\cos\theta} + \frac{\mathcal{I}_m c}{\sin\theta}$ in Theorem 3.4, we obtain the following result:

Corollary 3.10. *Let* $(\mathcal{X}, \boldsymbol{\xi})$ *be a C.V. metric space. Suppose there exists a function* $\hbar : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ *so that* $(\mathcal{X}, \boldsymbol{\xi})$ *be* $\hbar$*-complete and* $\digamma : \mathcal{X} \to \mathcal{CB}(\mathcal{X})$ *be a multivalued mapping such that there exists* $\tau > 0$ *and* $c \in \mathfrak{S}(\digamma\rho, \digamma\sigma)$ *with* $c \neq 0$ *satisfying*

$$\frac{\frac{\mathcal{R}_e c}{\cos\theta} + \frac{\mathcal{I}_m c}{\sin\theta}}{\frac{\mathcal{R}_e \boldsymbol{\xi}(\rho,\sigma)}{\cos\theta} + \frac{\mathcal{I}_m \boldsymbol{\xi}(\rho,\sigma)}{\sin\theta}} e^{\frac{\mathcal{R}_e c}{\cos\theta} + \frac{\mathcal{I}_m c}{\sin\theta} - (\frac{\mathcal{R}_e \boldsymbol{\xi}(\rho,\sigma)}{\cos\theta} + \frac{\mathcal{I}_m \boldsymbol{\xi}(\rho,\sigma)}{\sin\theta})} \leq e^{-\tau} \tag{13}$$

for all $\rho, \sigma \in \mathcal{X}$ *with* $\digamma\rho \neq \digamma\sigma$ *and* $\hbar(\rho, \sigma) \geq 1$ *where* $\theta \in (0, \frac{\pi}{2})$*. Moreover, let*

i) there exist $\rho_0 \in \mathcal{X}$ *and* $\rho_1 \in \digamma \rho_0$ *such that* $\hbar(\rho_0, \rho_1) \geq 1$*,*

ii) $\digamma$ *is* $\hbar$*-admissible,*

iii) $\mathcal{X}$ *is* $\hbar$*-regular.*

Then, $\digamma$ *has at least one fixed point* ρ^**.*

Before providing an example we need to give the following lemma:

Lemma 3.11. *Let* $(\mathcal{X}, \boldsymbol{\xi})$ *be a C.V. metric space,* $\mathrm{A}, \mathrm{B} \in \mathcal{N}(\mathcal{X})$ *and* $\mathrm{A} \subseteq \mathrm{B}$*. Then,*

$$\mathfrak{S}(\mathrm{A}, \mathrm{B}) = \bigcap_{b \in \mathrm{B} \setminus \mathrm{A}} \mathrm{S}(b, \mathrm{A}). \tag{14}$$

Proof. Let $a \in \mathrm{A}$. Then $\mathrm{S}(a, \mathrm{B}) = \bigcup_{b \in \mathrm{B}} \mathcal{S}(\boldsymbol{\xi}(a, b))$. Since $\mathrm{A} \subseteq \mathrm{B}$, so $a \in \mathrm{B}$. On the other hand $\mathcal{S}(\boldsymbol{\xi}(a, a)) = \mathcal{S}(0) = \mathbb{C}^{\succeq 0}$. Thus $\mathrm{S}(a, \mathrm{B}) = \mathbb{C}^{\succeq 0}$. Therefore,

$$\mathfrak{S}(\mathrm{A}, \mathrm{B}) = (\bigcap_{b \in \mathrm{B}} \mathrm{S}(b, \mathrm{A})) \cap (\mathbb{C}^{\succeq 0}) = \bigcap_{b \in \mathrm{B}} \mathrm{S}(b, \mathrm{A}). \tag{15}$$

Also for any $b \in \mathrm{B}$, we have either $b \in \mathrm{A}$ or $b \in \mathrm{B} \setminus \mathrm{A}$. For $b \in \mathrm{A}$, we have $\mathrm{S}(b, \mathrm{A}) = \bigcup_{a \in \mathrm{A}} \mathcal{S}(\boldsymbol{\xi}(b, a))$. On the other hand $\mathcal{S}(\boldsymbol{\xi}(b, b)) = \mathcal{S}(0) = \mathbb{C}^{\succeq 0}$. Thus $\mathrm{S}(b, \mathrm{A}) = \mathbb{C}^{\succeq 0}$. So by (15), we have

$$\mathfrak{S}(\mathrm{A}, \mathrm{B}) = \bigcap_{b \in \mathrm{B} - \mathrm{A}} \mathrm{S}(b, \mathrm{A}).$$

$\square$

Example 3.12. Let $A = \{S_n = \{\frac{n^2}{2} + \frac{n}{2}i| \quad n = 1, 2, \cdots\}$, $B = \{0\}$, $\mathcal{X} = A \cup B$ and define $\boldsymbol{\xi} : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ by

$$\boldsymbol{\xi}(\rho, \sigma) = \boldsymbol{\xi}(\sigma, \rho) = \begin{cases} \frac{n^2}{2} \cos\theta + (\frac{n}{2} \sin\theta)i, & \rho = S_n \in A, \sigma = 0 \in B, \\ |\frac{m^2}{2} - \frac{n^2}{2}| \cos\theta + (|\frac{m}{2} - \frac{n}{2}| \sin\theta)i, & \rho = S_m \in A, \sigma = S_n \in A, \\ 0, & \rho = \sigma = 0 \in B \end{cases}$$

and $\digamma : \mathcal{X} \to \mathcal{CB}(\mathcal{X})$ by

$$\digamma\rho = \begin{cases} \{S_1\}, & \rho = S_1, \\ \{S_1, \cdots, S_{n-1}\}, & \rho = S_n \in A, n \geq 2, \\ \{S_1\}, & \rho = 0 \in B. \end{cases}$$

Define a function $\hbar : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ by

$$\hbar(\rho, \sigma) = \begin{cases} 1, & \rho, \sigma \in A, \\ 0, & \text{otherwise}. \end{cases}$$

It is easy to check that $(\mathcal{X}, \boldsymbol{\xi})$ is an $\hbar$-complete C.V. metric space, $\mathcal{X}$ is $\hbar$-regular and $\digamma$ is $\hbar$-admissible. Also, for $\rho_0 = S_1, \rho_1 = S_1$, we have $\hbar(\rho_0, \rho_1) = \hbar(S_1, S_1) \geq 1$. We want to obtain the contraction in Corollary 3.10, for $\tau = 1$. For any $\rho, \sigma \in \mathcal{X}$ with $\hbar(\rho, \sigma) \geq 1$ and $\digamma\rho \neq \digamma\sigma$, we have two cases:

Case 1: $\rho = S_m, \sigma = S_n, \quad m > n > 1$. In this case, let $c = \boldsymbol{\xi}(S_{m-1}, S_{n-1})$. Let us first to show that $c \in \mathfrak{S}(\digamma S_m, \digamma S_n) = \mathfrak{S}(\digamma\rho, \digamma\sigma)$. Since $\digamma S_n \subseteq \digamma S_m$, by lemma 3.11, it is sufficient to show that for any $S_p \in \digamma S_m \setminus \digamma S_n = \{S_n, \cdots S_{m-1}\}$, $c \in \mathrm{S}(S_p, \digamma S_n)$. To see this, note that $n \leq p \leq m - 1$. Now

$$\begin{aligned} c = \; & \boldsymbol{\xi}(S_{m-1}, S_{n-1}) = (\tfrac{(m-1)^2}{2} - \tfrac{(n-1)^2}{2}) \cos\theta + (\tfrac{m-1}{2} - \tfrac{n-1}{2})(\sin\theta)i \\ & \succeq (\tfrac{p^2}{2} - \tfrac{(n-1)^2}{2}) \cos\theta + (\tfrac{p}{2} - \tfrac{n-1}{2})(\sin\theta)i = \boldsymbol{\xi}(S_p, S_{n-1}) \end{aligned}$$

yields that $c \in \mathcal{S}(\boldsymbol{\xi}(S_p, S_{n-1})) \subseteq \mathrm{S}(S_p, \digamma S_n)$ as required. Thus $c \in \mathfrak{S}(\digamma S_m, \digamma S_n) = \mathfrak{S}(\digamma \rho, \digamma \sigma)$. Now we shall show that (13) in Corollary 3.10 holds. To see this, we have

$$\begin{array}{rl} & \frac{\frac{\mathcal{R}e\,c}{\cos\theta}+\frac{\mathcal{I}m\,c}{\sin\theta}}{\frac{\mathcal{R}e\,\boldsymbol{\xi}(\rho,\sigma)}{\cos\theta}+\frac{\mathcal{I}m\,\boldsymbol{\xi}(\rho,\sigma)}{\sin\theta}} e^{\frac{\mathcal{R}e\,c}{cos\theta}+\frac{\mathcal{I}m\,c}{sin\theta}-(\frac{\mathcal{R}e\,\boldsymbol{\xi}(\rho,\sigma)}{\cos\theta}+\frac{\mathcal{I}m\,\boldsymbol{\xi}(\rho,\sigma)}{\sin\theta})} \\ = & \frac{(\frac{(n-1)^2}{2}-\frac{(m-1)^2}{2})+(\frac{n-1}{2}-\frac{m-1}{2})}{(\frac{n^2}{2}-\frac{m^2}{2})+(\frac{n}{2}-\frac{m}{2})} e^{(\frac{(n-1)^2}{2}-\frac{1}{2})+(\frac{n-1}{2}-\frac{1}{2})-((\frac{n^2}{2}-\frac{m^2}{2})+(\frac{n}{2}-\frac{m}{2}))} \\ = & \frac{n+m-1}{n+m+1} e^{-(n-m)} \\ \leq & e^{-1}. \end{array}$$

Case 2: $\rho = S_m, m > 1$ and $\sigma = S_1$. In this case, note that since $\digamma S_m \neq \digamma S_1$, thus $m \neq 2$ and so $m > 2$. let $c = \boldsymbol{\xi}(S_{m-1}, S_1)$. Let us first to show that $c \in \mathfrak{S}(\digamma S_m, \digamma S_1) = \mathfrak{S}(\digamma \rho, \digamma \sigma)$. Since $\digamma S_1 \subseteq \digamma S_m$, by lemma 3.11, it is sufficient to show that for any $S_p \in \digamma S_m \setminus \digamma S_1 = \{S_2, \cdots S_{m-1}\}$, $c \in \mathrm{S}(S_p, \digamma S_1)$. To see this, note that $2 \leq p \leq m-1$. Now

$$\begin{array}{rl} c = & \boldsymbol{\xi}(S_{m-1}, S_1) = (\frac{(m-1)^2}{2} - \frac{1}{2})\cos\theta + (\frac{m-1}{2} - \frac{1}{2})(\sin\theta)i \\ & \succeq (\frac{p^2}{2} - \frac{1}{2})\cos\theta + (\frac{p}{2} - \frac{1}{2})(\sin\theta) = \boldsymbol{\xi}(S_p, S_1) \end{array}$$

yields that $c \in \mathcal{S}(\boldsymbol{\xi}(S_p, S_1)) = \mathrm{S}(S_p, \digamma S_1)$ as required. Thus $c \in \mathfrak{S}(\digamma S_m, \digamma S_1) = \mathfrak{S}(\digamma \rho, \digamma \sigma)$. Now we shall show that (13) in Corollary 3.10 holds. To see this, we have

$$\begin{array}{rl} & \frac{\frac{\mathcal{R}e}{\cos\theta}+\frac{\mathcal{I}m\,c}{sin\theta}}{\frac{\mathcal{R}e\,\boldsymbol{\xi}(\rho,\sigma)}{\cos\theta}+\frac{\mathcal{I}m\,\boldsymbol{\xi}(\rho,\sigma)}{\sin\theta}} e^{\frac{\mathcal{R}e}{\cos\theta}+\frac{\mathcal{I}m\,c}{\sin\theta}-(\frac{\mathcal{R}e\,\boldsymbol{\xi}(\rho,\sigma)}{\cos\theta}+\frac{\mathcal{I}m\,\boldsymbol{\xi}(\rho,\sigma)}{\sin\theta})} \\ = & \frac{(\frac{(n-1)^2}{2}-\frac{1}{2})+(\frac{n-1}{2}-\frac{1}{2})}{(\frac{n^2}{2}-\frac{1}{2})+(\frac{n}{2}-\frac{1}{2})} e^{(\frac{(n-1)^2}{2}-\frac{1}{2})+(\frac{n-1}{2}-\frac{1}{2})-((\frac{n^2}{2}-\frac{1}{2})+(\frac{n}{2}-\frac{1}{2}))} \\ = & \frac{n^2-n-2}{n^2+n-2} e^{-n} \\ \leq & e^{-1}. \end{array}$$

Thus (13) is satisfied in Corollary 3.10. So, by Corollary 3.10, $\digamma$ has a fixed point. Here S_1 is a fixed point of $\digamma$. Now, we show that $\digamma$ is not a contractive type contraction. Let by contradiction, we suppose that $\digamma$ is a contractive type contraction. Then, there exists $\lambda \in \mathbb{C}$ with $0 < |\lambda| < 1$ so that

$$\lambda \boldsymbol{\xi}(\rho, \sigma) \in \mathfrak{S}(\digamma \rho, \digamma \sigma) \tag{16}$$

for all $\rho, \sigma \in \mathcal{X}$ with $\digamma\rho \neq \digamma\sigma$ and $\hbar(\rho, \sigma) \geq 1$. Taking $\rho = S_n (n > 2)$ and $\sigma = S_1$, we have $\digamma S_n \neq \digamma S_1$ and $\hbar(S_n, S_1) \geq 1$ and so

$$\lambda\boldsymbol{\xi}(S_n, S_1) \in \mathfrak{S}(\digamma S_n, \digamma S_1) = \bigcap_{S_p \in \digamma S_n \setminus \digamma S_1} \mathrm{S}(S_p, \digamma S_1).$$

Since $S_{n-1} \in \digamma S_n \setminus \digamma S_1 = \{S_2, \cdots S_{n-1}\}$ so $\lambda\boldsymbol{\xi}(S_n, S_1) \in \mathrm{S}(S_{n-1}, \digamma S_1) = \mathcal{S}(\boldsymbol{\xi}(S_{n-1}, S_1))$, that is, $\lambda\boldsymbol{\xi}(S_n, S_1) \succeq \boldsymbol{\xi}(S_{n-1}, S_1)$. Therefore $|\lambda||\boldsymbol{\xi}(S_n, S_1)| \geq |\boldsymbol{\xi}(S_{n-1}, S_1)|$ and so

$$\frac{|\boldsymbol{\xi}(S_{n-1}, S_1)|}{|\boldsymbol{\xi}(S_n, S_1)|} \leq |\lambda|. \tag{17}$$

On the other hand we have

$$\lim_{n\to\infty} \frac{|\boldsymbol{\xi}(S_{n-1}, S_1)|}{|\boldsymbol{\xi}(S_n, S_1)|} = \lim_{n\to\infty} \frac{\sqrt{(n^2 - 2n)^2 \cos^2\theta + (n-2)^2 \sin^2\theta}}{\sqrt{(n^2 - 1)^2 \cos^2\theta + (n-1)^2 \sin^2\theta}} = 1. \tag{18}$$

From (17), (18) we have $1 \leq |\lambda|$ which is a contradiction. Thus $\digamma$ is not a contractive type multivalued mapping in C.V. metric space $(\chi, \boldsymbol{\xi})$. Thus, in this example, we can't apply the contractive type results in C.V. metric spaces.

4 Conclusions and Future Works

In this paper, we proved some fixed point results for Wardowski type admissible multi-valued contractions on complex valued metric spaces and illustrated the new results with an example. We suggest the readers to investigate common and coupled fixed point results in this direction and apply these results to the solvability of integral inclusions.

References

[1] M. Abbas, T. Nazir, and S. Radenovic, Common fixed points of four maps in partially ordered metric spaces, *Appl. Math. Lett.*, 24 (2011), 1520-1526.

[2] J. Ahmad, C. Klin-Eam, and A. Azam, Common fixed points for multivalued mappings in complex valued metric spaces with applications, *Abstr. Appl. Anal.* , 2013 (2013), Article ID 854965.

[3] H. H. Alsulami, S. Chandok, M. A. Taoudi, and I. M. Erhan, Some common fixed points theorems for α_*-ψ-common rational type contractive and weakly increasing multi-valued mappings on ordered metric spaces, *Fixed Point Theory Appl.*, 2015 (2015), 97.

[4] A. Amini-Harandi, Coupled and tripled fixed point theory in partially ordered metric spaces with application to initial value problem, *Math. Comput. Modelling*, 57 (2013), 2343-2348.

[5] M. Asadi, E. Karapinar, and A. Kumar, α-ψ-Geraghty contractions on generalized metric spaces, *J. Inequal. Appl.*, 2014 (2014), 423.

[6] P. Amiri and H. Afshari, Common fixed point results for multi-valued mappings in complex-valued double controlled metric spaces and their applications to the existence of solution of fractional integral inclusion systems, *Chaos, Solitons and Fractals* , 154 (2022), 111622.

[7] A. Azam, B. Fisher, M. Khan, Common fixed point theorems in complex valued metric spaces, *Numer. Funct. Anal. Optim.*, 32(3) (2011), 24353.

[8] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integrales, *Fundam. Math.*, 3 (1922), 133-181.

[9] O. Ege and I. Karaca, Common fixed point results on complex valued Gb-metric spaces, *Thai Journal of Mathematics*, 16(3) (2018), 775-787.

[10] A. Farajzadeh, A. Kaewcharoen, and P. Lahawech, On Fixed point theorems for (ξ,α,η)-Expansive mappings in complete metric spaces, *Int. J. Pure Appl. Math.*, 102(1) (2015), 129-146.

[11] Y. Feng and S. Liu, Fixed point theorems for multi-valued increasing operators in partially ordered spaces, *Soochow J. Math*, 30 (4) (2004), 461-469.

[12] J. Hassanzadeh Asl, Sh. Rezapour, and N. Shahzad, *On fixed points of α-ψ-contractive multifunctions*, *Fixed Point Theory Appl.*, 2012 (2012), 212.

[13] G. Mani, A. J. Gnanaprakasam, O. Ege, N. Fatima, and N. Mlaiki, Solution of Fredholm integral equation via common fixed point theorem on bicomplex valued b-metric space, *Symmetry*, 15(2) (2023), 1-15.

[14] G. Mani, S. Haque, A. J. Gnanaprakasam, O. Ege, and N. Mlaiki , The study of bicomplex-valued controlled metric spaces with applications to fractional differential equations, Mathematics, 11(12) (2023), 1-19.

[15] B. Mohammadi, S. Rezapour, N. Shahzad, Some results on fixed points of $\alpha-\psi$-Ciric generalized multifunctions, *Fixed Point Theory Appl.*, 2013 (2013), 24.

[16] SB Jr., Nadler, Multi-valued contraction mappings, *Pacific J. Math.*, 30 (1969), 475-488.

[17] G. Nallasellia, A. J. Gnanaprakasama, G. Manib, and O. Ege, Solving integral equations via admissible contraction mappings, Filomat, 36(14) (2022), 4947-4961.

[18] A. Pansuwan, W. Sintunavarat, V. Parvaneh, and Y. J. Cho, Some fixed point theorems for (α, Θ, k)-contractive multi-valued mappings with some applications, *Fixed Point Theory Appl.*, 2015 (2015), 132.

[19] V. Parvaneh, N. Hussain, and Z. Kadelburg, Generalized Wardowski-type fixed point theorems via α-admissible FG-contractions in b-metric spaces, *Acta Mathematica Scientia*, 36(5) (2016), 1445-1456.

[20] B. Samet, C. Vetro, and P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, *Nonlinear Anal.*, 75 (2012), 2154-2165.

[21] T.L. Shateri, O. Ege, and M. de la Sen, Common fixed point on the b_v(s)-metric space of function-valued mappings, *AIMS Mathematics*, 6(1) (2021), 1065-1074.

[22] N. Ullah, M. S. Shagari, and A. Azam, Fixed point theorems in complex valued extended b-metric spaces, *Moroc. J. Pure Appl. Math.*, 5(2) (2019), 140-163.

[23] D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, *Fixed Point Theory Appl.*, 2012 (2012), 94.

[24] D. Wardowski and N. Van Dung, Fixed points of F-weak contractions on complete metric spaces, *Demonstratio Math.*, 1 (2014), 146-155.

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