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Karush-Kuhn-Tucker Types Optimality Conditions for Non-Smooth Semi-Infinite Vector Optimization Problems

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Abstract. In this paper we establish necessary and sufficient optimality conditions for a nondifferenriable, nonconvex semi-infinite vector optimization problem involving locally Lipschitz functions, whose constraints are required to depend continuously on an index j belonging to a compact set J.

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1. Introduction

A semi-infinite vector optimization problem (SIVOP for short) is the simultaneously minimization of finitely many scalar objective functions subject to an arbitrary (possibly infinite) set of constraint functions. To the best of our knowledge, there are only a very few works available dealing with optimality conditions for SIVOP; see, e.g., [2] in differentiable cases, [5] in convex cases, and [3, 7] in nonsmooth cases. In [3], a limiting constraint qualification in terms of Mordukhovich subdifferential is introduced. The authors in [7] considered three various constraint

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qualifications such as Abadie, basic, and regular constraint qualifications using Clarke subdifferential.

In this paper, we study a non-differentiable non-convex SIVOP with locally Lipschitz functions, and introduce a new constraint qualification for the problem. Then we establish a necessary and a sufficient optimality conditions for SIVOP.

The paper is organized as follows. In Section 2, we introduce some notations, basic definitions, and preliminaries, which are used throughout the paper. In Section 3, after defining a new constraint qualification, we prove an optimality result by terms of the Clarke subdifferential.

2. Notations and Preliminaries

In this section we present few definitions and auxiliary results that will be needed in the sequel.

Let A be a nonempty subset of $\mathbb{R}^n$, denote by $\bar{A}$, $conv(A)$, and $cone(A)$, the closure of A, the convex hull, and the convex cone (containing the origin) generated by A, respectively. Also, the polar cone and strict polar cone of A are defined respectively by:

$$A^0 := \big\{d \in \mathbb{R}^n \mid \langle x, d\rangle \leqslant 0 \quad \forall x \in A\big\},$$
$$A^- := \big\{d \in \mathbb{R}^n \mid \langle x, d\rangle < 0 \quad \forall x \in A\big\},$$

where $\langle ., .\rangle$ exhibits the standard inner product in $\mathbb{R}^n$. Notice that A^0 is always a closed convex cone. It is easy to show that if $A^- \neq \emptyset$ then $\overline{A^-} = A^0$. The bipolar Theorem states that $A^{00} = \overline{cone}(A)$; see [1, 6].

Let us recall the following theorems which will be used in the sequel.

Theorem 2.1. ([6,9]) *Let A be a nonempty compact subset of $\mathbb{R}^n$. Then*

(I) *$conv(A)$ is a closed set.*

(II) *$cone(A)$ is a closed cone, if $0 \notin conv(A)$.*

We recall that for $A \subseteq \mathbb{R}^n$ and $\hat{x} \in \overline{A}$, the contingent cone to A at $\hat{x}$ is defined by

$$T(A, \hat{x}) := \Big\{d \in \mathbb{R}^n \mid \exists (t_k, d_k) \to (0^+, d), \textit{such that}\ x + t_k d_k \in A \ \ \forall k \in \mathbb{N}\Big\}.$$

Notice that $T(A,\hat{x})$ is closed cone (generally nonconvex) in $\mathbb{R}^n$.
Let $\hat{x} \in \mathbb{R}^n$ and let $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a locally Lipschitz function. The Clarke directional derivative of φ at $\hat{x}$ in the direction $d \in \mathbb{R}^n$ introduced in [4] is given by

$$\varphi^0(\hat{x}; d) := \limsup_{y \to \hat{x},\ t \downarrow 0} \frac{\varphi(y+td) - \varphi(y)}{t},$$

and the Clarke subdifferential of φ at $\hat{x}$ is given by the set

$$\partial^c \varphi(\hat{x}) := \big\{ \xi \in \mathbb{R}^n \mid \langle \xi, d \rangle \leqslant \varphi^0(\hat{x}; d) \qquad \text{for all } d \in \mathbb{R}^n \big\}.$$

The Clarke subdifferential is a natural generalization of the derivative since it is known (see [4]) that when function φ is continuously differentiable at $\hat{x}$, then $\partial^c \varphi(\hat{x}) = \{\nabla \varphi(\hat{x})\}$. Moreover when a function φ is convex, the Clarke subdifferential coincides with the subdifferential in the sense of convex analysis.
In the following theorem we summarize some important properties of the Clarke directional derivative and the Clarke subdifferential from [4] which are widely used in what follows.

Theorem 2.2. *Let φ and ϕ be functions from $\mathbb{R}^n$ to $\mathbb{R}$ which are locally Lipschitz near $\hat{x}$. Then, the following assertions hold:*

(i) *One has always that*

$$\begin{aligned} &\varphi^0(\hat{x}x; v) = \max\big\{\langle \xi, v \mid \xi \in \partial^c \varphi(\hat{x})\big\}, \\ &\partial^c\big(\max\{\varphi, \phi\}\big)(\hat{x}) \subseteq conv\big(\partial^c\varphi(\hat{x}) \bigcup \partial^c \phi(\hat{x})\big), \\ &\partial^c(\lambda\varphi)(\hat{x}) = \lambda \partial^c \varphi(\hat{x}), \qquad \forall\, \lambda \in \mathbb{R}, \\ &\partial^c(\varphi + \phi)(\hat{x}) \subseteq \partial^c \varphi(\hat{x}) + \partial^c \phi(\hat{x}). \end{aligned}$$

(ii) *The function $v \longrightarrow \varphi^0(\hat{x}; v)$ is finite, positively homogeneous, and subadditive on $\mathbb{R}^n$, and*

$$\partial\big(\varphi^0(\hat{x}; .)\big)(0) = \partial^c \varphi(\hat{x}),$$

where ∂ denotes the subdifferential in sense of convex analysis.

(iii) *$\partial^c \varphi(\hat{x})$ is a nonempty, convex, and compact subset of $\mathbb{R}^n$.*

3. Necessary Conditions

In the rest of this paper, we consider the following semi-infinite vector optimization problem:

$$\begin{array}{ll} & \inf\big(f_1(x),\ldots,f_m(x)\big) \\ \text{s.t.} & g_j(x) \leqslant 0, \quad i \in J, \\ & x \in \mathbb{R}^n, \end{array} \tag{P}$$

where the functions $f_i : \mathbb{R}^n \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$ and $g_j : \mathbb{R}^n \to \mathbb{R}_\infty$, with $i \in I := \{1,\ldots,m\}$ and $j \in J$, are locally Lipschitz, and J is an arbitrary set, not necessarily finite (but nonempty). Denote by S the feasible region, i.e.,

$$S := \big\{x \in \mathbb{R}^n \mid g_j(x) \leqslant 0, \quad \forall j \in J\big\}.$$

For a given $\hat{x} \in S$, let $J(\hat{x})$ denotes the index set of all active constraints at $\hat{x}$,

$$J(\hat{x}) := \big\{j \in J \mid g_j(\hat{x}) = 0\big\}.$$

A point $\hat{x}$ is said to be a weakly efficient solution to problem (P) iff there is no $x \in S$ satisfying $f_i(x) < f_i(\hat{x}),\ i \in I$.
Recall the following definition from [7, Definition 3.2]:

We say that (P) satisfies the regular constraint qualification (RCQ, briefly) at $\hat{x} \in S$ if

$$\Big(\bigcup_{i=1}^{m} \partial^c f_i(\hat{x})\Big)^- \cap \Big(\bigcup_{j\in J(\hat{x})} \partial^c g_j(\hat{x})\Big)^0 \subseteq T(S,\hat{x}).$$

The theorem below is proved in [7, Theorem 3.4].

Theorem 3.1. (KKT Necessary Condition) *Let x_0 be a weakly efficient solution of (P) and RCQ holds at x_0. If in addition cone$\Big(\bigcup_{j\in J(\hat{x})} \partial^c g_j(\hat{x})\Big)$ is a closed cone, then there exist $\alpha_i \geqslant 0$ (for $i \in I$) with $\sum_{i=1}^m \alpha_i = 1$, and $\beta_j \geqslant 0$ (for $j \in J(x_0)$) with $\beta_j \neq 0$ for at most finitely many indexes, such that*

$$0 \in \sum_{i=1}^{m} \alpha_i \partial^c f_i(\hat{x}) + \sum_{j\in J(\hat{x})} \beta_j \partial^c g_j(\hat{x}). \tag{1}$$

At this point, we introduce a new qualification condition for (P).

Definition 3.2. *Let $\hat{x} \in S$. We say that (P) satisfies the Cottle constraint qualification (CCQ, in brief) at $\hat{x}$, if J is a compact subset of $\mathbb{R}^p$, and the function $(x, j) \to g_j(x)$ is upper semicontinuous on $\mathbb{R}^n \times J$, and $\partial^c g_j(x)$ is an upper semicontinuous mapping in j for each x, and* $\left(\bigcup_{j\in J(\hat{x})} \partial^c g_j(\hat{x})\right)^- \neq \emptyset$.

Lemma 3.3. *Let $\hat{x} \in S$. If CCQ holds at $\hat{x}$, then $conv\left(\bigcup_{j\in J(\hat{x})} \partial^c g_j(\hat{x})\right)$ and $cone\left(\bigcup_{j\in J(\hat{x})} \partial^c g_j(\hat{x})\right)$ are closed sets.*

Proof. Firstly, we claim that $\bigcup_{j\in J(\hat{x})} \partial^c g_j(\hat{x})$ is a compact set. Let $\{\xi_k\}_{k=1}^{\infty}$ be a sequence in $\bigcup_{j\in J(\hat{x})} \partial^c g_j(\hat{x})$. If $|\partial^c g_{j_*}(\hat{x}) \cap \{\xi_k\}_{k=1}^{\infty}| = \infty$ for some $j_* \in J(\hat{x})$, then there exists subsequence $\{\xi_{k_p}\}$ which converges to some $\hat{\xi} \in \partial^c g_{j_*}(\hat{x})$ (by compactness of $\partial^c g_{j_*}(\hat{x})$). If $|\partial^c g_j(\hat{x}) \cap \{\xi_k\}_{k=1}^{\infty}| < \infty$ for all $j \in J(\hat{x})$, then without loss of generality we can assume that $\xi_k \in \partial^c g_{j_k}(\hat{x})$ for all $k \in \mathbb{N}$, and hence, $j_{k_p} \to \hat{j} \in J(\hat{x})$ for some subsequence $\{j_{k_p}\}$ of $\{j_k\}$ (by compactness of $J(\hat{x})$). Since the mapping $j \to \partial^c g_j(\hat{x})$ is upper-semicontinuous, there exists a subsequence of $\{\xi_{k_p}\}$ which converges to $\hat{\xi} \in \partial^c g_{\hat{j}}(\hat{x})$. Therefore, our claim is proved, i.e., $\bigcup_{j\in J(\hat{x})} \partial^c g_j(\hat{x})$ is a compact set.

This implies that $conv\left(\bigcup_{j\in J(\hat{x})} \partial^c g_j(\hat{x})\right)$ is closed by Theorem **??**(I). Now, because of

$$\Big(conv\Big(\bigcup_{j\in J(\hat{x})} \partial^c g_j(\hat{x})\Big)\Big)^- = \Big(\bigcup_{j\in J(\hat{x})} \partial^c g_j(\hat{x})\Big)^- \neq \emptyset,$$

it follows that $0 \notin conv\left(\bigcup_{j\in J(\hat{x})} \partial^c g_j(\hat{x})\right)$. Thus, $cone\left(\bigcup_{j\in J(\hat{x})} \partial^c g_j(\hat{x})\right)$ is a closed set by Theorem **??**(II).

Now suppose that CCQ holds at $x \in S$ and define

$$G(x) := \max_{j\in J} g_j(x), \qquad \forall x \in S.$$

It follows readily that G is locally Lipschitz, since each g_j is (see [4, Theorem 2.8.2]). The proof of the estimate

$$G^0(\hat{x}; d) \leqslant \max_{j \in J(\hat{x})} g_j^0(\hat{x}; d) \qquad \forall d \in \mathbb{R}^n, \tag{2}$$

is presented in [4, Theorem 2.8.2, step 1]. Note that the function $j \to g_j^0(\hat{x}; d)$ is upper-semicontinuous and $J(\hat{x})$ is compact, so that the notation "max" is justified in (2). $\square$

Lemma 3.4. *If CCQ holds at $\hat{x} \in S$, then one has*

$$\partial^c G(\hat{x}) \subseteq conv\Big(\bigcup_{j \in J(\hat{x})} \partial^c g_j(\hat{x}) \Big).$$

Proof. Let $\xi \in \partial^c G(\hat{x})$. The inequality in (2) implies that

$$\max_{j \in J(\hat{x})} \hat{g}_j(d) \geqslant \big\langle \xi, d \big\rangle \qquad \forall d \in \mathbb{R}^n,$$

where $\hat{g}_j(d) := g_j^0(\hat{x}; d)$. Since each $\hat{g}_j(.)$ is convex and $\hat{g}_j(0) = 0$, we can conclude that $\xi \in \partial \widehat{G}(0)$, where $\widehat{G}$ defined for each d by $\widehat{G}(d) := \max_{j \in J(\hat{x})} \hat{g}_j(d)$. On the other hand, for every j, $\hat{g}_j$ is continuous at $\hat{d} := 0$, and for every d, the function $j \to \hat{g}_j(d)$ is upper-semicontinuous. So, the well-known Pshenichnyi-Levin-Valadire Theorem ([6, pp. 267]) can be applied to obtain that

$$\partial \widehat{G}(0) = \overline{conv}\Big(\bigcup_{j \in \widehat{J}(0)} \partial \hat{g}_j(0) \Big),$$

where, $\widehat{J}(0) := \big\{ j \in J(\hat{x}) \mid \hat{g}_j(0) = \widehat{G}(0) = 0 \big\}$. But this gives the announced result because $\widehat{J}(0) = J(\hat{x})$ and $\partial \hat{g}_j(0) = \partial^c g_j(\hat{x})$ and $conv\Big(\bigcup_{j \in J(\hat{x})} \partial^c g_j(\hat{x}) \Big)$ is closed by Lemma 3.4. $\square$

Theorem 3.5. *The CCQ implies RCQ at $\hat{x}$.*

Proof. Let $d \in \Big(\bigcup_{j \in J(\hat{x})} \partial^c g_j(\hat{x}) \Big)^-$. Since

$$\Big(\bigcup_{j \in J(\hat{x})} \partial^c g_j(\hat{x}) \Big)^- = \Big(conv\Big(\bigcup_{j \in J(\hat{x})} \partial^c g_j(\hat{x}) \Big) \Big)^-,$$

Lemma 3.4. leads to

$$d \in \Big(conv\Big(\bigcup_{j \in J(\hat{x})} \partial^c g_j(\hat{x}) \Big) \Big)^- \subseteq \big(\partial^c G(\hat{x}) \big)^-.$$

Hence, $G^0(\hat{x}; d) < 0$, and consequently, there exists a scalar $\delta > 0$ such that

$$G(\hat{x} + \beta d) < G(\hat{x}) \leqslant 0, \qquad \forall\ \beta \in (0, \delta].$$

Thus, for all $j \in J$ and for all $\beta \in (0, \delta]$, we conclude

$$g_j(\hat{x} + \beta d) < 0.$$

Therefore, for all $\beta \in (0, \delta]$ we have $\hat{x} + \beta d \in S$, which implies

$$d \in T(S, \hat{x}).$$

We have thus proved

$$\Big(\bigcup_{j \in J(\hat{x})} \partial^c g_j(\hat{x}) \Big)^- \subseteq T(S, \hat{x}).$$

Since $\Big(\bigcup_{j \in J(\hat{x})} \partial^c g_j(\hat{x}) \Big)^- \neq \emptyset$, we obtain that

$$\Big(\bigcup_{j \in J(\hat{x})} \partial^c g_j(\hat{x}) \Big)^0 = \overline{\Big(\bigcup_{j \in J(\hat{x})} \partial^c g_j(\hat{x}) \Big)^-} \subseteq \overline{T(S, \hat{x})} = T(S, \hat{x}),$$

and the proof is complete. $\square$

As an immediate consequence of Theorems 3.1 and 3.5, and Lemma 3.3, we can obtain the following theorem.

Theorem 3.6. (KKT Necessary Condition) *Let $\hat{x} \in S$ be a weakly efficient solution of (P) and CCQ holds at $\hat{x}$. Then there exist $\alpha_i \geqslant 0$ (for $i \in I$) with $\sum_{i=1}^m \alpha_i = 1$, and $\beta_j \geqslant 0$ (for $j \in J(\hat{x})$) with $\beta_j \neq 0$ for at most finitely many indexes, such that*

$$0 \in \sum_{i=1}^m \alpha_i \partial^c f_i(\hat{x}) + \sum_{j \in J(\hat{x})} \beta_j \partial^c g_j(\hat{x}).$$

The following example shows that the assumption of closedness of *cone* $\left(\bigcup_{j\in J(\hat{x})}\partial^c g_j(\hat{x})\right)$ can not be waived in Theorem 3.1.

Example 3.7. Consider the following problem;

$$\begin{array}{rl} (P_1) & \inf\big(f_1(x), f_2(x)\big) \\ s.t. & g_j(x) \leqslant 0, \quad j \in J := \mathbb{N}\cup\{0\} \\ & x \in \mathbb{R}^2, \end{array}$$

where $f_1(x) = f_2(x) = -x_1$ and $g_j(x)$ is the support function of the following set

$$U_j = \left\{x \in \mathbb{R}^2 \mid x_1^2 + (x_2 - 1 - j)^2 \leqslant (1+j)^2,\ x_1 \geqslant 0,\ x_2 \geqslant 0\right\}.$$

The set of feasible solutions for the problem (P) is

$$S = \{x \in \mathbb{R}^2 \mid g_j(x) \leqslant 0 \quad \forall j \in J\} = \{x \in \mathbb{R}^2 \mid x_1 \leqslant 0,\ x_2 \leqslant 0\}.$$

It is easy to verify that $\hat{x} = (0,0)$ is an optimal solution for (P_1). We observe that

$$T(S,\hat{x}) = S \quad , \quad \partial^c g_j(\hat{x}) = U_j \quad , \quad \partial^c f_1(\hat{x}) = f_2(\hat{x}) = \{(-1,0)\},$$

$$\Big(\bigcup_{j\in J(\hat{x})}\partial^c g_j(\hat{x})\Big)^0 = S.$$

Note that *cone* $\left(\bigcup_{j\in J(\hat{x})}\partial^c g_j(\hat{x})\right)$ is not closed. It should be observed that (RCQ) holds at $\hat{x}$. It is easy to see that there is no sequence of scalars as in Theorem 3.1. satisfying (1).
The following example shows that the assumption of compactness of J is not necessary in Theorem 3.6.

Example 3.8. Consider the following problem:

$$\begin{array}{rl} & \inf\big(f_1(x), f_2(x)\big) \\ s.t. & g_j(x) \leqslant 0 \quad j \in J := \mathbb{N}, \\ & x \in \mathbb{R}^2, \end{array} \qquad (P2)$$

where $f_1(x) = f_2(x) := |x_1| - |x_2|$, and $g_j(x) := x_1^2 + x_2^2 - j^2$ for all $j \in \mathbb{N}$. It is easy to verify that:

- $S = \{(x_1, x_2) \in \mathbb{R}^2 \quad x_1^2 + x_2^2 \leqslant 1\}$,
- $\hat{x} := (0, -1)$,
- $J(\hat{x}) = \{1\}$,
- $\partial^c f_1(x_0) = \partial^c f_2(x_0) = [-1, 1] \times [-1, 1]$,
- $\bigcup_{j \in J(\hat{x})} \partial^c g_j(\hat{x}) = \{(0, -2)\}$.

It is worth noting that CCQ holds and J is not compact. It is easy to verify that (1) in Theorem 3.6 holds.
The following concepts will be useful in the sequel. For more details, discussion, and applications of invexity and it's generalizations see [8] and its references.

Definition 3.9. *Let $\varphi := (\varphi_1, \varphi_2, \dots, \varphi_q) : \mathbb{R}^p \to \mathbb{R}^q$ be a locally Lipschitz mapping, and let $x_0 \in \mathbb{R}^p$. We shall say that φ is generalized η- pseudoinvex at x_0 if there exist functions $\eta : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$ and $\nu_l : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}_+ \backslash \{0\}$ for $l \in \{1, 2, \dots, q\}$ such that the condition*

$$\sum_{l=1}^{q} \nu_l(x, x_0)\big(\varphi_l(x) - \varphi_l(x_0)\big) < 0 \Rightarrow \sum_{l=1}^{q} \big\langle \xi_l, \eta(x, x_0) \big\rangle < 0,$$

holds for each $x \in \mathbb{R}^p$ and for all $\xi_l \in \partial^c \varphi_l(x_0)$.

Definition 3.10. *Let $\varphi := (\varphi_1, \varphi_2, \dots, \varphi_q) : \mathbb{R}^p \to \mathbb{R}^q$ be a locally Lipschitz mapping, and let $x_0 \in \mathbb{R}^p$. We shall say that φ is generalized η- quasiinvex at x_0 if there exist functions $\eta : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$ and $\theta_l : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}_+ \backslash \{0\}$ for $l \in \{1, 2, \dots, q\}$ such that the condition*

$$\sum_{l=1}^{q} \theta_l(x, x_0)\big(\varphi_l(x) - \varphi_l(x_0)\big) \leqslant 0 \Rightarrow \sum_{l=1}^{q} \big\langle \xi_l, \eta_l(x, x_0) \big\rangle \leqslant 0,$$

holds for each $x \in \mathbb{R}^p$ and for all $\xi_l \in \partial^c \varphi_l(x_0)$.

Theorem 3.11. (KKT Sufficient Condition) *Suppose that there exist a feasible solution $\hat{x} \in S$ for (P) and scalars $\alpha_i \geqslant 0$ with $\sum_{i=1}^{m} \alpha_i = 1$ and a finite set $J^* \subseteq J(\hat{x})$ and scalars $\beta_j \geqslant 0$ for $\in \{1, 2, \ldots, k\}$ such that*

$$0 \in \sum_{i=1}^{m} \alpha_i \partial^c f_i(\hat{x}) + \sum_{r=1}^{k} \beta_{j_r} \partial^c g_{j_r}(\hat{x}). \tag{3}$$

Moreover if the function $(\alpha_1 f_1, \alpha_2 f_2, \ldots, \alpha_m f_m)$ is generalized η- pseudoinvex at $\hat{x}$ and the function $(\beta_{j_1} g_{j_1}, \beta_{j_2} g_{j_2}, \ldots, \beta_{j_k} g_{j_k})$ is generalized η- quasiinvex at $\hat{x}$, then $\hat{x}$ is a weak efficient solution for (P).

Proof. From (3), it is clear that there exist $\xi_i^* \in \partial^c f_i(\hat{x})$ and $\varsigma_{j_r}^* \in \partial^c g_{j_r}(\hat{x})$ such that

$$\sum_{i=1}^{m} \alpha_i \xi_i^* + \sum_{r=1}^{k} \beta_{j_r} \varsigma_{j_r}^* = 0. \tag{4}$$

Suppose on the contrary that $\hat{x}$ is not a weak efficient solution for (P), then there exist $x \in S$ such that $f(x) < f(\hat{x})$. Since $(\alpha_1, \alpha_2, \ldots, \alpha_m) \gneqq 0$ and $\nu_i(x, \hat{x}) > 0$ for all $i \in I$, we obtain

$$\sum_{i=1}^{m} \nu_i(x, \hat{x}) \big(\alpha_i f_i(x) - \alpha_i f_i(\hat{x})\big) = \sum_{i=1}^{m} \alpha_i \nu_i(x, \hat{x}) \big(f_i(x) - f_i(\hat{x})\big) < 0.$$

By η-pseudoinvexity of $(\alpha_1 f_1, \alpha_2 f_2, \ldots, \alpha_m f_m)$ at $\hat{x}$ we get

$$\sum_{i=1}^{m} \big\langle \widehat{\xi_i}, \eta(x, \hat{x}) \big\rangle < 0, \qquad \forall\, \widehat{\xi_i} \in \partial^c(\alpha_i f_i)(\hat{x}). \tag{5}$$

On the other hand, since $\{j_1, j_2, \ldots, j_k\} \subseteq J(\hat{x})$ and $x \in S$, then

$$g_{j_r}(x) \leqslant 0 = g_{j_r}(\hat{x}), \qquad \forall r \in \{1, 2, \ldots, k\}.$$

Now, Since $\beta_{j_r} \geqslant 0$ and $\theta_{j_r}(x, \hat{x}) > 0$ for all $r \in \{1, 2, \ldots, k\}$, we obtain

$$\sum_{r=1}^{k} \theta_{j_r}(x, \hat{x}) \big(\beta_{j_r} g_{j_r}(x) - \beta_{j_r} g_{j_r}(\hat{x})\big) = \sum_{r=1}^{k} \beta_{j_r} \theta_{j_r}(x, \hat{x}) \big(g_{j_r}(x) - g_{j_r}(\hat{x})\big) \leqslant 0.$$

By η-quasiinvexity of $(\beta_{j_1} g_{j_1}, \beta_{j_2} g_{j_2}, \ldots, \beta_{j_k} g_{j_k})$ at $\hat{x}$ we get

$$\sum_{r=1}^{k} \left\langle \widehat{\varsigma}_{j_r}, \eta(x, \hat{x}) \right\rangle \leqslant 0, \qquad \forall\, \widehat{\varsigma}_{j_r} \in \partial^c(\beta_i g_{j_r})(\hat{x}). \tag{6}$$

Adding the inequalities (5) and (6), we get

$$\Big\langle \sum_{i=1}^{m} \widehat{\xi}_i + \sum_{r=1}^{k} \widehat{\varsigma}_{j_r}, \eta(x, \hat{x}) \Big\rangle < 0. \tag{7}$$

But, by (1), there exist $\widehat{\xi}_i \in \partial^c(\alpha_i f_i)(\hat{x})$ and $\widehat{\varsigma}_{j_r} \in \partial^c(\beta_i g_{j_r})(\hat{x})$, such that $\widehat{\xi}_i = \alpha_i \xi_i^*$ and $\widehat{\varsigma}_{j_r} = \beta_{j_r} \varsigma_{j_r}^*$ for all $(i, r) \in I \times \{1, 2, \ldots, k\}$. Hence the inequality (7) becomes

$$\Big\langle \sum_{i=1}^{m} \alpha_i \xi_i^* + \sum_{r=1}^{k} \beta_{j_r} \varsigma_{j_r}^*, \eta(x, \hat{x}) \Big\rangle < 0$$

which contradicts (3). This completes the proof. $\square$

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