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Beurling-Fourier Algebras on The Homogeneous Spaces

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Abstract. For a locally compact group G with a closed subgroup H, we define and study Beurling-Fourier algebras on the homogeneous space G/H, which consists of the left cosets of H in G. The cornerstone of our approach is the definition of Beurling-Fourier algebras in terms of the weight inverses. For G with closed subgroup H and weight $\omega : G \to [1, \infty)$, we study Beurling-Fourier algebras on G/H. We show that our construction on G/H, denoted by $A(G : H, \omega)$ and equipped with the norm $\|.\|_\omega$, forms a Banach algebra. In particular, we establish a version of Leptin theorem: if H is compact, then G is amenable, if and only if Beurling-Fourier algebra on G/H has a bounded approximate identity.

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1 Introduction

The study of the Fourier algebra on a topological group, not necessarily abelian, began in the early twentieth century. Fourier and Fourier-

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Stieltjes algebras on group G ($A(G)$ and $B(G)$ respectively), were born by Eymard [5], in his PhD thesis (1964). For a locally compact abelian group G the Fourier transform provides an isometrically isomorphism from $L^1(G)$ onto $A(\widehat{G})$, where $\widehat{G}$ is the Pontryagin dual of G. Another group algebra is the Fourier–Stieltjes algebra, which for a locally compact abelian group G the Fourier–Stieltjes transform provides an isometrically isomorphism to $M(\widehat{G})$, the Borel regular complex measure algebra on $\widehat{G}$.

There have been many generalizations of the Fourier algebra. Herz [12], Figa-Talamanca [7], Runde [19], Amini, etc, have all attempted to generalize the Fourier algebra from a different perspective. These generalizations include changing the underlying space from groups to semigroups, hypergroups, and quantum groups, as well as changing the techniques of Banach spaces to operator algebras and p-operator spaces. The complete order amenability of Fourier algebras was first studied in [21]. Later developments extended these ideas to p-operator spaces, revealing new functorial properties of p-analogs of Fourier-Stieltjes algebras [1, 2].

Homogeneous spaces play a significant role in various fields of mathematics, especially in geometry, topology, and algebra. Some of the main aspects of their importance include symmetry and group actions [11], geometric analysis of invariant structures [4], and the foundational theory of Lie groups and their homogeneous quotients [20]. In summary, homogeneous spaces are crucial for understanding symmetry, geometric structures, and algebraic properties across various branches of mathematics and its applications.

In [8], it has been introduced and investigated Fourier algebra $A(G : H)$ and Fourier-Stieltjes algebra $B(G : H)$ on homogeneous spaces G/H of left cosets of H in G, where H is a closed subgroup of locally compact group G. Also, assuming that H is a compact subgroup of G, it is shown that G is amenable if and only if $A(G : H)$ has a bounded approximate identity.

Weights and weighted function spaces play an important role in mathematical analysis, functional theory, and differential equations. In principle, weights allow us to study the behavior of functions around a certain point, to ignore their fluctuations at infinity, or, conversely, to enhance the asymptotic behavior of a function. This concept has numerous applications in numerical mathematics and is often used for precise applications (signal theory, Gabor analysis, sampling theory, $\cdots$), for example [10, 6]. On the other hand, weights appear naturally in analysis: in inequalities relating the norm of a function to the norm of its derivatives, and in extension theorems, $\cdots$ One area where weighted spaces are used intensively is the theory of boundary value problems for partial differential equations (see [9]).

Constructing the Fourier algebra using weight functions is very complicated due to the nature of the definition of these functions, and there are various approaches to it. In [17], [14], Beurling algebras and Beurling–Fourier algebras have been introduced, which are functions that exhibit a suitable average behavior under the influence of weight. In [14], Lee and Samei introduced the notion of a Beurling–Fourier algebra. If G is a locally compact group and $\omega : G \to [1, \infty)$ is a weight, then multiplication with ω defines a closed, densely defined operator on $L^2(G)$, which is bounded if and only if ω is bounded, i.e. $L^1(G, \omega)$ is trivial. Consequently, Lee and Samei define what they call a weight on the dual of G as a closed, densely defined operator on $L^2(G)$ affiliated with the group von Neumann algebra $VN(G)$. The resulting theory of Beurling–Fourier algebras is particularly tractable for what Lee and Samei call central weights on the duals of compact groups. These weights and their corresponding Beurling–Fourier algebras were also introduced and investigated by Ludwig, Spronk and Turowska [15]. In [17], their approach is not to define Beurling algebras in terms of weights, i.e., possibly continuous functions, but rather in terms of their inverses, which are bounded continuous functions. Motivating, we mention some examples in Section 3. Precisely, on $S^n = SO(n+1)/SO(n)$, weighted Fourier algebras analyze functions with rotational symmetry. Weights on $\mathbb{R}^n \cong \mathbb{R}^n/\{0\}$ model decay at infinity [9, 10].

In this paper, we aim is to replace the underlying space from a group to a homogeneous space, to construct a new Fourier–type algebra. Using the Oztop [17] technique, which involves the weight inverses, we introduce a new algebra called the Beurling–Fourier algebra on homogeneous spaces denoted by $A(G : H, \omega)$.

The paper is organized as follows: In Section 2, we review Beurling-

Fourier algebras on locally compact groups, weight inverses and Beurling–Fourier algebras [17]. In Section 3, we first recall the Fourier algebras on coset spaces, [8], and then introduce Beurling–Fourier algebras on homogeneous spaces, using the inverse of weights. In Proposition 3.6, we show that our construction is a Banach algebra. In Theorem 3.13, for Beurling-Fourier algebras on homogeneous spaces, we prove a Leptin–type theorem: if H is compact, then G is amenable if and only if Beurling-Fourier algebra on G/H has a bounded approximate identity.

2 Preliminaries

The structure of Beurling and Fourier-Beurling algebras will be explained in this section. This is discussed in detail in reference [17]. Beurling algebras have long been studied in harmonic analysis, especially for abelian groups (see, for example, [14] and [18]).

2.1 Fourier algebras on the locally compact groups

Suppose G is a locally compact group, and $C^*(G)$ represent the full group C^*-algebra, which is the enveloping C^*-algebra of $L^1(G)$. Additionally, suppose $\sum_G$ denote the set of equivalence classes of weakly continuous unitary representations of G. For a representation $\pi \in \sum_G$ and vectors $\xi, \eta \in \mathcal{H}_\pi$, where $\mathcal{H}_\pi$ is the Hilbert space related to π, the function defined by $u(x) = \langle \pi(x)\xi, \eta \rangle$ is referred to as a coefficient function of π. The dual space of $C^*(G)$ can be identified with $B(G)$, which encompasses all coefficient functions associated with G . The space $B(G)$ forms a commutative Banach algebra under the dual norm and pointwise multiplication, and it is known as the *Fourier–Stieltjes algebra* of G.

One of the well-known representation on a locally compact group G, is the *left regular representation* $(\lambda, L^2(G))$, which is defined through the following relations, for $x, y \in G$ and $\xi \in L^2(G)$,

$$\lambda : G \to \mathcal{B}(L^2(G)), \quad \lambda(x) : L^2(G) \to L^2(G), \quad \lambda(x)(\xi) \in L^2(G),$$
$$\lambda(x)(\xi) : G \to \mathbb{C}, \quad \lambda(x)(\xi)(y) = \xi(x^{-1}y).$$

The space of all coefficient functions of the left regular representation forms an algebra which is called *Fourier algebra*, denoted by $A(G)$, which can be considered as $\overline{\langle B(G) \cap C_c(G)\rangle}$ or $\{f * \tilde{g} : \ f, g \in L^2(G)\}$. The Fourier algebra $A(G)$ is a closed ideal in $B(G)$. Indeed, the Fourier algebra $A(G)$ consists of all coefficient functions of the left regular representation λ of G, i.e.

$$A(G) = \{w = (\lambda\xi, \eta) \ : \ \xi, \eta \in L^2(G)\}.$$

This is a regular, commutative Banach algebra with pointwise multiplication and the norm $\|w\| = \inf\{\|\xi\|\|\eta\| : w = (\lambda\xi, \eta)\}$.

Using integration, λ can be "extended" to a $*$-representation of the group algebra $L^1(G)$ on the Hilbert space $L^2(G)$, and we will continue to use the notation λ for this extension. The reduced group C^*-algebra, $C_r^*(G)$, and the group von Neumann algebra of G, $VN(G)$, are defined as follows:

$$C_r^*(G) := \overline{\lambda(L^1(G))}^{\|\cdot\|} \qquad \text{and} \qquad VN(G) := \overline{\lambda(L^1(G))}^{\text{weak}^*},$$

The Fourier algebra $A(G)$ serves as the predual of $VN(G)$ [5]. This duality is defined as follows:

$$\Phi : VN(G) \longrightarrow (A(G))^*$$
$$\Phi(T) := \Phi_T, \quad \Phi_T((f * \tilde{g})) = \langle T(f), g\rangle.$$

As the predual of the (left) group von Neumann algebra $VN(G)$, $A(G)$ has the canonical operator space structure.

Consider the unitary operator $W \in \mathcal{B}(L^2(G \times G))$, which is given by:

$$(W\xi)(x, y) := \xi(x, xy) \qquad (\xi \in L^2(G \times G), x, y \in G).$$

Then

$$\hat{\Gamma} : \mathcal{B}(L^2(G)) \longrightarrow \mathcal{B}(L^2(G \times G)), \qquad T \mapsto W^{-1}(T \otimes 1)W$$

is a co-multiplication, satisfying (will be shown in the following.):

$$\hat{\Gamma}\lambda_x = \lambda_x \otimes \lambda_x \qquad (x \in G). \tag{1}$$

So we have the co-multiplication

$$\hat{\Gamma} : VN(G) \longrightarrow VN(G)\bar{\otimes}VN(G) \cong VN(G \times G),$$

where $\bar{\otimes}$ means the tensor product of von Neumann algebras, and $\cong$ denoted von Neumann isometric isomorphism. Thus this co-multiplication turns $A(G)$ into a completely contractive Banach algebra. Indeed co–multiplication $\hat{\Gamma}$, encodes group operation and ensures convolution compatibility.

To see the equation 1 we have:

$$\hat{\Gamma}\lambda_x = W^{-1}(\lambda_x \otimes id)W \qquad (x \in G).$$

Now for $\xi_1, \xi_2 \in L^2(G)$, and $t, s \in G$, we have

$$\begin{aligned}(\lambda_x \otimes id)(W(\xi_1 \otimes \xi_2))(t,s) &= W(\xi_1 \otimes \xi_2)(x^{-1}t, s)\\ &= \xi_1(x^{-1}t)\xi_2(x^{-1}ts).\end{aligned}$$

On the other

$$\begin{aligned}W(\lambda_x \otimes \lambda_x)(\xi_1 \otimes \xi_2)(t,s) &= (\lambda_x \otimes \lambda_x)(\xi_1 \otimes \xi_2(t, ts))\\ &= L_x\xi_1(t)L_x\xi_2(ts)\\ &= \xi_1(x^{-1}t)\xi_2(x^{-1}ts).\end{aligned}$$

Therefore, $W^{-1}(\lambda_x \otimes id)W = \lambda_x \otimes \lambda_x$.

The existence of $\hat{\Gamma}$ on $VN(G)$ induces a product on $A(G)$, which is represented by $\hat{*}$. For any $f, g \in A(G)$ and $x \in G$, we have

$$\langle f \hat{*} g, \lambda_x\rangle = \langle f \otimes g, \hat{\Gamma}\lambda_x\rangle = \langle f \otimes g, \lambda_x \otimes \lambda_x\rangle = f(x)g(x),$$

i.e., $\hat{*}$ is pointwise multiplication.

For a von Neumann algebra N, its predual N_* naturally forms an N-bimodule through the relationship:

$$\langle x, yf\rangle := \langle xy, f\rangle = \langle y, fx\rangle \qquad (f \in N_*, x, y \in N). \tag{2}$$

Moreover, there exists a canonical, weak* continuous, complete contraction $\theta : VN(G \times G) \longrightarrow (A(G)\hat{\otimes}A(G))^*$, such that the preadjoint $(\theta\hat{\Gamma})_* : A(G)\hat{\otimes}A(G) \longrightarrow A(G)$ represents pointwise multiplication (here, $\hat{\otimes}$ denotes the operator space projective tensor product).

2.2 Weight inverses and Beurling–Fourier algebras

A weight on a locally compact group G is defined as a measurable and locally integrable function $\omega : G \longrightarrow [1, \infty)$ that satisfies the condition

$$\omega(xy) \leq \omega(x)\omega(y) \qquad (x, y \in G).$$

The associated Beurling algebra ([18], Definition 3.7.2) is defined as

$$L^1(G, \omega) := \left\{ f \in L^1(G) : \omega f \in L^1(G) \right\}.$$

This algebra is a subalgebra of $L^1(G)$ and forms a Banach algebra with respect to the norm $\|f\|_{L^1(G,\omega)} := \|\omega f\|_1$ for $f \in L^1(G, \omega)$. We can assume without loss of generality that ω is continuous (see [18], Theorem 3.7.5). Beurling algebras have been a significant focus in abstract harmonic analysis for an extended period, particularly in the case of abelian groups G (see [13] and [18], for example).
The generalization of this structure to Fourier algebras has been pursued by mathematicians. Using the concept of weight or the weight inverse of in [14] and [17] has been two approaches in this direction.
Oztop approach in [17] provides Beurling–Fourier algebra such that, if G is a locally compact abelian group with dual group $\widehat{G}$, then the achieved algebra correspond–via the Fourier transform–to the Beurling algebras on $\widehat{G}$.

Oztop's main idea is not to define the "dual" concept of weight but rather to define the inverse concept of weight. This approach allows for the definition of Beurling–Fourier algebras without relying on the theory of von Neumann algebras, which [14] heavily depends on.

If G is a locally compact group and $\omega : G \longrightarrow [1, \infty)$ is a weight, then ω is bounded if and only if $L^1(G, \omega) = L^1(G)$ with an equivalent norm. In other words, unless $L^1(G, \omega)$ is trivial, the multiplication operator induced by ω on $L^2(G)$ is unbounded. However, the inverse of ω–with respect to pointwise multiplication–is bounded on G. This means that the corresponding multiplication operator on $L^2(G)$ is bounded and thus lies in the multiplier algebra of $C_0(G)$, the C^*-algebra of all continuous functions on G vanishing at infinity (represented on $L^2(G)$ as multiplication operators). For a locally compact group G, we denote by $C_b(G)$ the C^*-algebra of all connected functions in G.

The following theorem discussed in [14] motivates use the weight inverses to construct Beurling–Fourier algebras.

Proposition 2.1 ([17]). *Let G be a locally compact group. Then the following are equivalent for non-negative $\alpha \in C_b(G)$ with $\|\alpha\|_\infty \leq 1$:*

(i) there is a weight $\omega : G \longrightarrow [1, \infty)$ such that $\alpha = \omega^{-1}$;

(ii) (a) the map

$$C_0(G) \longrightarrow C_0(G), \qquad f \mapsto \alpha f \tag{3}$$

has dense range;

(b) there is $\Omega \in L^\infty(G \times G)$ with $||\Omega|| \leq 1$ such that

$$\alpha(x)\alpha(y) = \alpha(xy)\Omega(x, y) \qquad (x, y \in G). \tag{4}$$

Moreover, if ω is as in (i), then

$$L^1(G, \omega) = \{\alpha f : f \in L^1(G)\}$$

and

$$\|\alpha f\|_\omega = \|f\|_1 \qquad (f \in L^1(G)).$$

The key takeaway from Proposition 2.1 is that Beurling algebras can be defined without referencing a weight– a possibly unbounded continuous function–. Instead, they can be defined using the inverses of weights, which are bounded continuous functions, i.e., multipliers of $C_0(G)$.

For a C^*-algebra A, an element α Belongs to $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is the Hilbert space related to A, is said to be a left (resp. right) multiplier of A if $\alpha A \subset A$ (resp. $A\alpha \subset A$). If α is a left and right multiplier of A, then it is called merely a multiplier. The set of all multipliers of A is denoted by $\mathcal{M}(A)$ and it is called the *multiplier algebra* of A.

Definition 2.2. *Let G be a locally compact group. An element ω^{-1} belonging to $\mathcal{M}(C_r^*(G))$ is termed a weight inverse, if it satisfies $||\omega^{-1}|| \leq 1$ and the following conditions hold:*

(a) *the maps*

$$C_r^*(G) \longrightarrow C_r^*(G), \qquad x \mapsto x\omega^{-1} \tag{5}$$

and

$$C_r^*(G) \longrightarrow C_r^*(G), \qquad x \mapsto \omega^{-1}x \tag{6}$$

both have dense ranges;

(b) *There exists an element $\Omega \in VN(G \times G)$ with $||\Omega|| \leq 1$, such that*

$$\omega^{-1} \otimes \omega^{-1} = (\hat{\Gamma}\omega^{-1})\Omega.$$

The associated Beurling–Fourier algebra is defined as follows:

$$A(G, \omega) := \{\omega^{-1}f : f \in A(G)\}.$$

This algebra with respect to the norm $\|\omega^{-1}f\|_{A(G,\omega)} := \|f\|_{A(G)}$ for $f \in A(G)$ is a Banach algebra.

Remark 2.3. The condition $\|\omega^{-1}\| \leq 1$ ensures contractivity of the map $f \mapsto \omega^{-1}f$ (Theorem 2.3). For $\|\omega^{-1}\| > 1$, one may define an equivalent norm $\|f\|'_\omega := \|\omega^{-1}f\|_{A(G)}$, but boundedness of Ω in Definition 2.2(b) may fail. Relaxing this requires further technical assumptions (e.g., positivity of ω^{-1}). It can be said that central weights on compact groups or polynomial weights on $\mathbb{R}^n$ satisfy this assumption:

1. Compact groups: For $G = SU(n)$, central weights $\omega(g) = (1 + \|g\|)^\alpha$ $(\alpha \geq 0)$ yield bounded ω^{-1} in $\mathcal{M}(C_r^*(G))$ via the operator norm.
2. $\mathbb{R}^n : \omega(x) = (1 + |x|)^\alpha$ $(\alpha > 0)$ gives $\omega^{-1} \in C_b(\mathbb{R}^n) \subset \mathcal{M}(C_r^*(\mathbb{R}^n))$.

Theorem 2.4. *(Theorem 2.6;[17]) Let G be a locally compact group, and let $\omega^{-1} \in \mathcal{M}(C_r^*(G))$ be a weight inverse. Then $A(G, \omega)$ is a dense subalgebra of $A(G)$. Moreover, if $A(G, \omega)$ is equipped with the unique operator space structure turning the bijection*

$$A(G) \longrightarrow A(G, \omega), \quad f \mapsto \omega^{-1}f.$$

into a complete isometry, then it is a completely contractive Banach algebra.

3 Beurling-Fourier algebras on coset spaces

A homogeneous space (or a homogeneous space of a topological group) refers to a space that can be uniformly covered using the action of a topological group on itself. In other words, if G is a topological group and X is a topological space, then X is homogeneous, if for any two points $x, y \in X$, there exists an element $g \in G$ such that $g \cdot x = y$. Here are several examples of homogeneous spaces:

1. **Lie Groups:** The space G/H , where G is a Lie group and H is a closed subgroup. This space is known as the homogeneous space of G with respect to the action of G on itself G/H .

2. **Euclidean Space:** The space $\mathbb{R}^n$ is considered as a homogeneous space because any point can be moved to any other point using vector addition and scalar multiplication.

3. **Sphere:** The sphere S^n is regarded as a homogeneous space, because any point can be moved to any other point using rotations (by act of the the group $SO(n)$).

4. **Projective Space:** The projective space $\mathbb{P}^n(\mathbb{R})$ is also a homogeneous space, because it can be acted upon by general linear groups (by act of the the group $PGL(n+1, \mathbb{R})$).

5. **Hyperbolic Space:** Hyperbolic space is another example of a homogeneous space, as it can be acted upon by hyperbolic groups (by act of the the group $PSL(2, \mathbb{R})$).

These examples illustrate the diversity of homogeneous spaces in mathematics and topology, each possessing its own unique characteristics.
In this section, we aim to define Beurling–Fourier algebras on coset spaces and study some properties of these algebras. This section employs the techniques described in [17]. To gain a better understanding, we first remind the Fourier–Stieltjes and Fourier algebras on coset spaces ($B(G:H)$ and $A(G:H)$ respectively), defined by Forrest [8].

Let H be a closed subgroup of G. The symbol G/H will denote the homogeneous space formed by the left cosets of H. $\tilde{x}$ denotes the left coset of xH as an element of G/H. The canonical map from G to

G/H is denoted by φ. A continuous function $\tilde{u}$ defined on G/H, can be associated with the continuous function u on G given by $u = \tilde{u} \circ \varphi$. This establishes an isomorphism between the space of continuous functions $C(G/H)$ and the subalgebra $C(G : H)$ of $C(G)$, which consists of functions that are constant on the cosets of H in G.

To define and describe the basic properties of Fourier algebras on coset spaces, we refer to [8] and begin by defining the Fourier–Stieltjes algebra and the Fourier algebra on coset spaces, denoted as $B(G : H)$ and $A(G : H)$, respectively.

Definition 3.1. *Suppose that H is a closed subgroup of a locally compact group G.*
Put $B(G : H) = \{u \in B(G) \mid u(xh) = u(x) \text{ for every } x \in G,\ h \in H\}$. Additionally, put

$$A(G : H) = \{u \in B(G : H) \mid \varphi(\text{supp } u) \text{ is compact in } G/H\}^{-\|\cdot\|_{B(G)}}.$$

Proposition 3.2. *(Proposition 3.1; [8])*

(i) Both $B(G : H)$ and $A(G : H)$ are closed subalgebras of $B(G)$. Furthermore, $A(G : H)$ is a closed ideal within $B(G : H)$.

(ii) $B(G : H)$ is unital.

(iii) $A(G : H) \cap A(G) \neq \{0\}$ if and only if H is compact.

(iv) $A(G : H) = B(G : H)$ if and only if G/H is compact.

Proposition 3.3. *(Proposition 3.2; [8]) Let H be a closed normal subgroup of G. Then $B(G : H)$ and $A(G : H)$ are isometrically isomorphic to $B(G/H)$ and $A(G/H)$, respectively.*

Proposition 3.4. *(Corollary 3.4; [8]) Let G be a locally compact group, and let H be its compact subgroup. The mapping P_H defined as*

$$P_H : B(G) \longrightarrow B(G : H), \qquad P_H(\phi)(x) = \int_H \phi(xh)dh, \tag{7}$$

is a continuous projection on $B(G)$ onto $B(G : H)$. The restriction P_H to $A(G)$ is a projection from $A(G)$ onto $A(G : H)$.

Now we are ready to define our new space of Fourier-type on a coset space, using weight inverses.

Definition 3.5. *Let H be a closed subgroup of a locally compact group G, and let ω^{-1} be a weight inverse. We define*

$$A(G:H,\omega) := \{\omega^{-1}f : f \in A(G:H)\}$$

and

$$\|\omega^{-1}f\|_\omega := \|f\| \qquad (f \in A(G:H)),$$

where, $\|.\|$ refers to $\|.\|_{A(G)}$.

We recall that the notation $\omega^{-1}f$, refers to the module action defined in 2.

In the following we show that $A(G:H,\omega)$ is a Banach algebra and we call it *Beurling-Fourier algebra* on the homogeneous space G/H.

As we see in the following proposition, $A(G:H,\omega)$ is multiplicatively closed because the co-multiplication $\hat{\Gamma}$ and the operator Ω (from Definition 2.2(b)) 'absorb' the weight inverse, preserving the coset invariance. For instance, if ω^{-1} is central, $\Omega = 1$, simplifying the product to pointwise multiplication.

Proposition 3.6. *Let H be a closed subgroup of G and ω^{-1} be a weight inverse. Then $A(G:H,\omega)$ with respect to the norm $\|.\|_\omega$ is a Banach algebra.*

Proof. First we show $\omega^{-1}f + \omega^{-1}g = \omega^{-1}(f+g)$. Let $f, g \in A(G:H)$ and $x \in G$ and λ_x the translation operator on $L^2(G)$, which belongs to $VN(G)$; then we have

$$\begin{aligned}
(\omega^{-1}f + \omega^{-1}g)(x) &= \langle \omega^{-1}f + \omega^{-1}g, \lambda_x\rangle \\
&= \langle \omega^{-1}f, \lambda_x\rangle + \langle \omega^{-1}g, \lambda_x\rangle \\
&= \langle \lambda_x\omega^{-1}, f\rangle + \langle \lambda_x\omega^{-1}, g\rangle \\
&= \lambda_x\omega^{-1}(f+g) = \langle \omega^{-1}(f+g), \lambda_x\rangle \\
&= \omega^{-1}(f+g)(x).
\end{aligned}$$

In the follwing we use the notation $f \otimes g$ for $f, g \in A(G)$, to denote the canonical element belongs to $A(G \times G)$, and $\omega^{-1} \otimes \omega^{-1}$ is the tensor product of two operator in $VN(G)$, that belongs to $VN(G \times G)$.

To see that $A(G : H, \omega)$ is multiplicatively closed, notice that we have:

$$\begin{aligned}
\langle (\omega^{-1} f)(\omega^{-1} g), \lambda_x \rangle &= \langle (\omega^{-1} \otimes \omega^{-1})(f \otimes g), \lambda_x \otimes \lambda_x \rangle \\
&= \langle (\hat{\Gamma}\omega^{-1})\Omega(f \otimes g), \hat{\Gamma}\lambda_x \rangle \\
&= \langle \Omega(f \otimes g), \hat{\Gamma}(\lambda_x \omega^{-1}) \rangle \\
&= \langle (\hat{\Gamma})_*(\Omega(f \otimes g)), \lambda_x \omega^{-1} \rangle \\
&= \langle \omega^{-1}(\hat{\Gamma})_*(\Omega(f \otimes g)), \lambda_x \rangle;
\end{aligned}$$

Now we show that $(\hat{\Gamma})_*(\Omega(f \otimes g)) \in A(G : H)$.
It is enough to say $\langle (\hat{\Gamma})_*\Omega(f \otimes g), \lambda_{xh} \rangle = \langle (\hat{\Gamma})_*\Omega(f \otimes g), \lambda_x \rangle$ for $x \in G$ and $h \in H$.
According to the Definition 2.2, Ω is an element in $VN(G \times G)$ with $||\Omega|| \leq 1$ such that $\omega^{-1} \otimes \omega^{-1} = (\hat{\Gamma}\omega^{-1})\Omega$.
So we have

$$\begin{aligned}
\langle (\hat{\Gamma})_*\Omega(f \otimes g), \lambda_{xh} \rangle &= \langle \Omega(f \otimes g), \lambda_{xh} \otimes \lambda_{xh} \rangle \\
&= \langle \Omega, (f \otimes g)(\lambda_{xh} \otimes \lambda_{xh}) \rangle \\
&= \langle \Omega, (f \otimes g)(\lambda_x \otimes \lambda_x) \rangle \\
&= \langle \Omega(f \otimes g), \lambda_x \otimes \lambda_x \rangle \\
&= \langle (\hat{\Gamma})_*\Omega(f \otimes g), \lambda_x \rangle.
\end{aligned}$$

The reason of the third equality is the functions f and g are constant on the cosets. So $(\omega^{-1} f)(\omega^{-1} g) = \omega^{-1}(\hat{\Gamma})_*(\Omega(f \otimes g)) \in A(G : H, \omega)$. Therefore, $A(G : H, \omega)$ is closed under multiplication.

It is clear that $\| \cdot \|_\omega$ is a norm. To show that $A(G : H, \omega)$ is complete, let $\{\omega^{-1} f_n\}$ be a Cauchy sequence in $A(G : H, \omega)$. We must show there exists f in $A(G : H)$, such that $\omega^{-1} f_n$ converges to $\omega^{-1} f$. Since $\{\omega^{-1} f_n\}$ is a Cauchy sequence in $A(G : H, \omega)$, then according to the given norm definition, $\{f_n\}$ is a Cauchy sequence, and since $A(G : H)$ is complete, so there exists f belongs to $A(G : H)$ such that $\|f_n - f\|_{A(G)} \longrightarrow 0$. Therefore, it is clear that $\|\omega^{-1} f_n - \omega^{-1} f\|_\omega \longrightarrow 0$. $\square$

Remark 3.7. Unlike weighted L^p-spaces, $A(G:H,\omega)$ encodes non-commutative harmonic analysis via the group von Neumann algebra. For $H=\{e\}$, it generalizes Figà–Talamanca-–Herz algebras[12, 7], but differs by using weight inverses to ensure bounded multipliers. For abelian group G, $A(G,\omega)$ Fourier-dualizes to $L^1(\widehat{G},\check{\omega})$ [17, Theorem. 2.6].

Remark 3.8. 1. *$A(G:H,\omega)$ embeds contractively in $A(G)$. Since, for $f\in A(G:H)$, $\|\omega^{-1}f\|\leq\|\omega^{-1}f\|_\omega$:*

$$\begin{aligned}\|\omega^{-1}f\| &= \sup_{\substack{\Phi\in VN(G)\\ \|\Phi\|\leqslant 1}}\|\Phi(\omega^{-1}f)\| \leq \sup_{\substack{\Phi\in VN(G)\\ \|\Phi\|\leqslant 1}}\|f\|\|\Phi\omega^{-1}\|\\ &\leq \sup_{\substack{\Phi\in VN(G)\\ \|\Phi\|\leqslant 1}}\|f\|\|\Phi\|\|\omega^{-1}\|\leq\|f\|,\end{aligned}$$

so, by the definition of the norm $\|.\|_\omega$, we have $\|\omega^{-1}f\|\leq\|\omega^{-1}f\|_\omega$.

2. *$A(G:H,\omega)$ is dense in $A(G)$. For that, let $\lambda_x\in VN(G)$ be such that $\langle f,\lambda_x\rangle=0$ for $f\in A(G:H,\omega)$, i.e., $\langle\omega^{-1}f,\lambda_x\rangle=\langle f,\lambda_x\omega^{-1}\rangle=0$ for $f\in A(G:H)$. Therefore $\lambda_x\omega^{-1}=0$. As (6) has dense range in $C_r^*(G)$, the set $\{\omega^{-1}\lambda_y|\lambda_y\in VN(G)\}$ is weak*-dense in $VN(G)$, so that $\lambda_x VN(G)=\{0\}$. Since $VN(G)$ is unital, we conclude that $\lambda_x=0$, so that $A(G:H,\omega)$ is dense in $A(G)$ and also in $A(G:H)$ by the Hahn–Banach theorem.*

Let H be a closed subgroup of G. Let $VN_H(G)$ denote the weak-star-closure of the linear span of the set $\{\lambda_h|h\in H\}$, where λ is the left regular representation of G. Then $VN_H(G)$ is a von Neumann algebra, contained in $VN(G)$.

Proposition 3.9. *Let H be a closed subgroup of G, and let $\omega^{-1}\in VN_H(G)$ be a weight inverse. Then $A(G:H,\omega)$ with respect to the norm $\|.\|_{A(G)}$ is a dense subalgebra of $A(G:H)$.*

Proof. We establish two claims:
i: $A(G:H,\omega)\subseteq A(G:H)$.
ii: $A(G:H,\omega)$ is dense in $A(G:H)$.

Proof of i: Let v be an element in $A(G:H,\omega)$. We show that v belongs to $A(G:H)$. By the definition $v=\omega^{-1}f$ for some $f\in A(G:H)$, we verify two properties:

(a) v is constant on H-cosets: $v(gh)=v(g)$ for all $g\in G$, $h\in H$.

(b) $v\in\{u\in B(G:H)\mid\varphi(\text{supp }u)\text{ is compact in }G/H\}^{-\|\cdot\|_{B(G)}}$.

Proof of (a): Since $\omega^{-1}\in VN_H(G)$, it is generated by left translations $\{\lambda_h:h\in H\}$. Consider two cases:

Case 1. $\omega^{-1}=\sum\limits_{i=1}^{n}\alpha_i\lambda_{h_i}$ for some $h_i\in H$ and $i=1,...,n$.

In this case for $g\in G$ and $h,h_i\in H$ and $i=1,...,n.$ we have

$$\begin{aligned} v(gh)=\omega^{-1}f(gh)=\langle\omega^{-1}f,\lambda_{gh}\rangle\langle f,\sum_{i=1}^{n}\alpha_i\lambda_{h_igh}\rangle \\ =\langle\sum_{i=1}^{n}\alpha_i\lambda_{h_i}f,\lambda_g\rangle=\langle\omega^{-1}f,\lambda_g\rangle=\omega^{-1}f(g). \end{aligned}$$

Case 2. $\omega^{-1}=\lim_{n\to\infty}T_n$, where $\{T_n\}$ is a sequence as case1 and converges is in weak operator topology to ω^{-1} , then

$$\begin{aligned} v(gh)=\omega^{-1}f(gh)=\langle\omega^{-1}f,\lambda_{gh}\rangle=\langle f,\lim_{n\to\infty}\sum_{i=1}^{n}\alpha_i\lambda_{h_i}\lambda_{gh}\rangle \\ =\langle\lim_{n\to\infty}\sum_{i=1}^{n}\alpha_i\lambda_{h_i}f,\lambda_g\rangle=\langle\omega^{-1}f,\lambda_g\rangle=\omega^{-1}f(g). \end{aligned}$$

Proof of (b): We show $v=\omega^{-1}f$ satisfies the support condition:
(i) let $f\in A(G:H)$ and $f\in C_c(G:H)$. We show that supp $\omega^{-1}f\subseteq$ supp f. Let $f(g)=0$ for some $g\in G$, it is enough to show that $\omega^{-1}f(g)=0$.
We follow some cases;

Case 1. $\omega^{-1} = \sum_{i=1}^{n} \alpha_i \lambda_{h_i}$ for some $h_i \in H, i = 1, ..., n$.

$$\omega^{-1} f(g) = \langle \omega^{-1} f, \lambda_g \rangle = \langle f, \sum_{i=1}^{n} \alpha_i \lambda_{h_i} \lambda_g \rangle$$
$$= \sum_{i=1}^{n} \alpha_i f(h_i g) = \sum_{i=1}^{n} \alpha_i f(g) = 0.$$

Case 2. $\omega^{-1} = \lim_{n\to\infty} T_n$ and where $\{T_n\}$ is sequence as case1 and to the convergence is in W.O.T or weak*-Topology,

$$\omega^{-1} f(g) = \langle \omega^{-1} f, \lambda_g \rangle = \langle f, \lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i \lambda_{h_i} \lambda_g \rangle$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i f(h_i g) = \lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i f(g) = 0.$$

So, for $f \in A(G : H)$ and $f \in C_c(G : H)$, we have the equation $v = \omega^{-1} f \in A(G : H)$.

(ii) Let $f \in A(G : H)$ and there exists a sequence $\{f_n\}$ belongs to $C_c(G)$ such that $f_n \xrightarrow{\|\cdot\|_{A(G)}} f$. Since $\{\omega^{-1} f_n\}$ satisfies condition (a), it follows that $\{\omega^{-1} f_n\}$ belongs to $A(G : H)$. Thus, we have $\omega^{-1} f \in \{u \in B(G : H) \mid \varphi(\text{supp } u) \text{ is compact in } G/H\}^{-\|\cdot\|_{B(G)}}$. Therefore, we conclude that $v = \omega^{-1} f \in A(G : H)$.

The density comes from Remark 3.8. □

Theorem 3.10. *Let G be a locally compact group, and let $\omega^{-1} \in \mathcal{M}(C_r^*(G))$ be a weight inverse. Then the following are equivalent:*

(i) the inclusion map from $A(G : H, \omega)$ into $A(G : H)$ is surjective;

(ii) ω^{-1} is left invertible in $\mathcal{M}(C_r^*(G))$.

Proof. (ii) $\Rightarrow$ (i) hold trivially.
(i) $\Rightarrow$ (ii):Assume that the inclusion map

$$\Phi : A(G:H,\omega) \longrightarrow A(G:H), \quad \omega^{-1}f \mapsto f$$

is surjective. So the composition of this map with $A(G:H) \to A(G:H,\omega)$; $\quad f \to \omega^{-1}f$, gives a bijection from $A(G:H)$ to itself. So going to the dual spaces we have the bijection $VN_H(G)$ to itself, that takes each element x to $x\omega^{-1}$. Since $VN_H(G)$ is a unital algebra we have the result. $\square$

Lemma 3.11. *Let $f \in A(G)$ and $f \in B(G:H)$. Then f belongs to $A(G:H)$.*

Proof. Since $A(G) = \overline{\langle B(G) \cap C_c(G) \rangle}$, there exists a sequence $\{u_n\}_{n\in\Lambda}$ belong to $B(G) \cap C_c(G)$ such that $\|u_n - f\|_{B(G)} \longrightarrow 0$. For each n, since u_n belongs to $C_c(G)$, the support of u_n is compact. Therefore, for every n, $\varphi(\text{supp } u_n)$ is compact. This implies that f belongs to the set $\{u \in B(G:H) \mid \varphi(\text{supp } u) \text{ is compact in } G/H\}^{-\|\cdot\|_{B(G)}}$. $\square$

Lemma 3.12. *Let H be a closed subgroup of G. Then $L^2(G)*L^2(G:H)$ is a subset of $A(G:H)$.*

Proof. Let $f \in L^2(G)$ and $g \in L^2(G:H)$. Now, for $x \in G$ and $h \in H$,

$$f*g(xh) = \int f(y)g(y^{-1}xh)dy = \int f(y)g(y^{-1}x)dy = f*g(x).$$

On the other hand, since $f*g \in A(G)$, according to the above lemma $f*g \in A(G:H)$, so $L^2(G)*L^2(G:H)$ is a subset of $A(G:H)$. $\square$

Now we are ready to give the main result, that is a version of Leptin theorem, for our construction.

Theorem 3.13. *Let H be a compact subgroup of G and $\omega^{-1} \in \mathcal{M}(C_r^*(G))$ be a weight inverse. Then the following are equivalent:*
(i) G is amenable;
(ii) Beurling-Fourier algebra on homogeneous space G/H, $A(G:H,\omega)$ has a bounded approximate identity.

Proof. (i) $\Rightarrow$ (ii): technique is similar to that outlined in Proposition 4.5 from [17]. Since G is amenable, it has Reiter's property (P_1) ([16], Proposition 6.12), meaning there exists a (P_1)-net $(\xi_\alpha)_{\alpha \in_I}$ in $L^1(G)$. Let the net $(\overline{e_\alpha}(x))_{\alpha \in I}$ be defined as

$$\overline{e}_\alpha(x) := \langle \lambda_{x^{-1}}(P\xi_\alpha)^{1/2}, \widetilde{(P\xi_\alpha)^{1/2}} \rangle \qquad (x \in G, \alpha \in I),$$

where P is defined as it is in equation (7). By Lemma 3.12, the net $(\overline{e}_\alpha(x))_{\alpha \in_I}$ belongs to $A(G : H)$. According to ([17], Proposition 4.5), the net $(\langle \omega^{-1}, 1 \rangle^{-1} \omega^{-1} \overline{e}_\alpha)_{\alpha \in I}$ in $A(G : H, \omega)$ serves as a weak approximate identity for $A(G : H, \omega)$. A well-known result in the theory of Banach algebras (see [3], Proposition 11.4) states that this implies the existence of a bounded approximate identity for $A(G : H, \omega)$.

(ii) $\Rightarrow$ (i): According to Proposition 3.8, since $A(G : H, \omega)$ is dense in $A(G : H)$ and the inclusion is (completely) contractive, any bounded approximate identity for $A(G : H, \omega)$ will also serve as an approximate identity for $A(G : H)$. As a result, we conclude that G must be amenable, as shown in [8, Theorem 4.2]. □

Remark 3.14. It would be interesting to generalize the results to the quantum state. For a quantum group $\mathbb{G}$ with co-subgroup $\mathbb{H}$, one might define $A(\mathbb{G} : \mathbb{H}, \omega)$ using the weight inverse $\omega^{-1} \in \mathcal{M}(C_r^*(\mathbb{G}))$ and a co-multiplication $\hat{\Gamma}$. The condition $\omega^{-1} \otimes \omega^{-1} = (\hat{\Gamma}\omega^{-1})\Omega$ (Definition 2.2(b)) naturally generalizes to this setting via Hopf algebra constructions [23, 22].

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