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Original Research Paper

Z^+-Initial Value Problem with Extended Triangular Distribution

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Abstract. Z-differential equations are used to model phenomena under uncertainty and partial reliability in scientific and engineering fields. Most existing methods for z-differential equations that involve z-numbers are based on discrete forms; however, the continuous form of z-numbers is more representative of the behavior of many phenomena. In this work, we examine z^+-numbers with triangular distributions as initial conditions in uncertain differential equations. The numerical method, called the Modified Euler method, is generalized to solve z^+-initial value problems, with proofs provided for its convergence and stability. Several examples are provided to demonstrate the accuracy and efficiency of the proposed method.

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1 Introduction

Information is essential in today's world. In decision, information must be reliable but in real-world information is typically identified by uncer-

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tainty and partial reliability. Z-numbers are powerful tools for defining imprecise and partially reliable information. The idea of z-numbers was originally presented by Zadeh, in 2011 [22]. Zadeh aimed to extend the ability to model uncertainty in information by incorporating both the value of the information and a measure of confidence in that value. This dual-component structure allows z-numbers to represent real-world situations more realistically, particularly in scenarios where uncertainty is pervasive. Each z-number consists of two parts: one that captures the uncertain value (often expressed as a fuzzy set) and another that indicates the reliability or confidence in that value. Z-numbers have since been applied in various fields, such as decision-making, risk assessment, and artificial intelligence, where managing uncertainty is essential.

Recently, efforts have been made by researchers to create a framework for mathematical computations utilizing z-numbers, as referenced in [21, 2, 3, 4, 12, 13, 17, 18, 19]. The initial approach to incorporating z-numbers into uncertain differential equations emerged in 2015 [6], introducing a Hukuhara derivative for z-valued function. Following this, further studies have focused on developing solutions for these uncertain differential equations, employing techniques such methods include the Sumudu transform [14] and artificial neural networks [15].

Z^+-number is an extension of z-numbers and add additional parameters to capture more complex forms of uncertainty. Particularly in fields that require highly detailed models of uncertainty, such as risk analysis, predictive modeling, and artificial intelligence. z^+-number aim to refine decision-making processes by giving a more comprehensive picture of the reliability and variability of data, especially in situations where conventional probabilistic methods may fall short.

M. Lordejani and et al.[9], presented the parametric form of z^+-numbers and the primary algebraic operations applied to them. Subsequently, by defining a metric on z^+-number space, they explored the concepts of limit, continuity, differentiability (strongly generalized Hukuhara differentiability) and integrability of a z^+-valued function in their work. Consequently, they provided a analytical method to solve z^+-differential equation by z^+-Laplace transforms.

Li and et al.[16], analyzed the complexity of continuous z-number calculations in their paper. The first one is using normal distribution. The use of the complex form of normal distributions makes it difficult to calculate exact hidden pdfs for continuous z^+-numbers. To overcome this, researchers often discretize the second fuzzy number, leading to discretization and estimation errors, causing potential information loss. Approaches like those by Aliev et al. [5], have introduced arithmetic operations for continuous z^+-numbers by discretizing fuzzy numbers, though with challenges in accuracy. The second one is, inconsistency in z^+-number definition. There is an inconsistency between the intended meaning of z^+-numbers (representing reliability or certainty) and their mathematical definition. According to Zadeh, a z^+-number reliability implies certainty in the variable's value, ideally corresponding to a probability of 1. However, the actual distribution does not align perfectly with this, especially for continuous z^+-numbers, leading researchers to often approximate z^+-numbers as fuzzy numbers due to difficulties in defining exact probability distributions. To overcome these difficulty, they extend the triangular distribution to serve as the hidden pdf of triangular z-numbers and apply it to perform operations on z^+-numbers. In this paper, we consider uncertain differential equations with z^+-numbers as initial values. We utilize the extended triangular distribution as the hidden probability density function of z^+-numbers. By generalizing the Modified Euler method [1], we obtain the solution of the z^+-differential equation in the form of a z^+-number. The Modified Euler method for the z^+-initial value problem is proposed in three cases, and its acceptable accuracy is illustrated through several examples.

The paper is organized as follows: Section 2 introduces some basic definitions and theorems. The proposed method is described in Section 3. Numerical examples are provided in Section 4, and finally, a conclusion is drawn.

2 Preliminaries

In this section, we present the necessary definitions, theorems and lemma. The space of fuzzy numbers on the real line is represented by $\mathbb{R}_{\mathcal{F}}$. For two fuzzy numbers $m,\ n \in \mathbb{R}_{\mathcal{F}}$ and $k \in \mathbb{R}$, the addition and scalar multiplication operations are defined as follows.:

$[m+n]^\alpha = [m]^\alpha + [n]^\alpha$,
$[km]^\alpha = k[m]^\alpha$,
where for $0 < \alpha \leq 1$, α-level set is defined by $[m]^\alpha = \Big\{t \in \mathbb{R} \Big| m(t) \geq \alpha\Big\}$,
and $[m]^0 = cl\Big\{t \in \mathbb{R} \Big| m(t) > 0\Big\}$.

Definition 2.1. (see.[7])For two fuzzy numbers m and n the Hausdorff distance $d_\infty : \mathbb{R}_\mathcal{F} \times \mathbb{R}_\mathcal{F} \longrightarrow \mathbb{R}^+ \cup \{0\}$ is defined as follows

$$d_\infty(m,\ n) = \sup_{\alpha \in [0,\ 1]} \max\Big\{|\underline{m}(\alpha) - \underline{n}(\alpha)|,\ |\overline{m}(\alpha) - \overline{n}(\alpha)|\Big\}.$$

Definition 2.2. (see.[7]) The Hukuhara difference of two fuzzy number m and n is denoted by $m \ominus n$, and it exists, if there exist $s \in \mathbb{R}_\mathcal{F}$ such that $m = n \oplus s$.

Definition 2.3. (see.[9]) φ_B mapping $t \to \varphi_B(t)$ be a fuzzy-valued function where $\varphi_B : [a,b] \subseteq \mathbb{R} \to \mathbb{R}_\mathcal{F}$, and we have

$$\forall t \in [a,b]; \quad \varphi_B(t) \in \mathbb{R}_\mathcal{F}. \tag{1}$$

The $\alpha-$level set of $\varphi_B(t)$ is shown by $[\underline{\varphi_B}(t,\alpha), \overline{\varphi_B}(t,\alpha)]$, where $t \in [a,b]$ and $\alpha \in [0,1]$.

Definition 2.4. (see.[9]) Let $\varphi_B : [a,b] \subseteq \mathbb{R} \to \mathbb{R}_\mathcal{F}$ and $t_0 \in (a,b)$. φ_B is strongly Hukuhara differentiable (sgH-differentiable) at t_0, if for all $\kappa > 0$ sufficiently closed to zero, there exists an element $\varphi'_{B_{sgH}}(t_0) \in \mathbb{R}$ where

1. the H-differences $\varphi_B(t_0+\kappa) \ominus \varphi_B(t_0)$, $\varphi_B(t_0) \ominus \varphi_B(t_0-\kappa)$ exist and limits (in the metric d_∞)

$$\lim_{\kappa \to 0^+} \frac{\varphi_B(t_0+\kappa) \ominus \varphi_B(t_0)}{\kappa} = \lim_{\kappa \to 0^+} \frac{\varphi_B(t_0) \ominus \varphi_B(t_0-\kappa)}{\kappa} = \varphi'_{B_{sgH}}(t_0), \tag{2}$$

or

2. the H-differences $\varphi_B(t_0) \ominus \varphi_B(t_0+\kappa)$, $\varphi_B(t_0-\kappa) \ominus \varphi_B(t_0)$ exist and limits (in the metric d_∞)

$$\lim_{\kappa \to 0^+} \frac{\varphi_B(t_0) \ominus \varphi_B(t_0+\kappa)}{-\kappa} = \lim_{\kappa \to 0^+} \frac{\varphi_B(t_0-\kappa) \ominus \varphi_B(t_0)}{-\kappa} = \varphi'_{B_{sgH}}(t_0), \tag{3}$$

or

3. the H-differences $\varphi_B(t_0+\kappa)\ominus\varphi_B(t_0)$, $\varphi_B(t_0-\kappa)\ominus\varphi_B(t_0)$ exist and limits (in the metric d_∞)

$$\lim_{\kappa\to 0^+}\frac{\varphi_B(t_0+\kappa)\ominus\varphi_B(t_0)}{\kappa}=\lim_{\kappa\to 0^+}\frac{\varphi_B(t_0-\kappa)\ominus\varphi_B(t_0)}{-\kappa}=\varphi'_{B_{sgH}}(t_0), \quad (4)$$

or

4. the H-differences $\varphi_B(t_0)\ominus\varphi_B(t_0+\kappa)$, $\varphi_B(t_0)\ominus\varphi_B(t_0-\kappa)$ exist and limits (in the metric d_∞)

$$\lim_{\kappa\to 0^+}\frac{\varphi_B(t_0)\ominus\varphi_B(t_0+\kappa)}{-\kappa}=\lim_{\kappa\to 0^+}\frac{\varphi_B(t_0)\ominus\varphi_B(t_0-\kappa)}{\kappa}=\varphi'_{B_{sgH}}(t_0). \quad (5)$$

Definition 2.5. (see.[9]) We consider $\varphi_B : [a,b]\subseteq\mathbb{R}\to\mathbb{R}_{\mathcal{F}}$, with parametric form $(\underline{\varphi}(t,\alpha),\overline{\varphi}(t,\alpha))$, where both differentiable at $t_0\in[a,b]$,

1. if φ_B is $[(i)-sgH]$-differentiable at t_0,
$\varphi'_{B_{i.sgH}}(t_0)=(\underline{\varphi}_B(t_0,\alpha),\overline{\varphi}_B(t_0,\alpha))$,
2. if φ_B is $[(ii)-sgH]$-differentiable at t_0,
$\varphi'_{B_{ii.sgH}}(t_0)=(\overline{\varphi}_B(t_0,\alpha),\underline{\varphi}_B(t_0,\alpha))$.

Definition 2.6. (see.[20]) Let $(\Omega,\mathcal{F},P)$ be a probability space

- Ω: Sample space (set of all possible outcomes)
- $\mathcal{F}$: σ-algebra of measurable events
- P: Probability measure satisfying $\mathbb{P}(\Omega)=1$.

A random variable is a measurable function $X\colon(\Omega,\mathcal{F})\to(\mathbb{R},\mathcal{B}(\mathbb{R}))$ where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$.

Definition 2.7. (see.[20]) Let $\varphi_R(t)$ be a random process where φ_N : $[a,b]\subseteq\mathbb{R}\to\mathbb{R}$ and $t_0\in[a,b]$. Then we say $\lim_{t\to t_0}\varphi_N(t)=L$ if for each $\epsilon>0$ there is $\delta>0$ such that $d_{ms}(\varphi_N(t),L)<\epsilon$, whenever $t\in[a,b]$ and $|t-t_0|<\delta$,
d_{ms} is mean-square metric and defined as follows:

$$d_{ms}(X_1,X_2)=\sqrt{\mathbb{E}[(X_1-X_2)^2]},$$

where X_1 and X_2 represent random variables.

Definition 2.8. (see.[20]) A random variable X is said to be second order random variable if and only if $d_{ms}(X,0) < \infty$.

Definition 2.9. (see.[20]) A random process $\varphi_N(t)$ defined on the probability space $(\Omega, \mathcal{F}, P)$, is called a second order random process if for each t, $\varphi_N(t)$ is second order random variable.

Definition 2.10. (see.[20])The mean square derivative of random process $\varphi_N(t)$ is denoted by $D_{ms}\varphi_N(t)$ and express as follows

$$d_{ms}(\frac{\varphi_N(t+\Delta t)-\varphi_N(t)}{\Delta t}, \varphi'_N(t)), as \quad \Delta t \to 0.$$

Lemma 2.11. *(see.[11]) Let $\varphi_N(t)$ be a second order random process, mean square continuous on $I=[t_0,T]$. Then there exists $\eta \in I$ such that*

$$\int_{t_0}^{t} \varphi(s)ds = \varphi(\eta)(t-t_0), \quad t_0 < t < T.$$

Definition 2.12. (see.[16]) Let X_1 and X_2 are two independent continuous random variables where $pdf_1(x_1)$ and $pdf_2(x_2)$ are the pdfs of X_1 and X_2, respectively. the pdf of the addition X_3, is

$$pdf_3(x_3) = \int_{-\infty}^{\infty} pdf_1(x_1)pdf_2(x_3-x_1)dx_1 = \int_{-\infty}^{\infty} pdf_1(x_3-x_2)pdf_2(x_2)dx_2,$$

the pdf of the subtraction X_3, is

$$pdf_3(x_3) = \int_{-\infty}^{\infty} pdf_1(x_1)pdf_2(x_1-x_3)dx_1 = \int_{-\infty}^{\infty} pdf_1(x_3+x_2)pdf_2(x_2)dx_2,$$

the pdf of the multiplication X_3, is

$$pdf_3(x_3) = \int_{-\infty}^{\infty} \frac{1}{|x_1|} pdf_1(x_1)pdf_2(\frac{x_3}{x_1})dx_1,$$

and the pdf of the division X_3, is

$$pdf_3(x_3) = \int_{-\infty}^{\infty} |x_1| pdf_1(x_1)pdf_2(\frac{x_1}{x_3})dx_1.$$

Definition 2.13. (see[9]) The $z-$number $z = (B, N)$, consists of two continuous fuzzy numbers, the first one B, is a fuzzy constraint on the variable X, with membership function $\mu_B(x) : R \to [0, 1]$ and the second fuzzy number N is a fuzzy constraint on probability measure of B having a membership function $\mu_N(\nu) : [0, 1] \to [0, 1]$.

Definition 2.14. (see[9]) The z^{+}-number denoted as $z^{+} = (B, N_x)$, is formed by fuzzy number B and random variable N. N_x denotes the probability distribution of a random variable N which can be interpreted as the hidden probability distribution of X and described by $pdf(x)$ such that $\int_{-\infty}^{\infty} \mu_B(x)pdf(x)dx \in supp(N)$. Furthermore by omitting the subscript x form N_x, we denoted z^{+}-number by $z^{+} = (B, N)$.

The set of $z^{+}-$numbers is denoted as $\mathbb{R}_{z^{+}}$ in this study.

Definition 2.15. (see.[9]) For arbitrary z^{+}-number $z^{+} = (B, N)$, the parametric form represent as:
$z^{+} = (\underline{z}^{+}(\alpha), \overline{z}^{+}(\alpha)) = ((\underline{B}(\alpha), N), (\overline{B}(\alpha), N))$, $\alpha \in [0, 1]$,
with following requirements:

1. for $\alpha \in [0, 1]$, $\underline{B}(\alpha) \leq \overline{B}(\alpha)$,
2. $\underline{B}(\alpha)$ is characterized as a left-continuous, non-decreasing, and bounded function on the interval $[0, 1]$,
3. $\overline{B}(\alpha)$ is characterized as a left-continuous, non-increasing, and bounded function on the interval $[0, 1]$.

Definition 2.16. (see.[9]) Let $z_1^{+} = ((\underline{B}_1(\alpha), N_1), (\overline{B}_1(\alpha), N_1))$ and $z_2^{+} = ((\underline{B}_2(\alpha), N_2), (\overline{B}_2(\alpha), N_2))$ are two z^{+}-numbers, the distance $d_{z^{+}} : \mathbb{R}_{z^{+}} \times \mathbb{R}_{z^{+}} \to \mathbb{R}_{+} \cup \{0\}$, is defined as follows

$$d_{z^{+}}(z_1^{+}, z_2^{+}) = d_{\infty}(B_1, B_2) + d_{ms}(N_1, N_2). \tag{6}$$

Let $z_1^{+}, z_2^{+}, z_3^{+}$ and z_4^{+} are z^{+}-numbers, the $d_{z^{+}}$ satisfies the following properties:

1. $d_{z^{+}}(z_1^{+} + z_3^{+}, z_2^{+} + z_3^{+}) = d_{z^{+}}(z_1^{+}, z_2^{+})$,
2. $d_{z^{+}}(\lambda z_1^{+}, \lambda z_2^{+}) = |\lambda| d_{z^{+}}(z_1^{+}, z_2^{+})$,

3. $d_{z^+}(z_1^+ + z_3^+, z_2^+ + z_4^+) \leq d_{z^+}(z_1^+, z_2^+) + d_{z^+}(z_3^+, z_4^+)$,

Lemma 2.17. *(see.[9])* $(\mathbb{R}_{z^+}, d_{z^+})$ *denotes a complete metric space.*

Definition 2.18. (see.[9]) The φ mapping $t \to \varphi(t)$ be z^+-valued function where $\varphi_{z^+} = [\varphi_B, \varphi_N] : [a,b] \subseteq \mathbb{R} \to \mathbb{R}_{z^+}$. For each $t \in [a,b]$, we have $\varphi_{z^+}(t) \in \mathbb{R}_{z^+}$.

Definition 2.19. (see.[9]) The limit of z^+-valued function $\varphi : [a,b] \subseteq \mathbb{R} \to \mathbb{R}_{z^+}$ for $t_0 \in [a,b]$ is defined as follows

1. $\lim_{t \to t_0^-} \varphi(t) = z^+$, if $\forall \epsilon > 0, \exists \delta > 0$, such that $-\delta < t - t_0 < 0 \Rightarrow d_{z^+}(\varphi(t), z^+) < \epsilon$,
2. $\lim_{t \to t_0^+} \varphi(t) = z^+$, if $\forall \epsilon > 0, \exists \delta > 0$, such that $0 < t - t_0 < \delta \Rightarrow d_{z^+}(\varphi(t), z^+) < \epsilon$,
3. $\lim_{t \to t_0} \varphi(t) = z^+$, if $\forall \epsilon > 0, \exists \delta > 0$, such that $0 < |t - t_0| < \delta \Rightarrow d_{z^+}(\varphi(t), z^+) < \epsilon$.

Definition 2.20. (see.[9]) The z^+-valued function $\varphi : [a,b] \subseteq \mathbb{R} \to \mathbb{R}_{z^+}$ is continuous on $[a,b]$, if
$\forall t_0 \in (a,b)$, $\lim_{t \to t_0} \varphi(t) = t_0$, $\lim_{t \to a^+} \varphi(t) = \varphi(a)$ and $\lim_{t \to b^-} \varphi(t) = \varphi(b)$.

Theorem 2.21. *(see.[9]) The z^+-value function, $\varphi_{z^+} : [a,b] \subseteq \mathbb{R} \to \mathbb{R}_{z^+}$, is said to be z^+-differentiable at $t \in [a,b]$, if and only if*

1. *the fuzzy function φ_B is differentiable with respect to t at that point by definition (2.4) and,*

2. *the random function φ_N is mean square differentiable at the point t.*

when these conditions are satisfied, the derivative of φ_{z^+} at t is denoted as:

$$\varphi'_{z^+}(t) = (\varphi'_{B_{sgH}}(t), \varphi'_N(t)).$$

Theorem 2.22. *Consider* $\varphi_{z^+} = [\varphi_B, \varphi_N] : [a, b] \subseteq \mathbb{R} \to \mathbb{R}_{z^+}$, *where* φ_B *is sgH-differentiable such that type of differentiability does not change over the interval* $[a, b]$, *and* φ_N *is differentiable. Then for* $a \leq s \leq b$,
i) If φ_B *is* $[(i) - sgH]-$*differentiable the* $\varphi'_{B_{i.sgH}}$ *is* FR*-integrable over* $[a, b]$, *and*

$$\varphi_{z^+}(s) = \varphi_{z^+}(a) \oplus \int_a^s \varphi'_{z^+}(t)dt, \tag{7}$$

ii) If φ_B *is* $[(ii) - sgH]-$*differentiable the* $\varphi'_{B_{ii.sgH}}$ *is* FR*-integrable over* $[a, b]$, *and*

$$\varphi_{z^+}(a) = \varphi_{z^+}(s) \oplus (-1) \int_a^s \varphi'_{z^+}(t)dt. \tag{8}$$

Proof: Let φ_B is $[(i) - sgH]-$differentiable, according to Theorem 3.1, which is proven in [8], we have

$$\varphi_{B_{i.sgH}}(s) = \varphi_{B_{i.sgH}}(a) \oplus \int_a^s \varphi'_{B_{i.sgH}}(t)dt,$$

and by using Lemma(2.11), we have

$$\varphi_N(s) = \varphi_N(a) \oplus \int_a^s \varphi'_N(t)dt,$$

therefore it can be easily concluded that

$$\varphi_{z^+}(s) = \varphi_{z^+}(a) \oplus \int_a^s \varphi'_{z^+}(t)dt,$$

Now, if φ_B is $[(ii) - sgH]-$differentiable, by using Theorem 3.1, in [8], we obtain

$$\varphi_{B_{ii.sgH}}(a) = \varphi_{B_{ii.sgH}}(s) \oplus (-1) \int_a^s \varphi'_{B_{ii.sgH}}(t)dt,$$

also by Lemma(2.11),

$$\varphi_N(a) = \varphi_N(s) + (-1) \int_a^s \varphi'_N(t)dt,$$

finally we concluded that

$$\varphi_{z^+}(a) = \varphi_{z^+}(s) \oplus (-1) \int_a^s \varphi'_{z^+}(t)dt,$$

and the proof is complete.

3 Proposed Method

The aim of this paper is to introduce a numerical method for solving z^+-initial value problem. To achieve this, first introduces the z^+-numbers which is utilized in this study. Then, the existence and uniqueness conditions of a initial value problem under z^+-numbers are presented. Finally, the numerical method for solving the z^+-initial value problem is introduced, and the necessary theorems are stated and proven.

3.1 z^+-number with extended triangular distribution

Initially, we will explain the type of z^+-number and traditional arithmetic of continuous z^+-numbers. For this purpose we need the following definitions.

Definition 3.1. (see.[16])The probability density function(pdf) of the triangular distribution forms, specified by three parameters a, b and m where a, b are the lower limit, upper limit and m is the mode. The pdf is denoted by $T(a, m, b)$ and can be formulated as a piecewise function as follows:

$$pdf(x) = \begin{cases} \frac{(x-a)}{(m-a)}(\frac{2}{b-a}), & a \leq x \leq m, \\ \frac{(b-x)}{(b-m)}(\frac{2}{b-a}), & m \leq x \leq b, \\ 0, & else. \end{cases} \tag{9}$$

In [16], Li and et al. enhanced the triangular distribution by introducing a new parameter β which modifies its convexity and concavity of piecewise function (9). This led to the development of the extended triangular distribution, definded as follows:

Definition 3.2. (see.[16]) The pdf of the extended triangular distribution is described by four parameters: the lower and upper limits a, b, the mode m and the minimum height β, which modifies the concavity and convexity of the distribution (9). The pdf of the extended triangular distribution can be expressed as a piecewise function $pdf(x) = \tau(a, m, b, \beta)$,

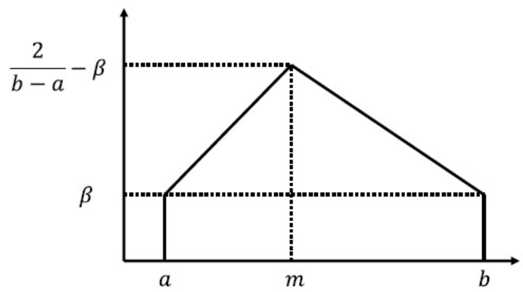

Figure 1: Extended triangular density when $\beta < \frac{1}{b-a}$.

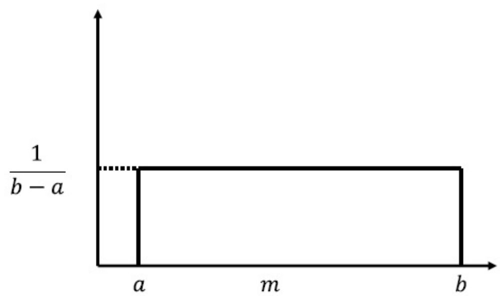

Figure 2: Extended triangular density when $\beta = \frac{1}{b-a}$.

where β lies within the interval $[0, \frac{2}{b-a}]$, and is defined as follows:

$$pdf(x) = \begin{cases} \beta + \frac{x-a}{m-a}(\frac{2}{b-a} - 2\beta), & a \leq x \leq m, \\ \beta + \frac{b-x}{b-m}(\frac{2}{b-a} - 2\beta), & m \leq x \leq b, \\ 0, & else. \end{cases} \tag{10}$$

When $0 \leq \beta < \frac{1}{b-a}$, the pdf (10) is convex (see Fig 1). Within this interval, smaller values of β correspond to a more pronounced convexity of the distribution.
If $\beta = \frac{1}{b-a}$, the pdf (10), reduces to a uniform distribution (see Fig 2). For $\frac{1}{b-a} < \beta \leq \frac{2}{b-a}$, the pdf (10) becomes concave (see Fig 3). Within this range, larger values of β lead to greater concavity in the distribution. To address this issue, Li and et al. in [16], introduced the value for the parameter β based on the concept of z^+-numbers with extended triangular distribution. They proposed the hidden pdf for $z^+ = (B, N)$

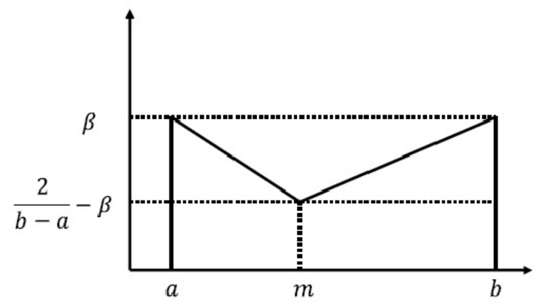

Figure 3: Extended triangular density when $\beta > \frac{1}{b-a}$,

as follows:

$$\tau(a_B, m_B, b_B, \frac{2(1-\nu)}{b_B - a_B}), \quad \nu \in [0,1],$$

where a_B, b_B are the lower limit, upper limit and m_B is the mode of fuzzy number B, and ν represents the discretization points of the support N.

The arithmetic of continuous z^+-numbers with triangular distribution, is derived from the arithmetic of fuzzy numbers and normal probability density functions (pdfs).
Consider two continuous z^+-numbers $z_1^+ = (B_1, N_1)$ and $z_2^+ = (B_2, N_2)$, which represent incomplete or imprecise information about the random variables X_1 and X_2. The objective is to calculate $z_{12}^+ = z_1^+ * z_2^+$, where $*$ denotes an operation from the set $\in \{+, -, \times, /\}$.
Step 1: $B'_{12} = B_1 * B_2$ by using the operations of continuous fuzzy numbers.
Step 2: Discretize $N_i, i = 1,2$ and derive the hidden pdfs. $supp(N_i)$ is divided into discrete points $\nu_{il}, l = 1, ..., m$, with equal spacing. As a result, N_i can be written as:
$N_i = \frac{\mu_{N_i}(\nu_{i1})}{\nu_{i1}} + \frac{\mu_{N_i}(\nu_{i2})}{\nu_{i2}} + ... + \frac{\mu_{N_i}(\nu_{im})}{\nu_{im}}$.
Step 3: Compute $pdf_{12} = pdf_1 * pdf_2$ of pdfs using definition(2.12).
Step 4: Compute the discretized base values of N_{12}.
$\nu_{12} = \int_{-\infty}^{\infty} \mu_{B_{12}}(x) pdf_{12}(x)dx$,
Step 5: Compute the membership function of N_{12}, $\mu_{N_{12}}(\nu_{12}) = \max(\mu_{N_1}(\nu_1) \wedge$

$\mu_{N_2}(\nu_2))$.

3.2 Z^{+}-Initial Value Problem

Having established the essential prerequisites, we will now proceed to introduce $z^{+}-$initial value problem as follows:

$$\begin{cases} z'_{sgH}(t) = \varphi(t, z^{+}(t)), \\ z^{+}(t_0) = z_0^{+}, \end{cases} \tag{11}$$

where $\varphi_{z^+} = [\varphi_B, \varphi_N]$ is z^{+}-valued function, $\varphi_{z^+} : [0,T] \subseteq \mathbb{R} \to \mathbb{R}_{z^+}$ consists of two parts, $\varphi_N : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ and $\varphi_B : [a,b] \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$. The initial value z_0 is z^{+}-number and based on equation (10), the pdf of the extended triangular distribution can be present as a segmentation function $pdf(x) = \tau(z_{0l}, z_{0c}, z_{0r}, \beta)$, where z_{0l}, z_{0c}, z_{0r} are the lower limit, mod and upper limit of the z_0^{+}.

3.3 Modified Euler method for Z^{+}-Initial Value Problem

Modified Euler method is powerful method for solving fuzzy differential equations (FDEs), that was proposed by N. Ahmady and et al. in 2020 [1]. This method is estimate the solution of FDEs by using a two-stage predictor–corrector algorithm with local truncation error of order two. In this work, we applied Modified Euler method for solving z^{+}-Initial Value Problem (11). For this purpose, we consider three case, based on the type of differentiability of the solution (11).

Case 1.

Let $z^{+}(t)$ be the solution of z^{+}-initial value problem (11), where $z^{+}(t)$ consist $z_B(t) : [0,T] \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ with hidden extended triangular distribution τ_z. Suppose that $z_B(t) \in \mathcal{C}^4_{sgH}([0,T],\ \mathbb{R}_{\mathcal{F}})$, and $z_B^{(i)}(t),\quad i = 1, ..., 4$ are $[(i) - sgH]$-differentiable.

$$z^{+}(t_{k+1}) = z^{+}(t_k) \oplus \int_{t_k}^{t_{k+1}} \varphi(t, z^{+}(t))dt, \tag{12}$$

Using Lagrange interpolation of points $\varphi(t_i, z^+(t_i))$ and $\varphi(t_{i+1}, z^+(t_{i+1}))$, we obtain

$$\varphi_{z^+}(t, z(t)) = \sum_{j=i}^{j=i+1} \ell_j(t) \odot \varphi(t_j, z^+(t_j)), \tag{13}$$

where $\ell_i(t)$ and $\ell_{i+1}(t)$ are positive for $t_i \leq t \leq t_{i+1}$. By putting Eq.(13) in Eq.(12), we have

$$z^+(t_{k+1}) = z^+(t_k) \oplus \frac{h}{2} \odot \Big(\varphi(t_{k+1}, z^+(t_{k+1})) \oplus \varphi(t_k, z^+(t_k))\Big) \tag{14}$$

where

$$z_B(t_{k+1}) = z_B(t_k) \oplus \frac{h}{2} \odot \Big(\varphi(t_{k+1}, z_B(t_{k+1})) \oplus \varphi(t_k, z_B(t_k))\Big) \tag{15}$$

To derive a numerical method for equation (15), the value of $z_B(t_{k+1})$ appearing on the right-hand side is unknown. To manage this, the value of $z_B(t_{k+1})$ is initially approximated using the Euler method and the predicted value is then utilized in Eq.(15).

Thus, the Modified Euler method can be expressed as follows:

$$\begin{cases} z_B^*(t_{k+1}) = z_B(t_k) \oplus h \odot \varphi(t_k, z_B(t_k)), \\ z_B(t_{k+1}) = z_B(t_k) \oplus \frac{h}{2} \odot \Big(\varphi(t_{k+1}, z_B^*(t_{k+1})) \oplus \varphi(t_k, z_B(t_k))\Big) \\ z_N(t_{k+1}) = pdf(z_B(t_{k+1})), k = 0, 1, ..., N-1., \end{cases} \tag{16}$$

Now , for $z_N(t_{k+1})$, we should write pdf of $z_B(t_{k+1})$ at t_{k+1}, for this purpose we denote $z_B(t_{k+1}) = z_{B_{k+1}} = ((z_{B_{k+1}})_l, (z_{B_{k+1}})_c, (z_{B_{k+1}})_r)$ and by Eq. (10), the pdf of $z_{B_{k+1}}$ is written as follows:

$$pdf(z_{B_{k+1}}(x)) = \begin{cases} \frac{x - z_{B_l}}{z_{B_c} - z_{B_l}}(\frac{2}{z_{B_r} - z_{B_l}} - 2\beta) + \beta, & z_{B_l} \leq x \leq z_{B_c}, \\ \frac{z_{B_r} - x}{z_{B_r} - z_{B_c}}(\frac{2}{z_{B_r} - z_{B_l}} - 2\beta) + \beta, & z_{B_c} \leq x \leq z_{B_r}, \\ 0, & else. \end{cases} \tag{17}$$

where $\beta = \frac{2(1-\nu)}{z_{B_r} - z_{B_l}}$, $\nu \in [0, 1]$, and ν represents the discretization points of the support $z_N(t_{k+1})$.

Case 2. Suppose that $z^+(t)$ be the $[(ii)-sgH]$-differentiable solution of z^+-initial value problem (11), and the type of differentiability remains unchanged, then

$$z^+(t_{i+1}) = z^+(t_i) \ominus (-1) \int_{t_i}^{t_{i+1}} \varphi(t, z^+(t))dt, \tag{18}$$

by putting Eq.(13) in Eq.(18) we have

$$z^+(t_{i+1}) = z^+(t_i) \ominus (-1) \int_{t_i}^{t_{i+1}} \sum_{j=i}^{j=i+1} \ell_j(t) \odot \varphi(t_j, z^+(t_j))dt, \tag{19}$$

By integrating yields:

$$z^+(t_{i+1}) = z^+(t_i) \ominus (-1)\frac{h}{2} \odot (\varphi(t_i, z^+(t_i)) \oplus \varphi(t_{i+1}, z^+(t_{i+1}))), \tag{20}$$

therefore the Modified Euler method for this case can be expressed as follows:

$$\begin{cases} z^*_B(t_{k+1}) = z_B(t_k) \ominus (-1)h \odot \varphi(t_k, z_B(t_k)), \\ z_B(t_{k+1}) = z_B(t_k) \ominus (-1)\frac{h}{2} \odot \Big(\varphi(t_{k+1}, z^*_B(t_{k+1})) \oplus \varphi(t_k, z_B(t_k))\Big), \\ z_N(t_{k+1}) = pdf(z_B(t_{k+1})), k = 0, 1, ..., N-1. \end{cases} \tag{21}$$

where $pdf(z_B(t_{k+1}))$ is obtained by (17).

Case 3.

We Consider partition of $[0, T]$ as follows

$$t_0 = 0, t_1, ..., t_j, \gamma, t_{j+1}, ..., t_N = T. \tag{22}$$

If the assumptions of Case 1 hold for $z_B(t)$ where $t \in [0, t_j]$ and the assumptions of Case 2 hold for $z^+(t)$ where $t \in [t_{j+1}, T]$ (with $\gamma \in [0, T]$ being a Type I switching point I) then the Modified Euler method can be expressed as follows:

$$\begin{cases} z_B^*(t_{k+1}) = z_B(t_k) \oplus h \odot \varphi(t_k, z_B(t_k)), \\ z_B(t_{k+1}) = z_B(t_k) \oplus \frac{h}{2} \odot \Big(\varphi(t_{k+1}, z_B^*(t_{k+1})) \oplus \varphi(t_k, z_B(t_k))\Big), & k = 0, 1, ..., j. \\ \\ z_B^*(t_{k+1}) = z_B(t_k) \ominus (-1)h \odot \varphi(t_k, z_B(t_k)), \\ z_B(t_{k+1}) = z_B(t_k) \ominus (-1)\frac{h}{2} \odot \Big(\varphi(t_{k+1}, z_B^*(t_{k+1})) \oplus \varphi(t_k, z_B(t_k))\Big), & k = j+1, 1, ..., N-1. \\ z_N(t_{k+1}) = pdf(z_B(t_{k+1})), k = 0, 1, ..., N-1. \end{cases} \tag{23}$$

Case 4.

If the assumptions of Case 2 hold for $z_B(t)$ where $t \in [0, t_j]$ and the assumptions of Case 1 hold for $z^+(t)$ where $t \in [t_{j+1}, T]$ (with $\gamma \in [0, T]$ being a Type I switching point II) then the Modified Euler method can be written as follows:

$$\begin{cases} z_B^*(t_{k+1}) = z_B(t_k) \ominus (-1)h \odot \varphi(t_k, z_B(t_k)), \\ z_B(t_{k+1}) = z_B(t_k) \ominus (-1)\frac{h}{2} \odot \Big(\varphi(t_{k+1}, z_B^*(t_{k+1})) \oplus \varphi(t_k, z_B(t_k))\Big), & k = 0, 1, ..., j. \\ \\ z_B^*(t_{k+1}) = z_B(t_k) \oplus h \odot \varphi(t_k, z_B(t_k)), \\ z_B(t_{k+1}) = z_B(t_k) \oplus \frac{h}{2} \odot \Big(\varphi(t_{k+1}, z_B^*(t_{k+1})) \oplus \varphi(t_k, z_B(t_k))\Big), & k = j+1, 1, ..., N-1. \\ z_N(t_{k+1}) = pdf(z_B(t_{k+1})), k = 0, 1, ..., N-1. \end{cases} \tag{24}$$

Theorem 3.3. *Let $\varphi_{z^+} : [a, b] \subseteq \mathbb{R} \to \mathbb{R}_{z^+}$ where $\varphi(t, z_B)$ satisfy in Lipschitz condition on the $\{(t, z_B(t)) | t \in [0, p], z_B \in \overline{B}(z_{B_0}, q), p, q > 0\}$ and φ_B''' exist, then the Modified Euler method converges to the solution of z^+-initial value problem (11).*

Proof. Let $z^+(t) = (z_B(t), z_N(t))$ be the $[(i) - sgH]-$solution of (11). By theorem (3.3) in [1], we have

$$\lim_{h \to 0} d_\infty(z_{B_{k+1}}, z_{B_k}) \to 0, \tag{25}$$

where h represent the step length. Also $z_{N_{k+1}}$ and z_{N_k} have triangular probability density distributions of $\tau((z_{B_{k+1}})_l, (z_{B_{k+1}})_c, (z_{B_{k+1}})_r, \beta)$ and

$\tau((z_{B_k})_l, (z_{B_k})_c, (z_{B_k})_r, \beta)$, respectively. By using (25) it can be easily concluded that
$\tau((z_{B_{k+1}})_l, (z_{B_{k+1}})_c, (z_{B_{k+1}})_r, \beta) \rightarrow \tau((z_{B_k})_l, (z_{B_k})_c, (z_{B_k})_r, \beta)$ when h goes to zero. Therefore $\lim_{h\to 0} d_{ms}(z_{N_{k+1}}, z_{N_k}) \to 0$, and consequently $\lim_{h\to 0} d_{z^+}(z^+_{k+1}, z^+_k) \to 0$, and the proof is complete. If $z^+(t) = (z_B(t), z_N(t))$ be the $[(ii) - sgH]-$solution of (11), the proof is similar.

Theorem 3.4. *The Modified Euler method for solving z^+-initial value problem (11) is stable.*

Proof. Assume that $z^+_{k+1}(t) = (z_{B_{k+1}}(t), z_{N_{k+1}}(t))$ is the $[(i) - sgH]-$solution of (11) with initial condition z^+_0 and let $\rho^+_{k+1}(t) = (\rho_{B_{k+1}}(t), \rho_{N_{k+1}}(t))$ be the $[(i) - sgH]-$solution of (11) with perturbed z^+-initial condition $\rho^+_0 = z^+_0 \oplus \epsilon \in \mathbb{R}_{z^+}$.
By using theorem (3.4) in [1], we have $d_\infty(z_{B_{k+1}}, \rho_{B_{k+1}}) \leq k d_\infty(z_{B_0}, \rho_{B_0})$, where $k = e^{-TL}$ for $kh \leq (k+1)h \leq b$. It can easily be concluded $d_{ms}(z_{N_{k+1}}, \rho_{N_{k+1}}) \to 0$, because when two triangular probability density functions are very close to each other d_{ms} approaches zero. Then finally $d_{z^+}(z^+_{k+1}, \rho^+_{k+1}) \leq k d_{z^+}(z^+_0, \rho^+_0)$, and the proof is complete. If $z^+_{k+1}(t) = (z_{B_{k+1}}(t), z_{N_{k+1}}(t))$ is the $[(ii) - sgH]-$solution of (11), proof is similar.

4 Numerical Example

Example 4.1. Let we consider the initial value problem

$$\begin{cases} (z^+)'_{gH} = (1-t) z^+(t), \quad 0 \leq t \leq 2, \\ z^+(0) = (z_B(0), z_N(0)) \end{cases} \tag{26}$$

Where $z_B(0) = (\alpha, 2-\alpha), z_N(0) = \tau(0, 1, 2, \beta)$.
To solve this problem, the Modified Euler method has been used. The solution of the problem (26) at $t = 1$ and the hidden extended triangular pdf at this point for different values of $\nu, (\nu = 0.9, 0.92, 0.94, 0.96, 0.98, 1)$ are shown in Figure 4. Additionally, the solution at $t = 2$, along with its hidden extended triangular pdf for different values of $\nu, (\nu = 0.9, 0.92, 0.94, 0.96, 0.98, 1)$ are shown in Figure 5.

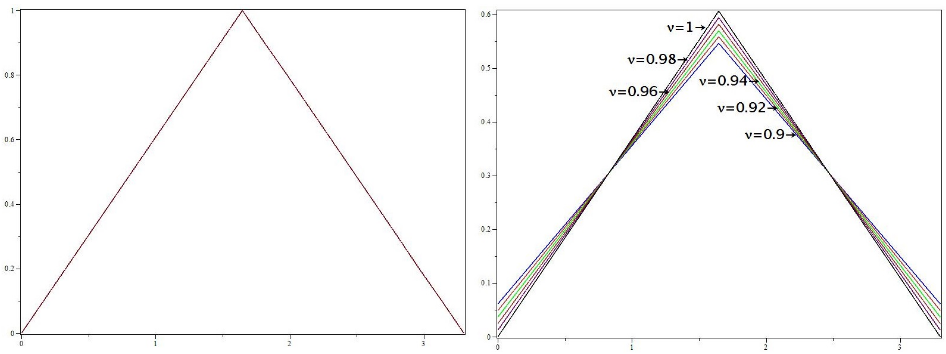

Figure 4: The hidden extended triangular pdf for $\nu = 0.9, 0.92, 0.94, 0.94, 0.96, 0.98, 1$, at $t = 1$, in Example (4.1)

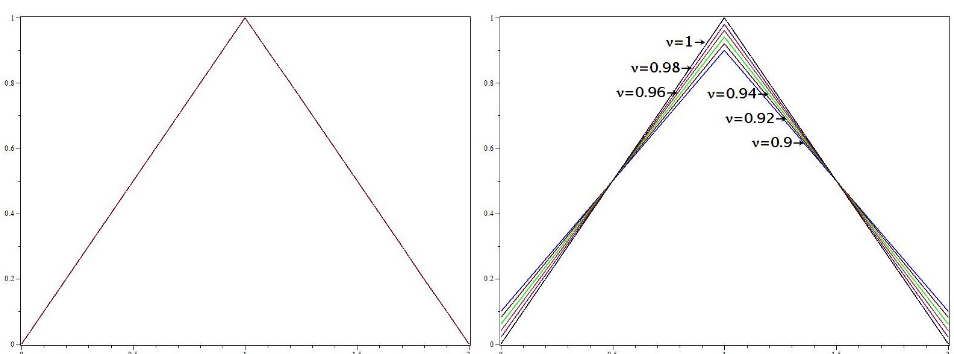

Figure 5: The hidden extended triangular pdf for $\nu = 0.9, 0.92, 0.94, 0.94, 0.96, 0.98, 1$, at $t = 2$, in Example (4.1)

Table 1: The global truncation errors for $t = 1$, in Example 4.1

α	$\underline{z}_B(t,\alpha)$	$\overline{z}_B(t,\alpha)$	$\underline{Z}_B(t,\alpha)$	$\overline{Z}_B(t,\alpha)$	Max Error
0	0	3.297439071	0	3.297442542	3.471×10^{-6}
0.2	0.3297439070	2.967695164	0.3297442442	2.967698288	3.124×10^{-6}
0.4	0.6594878140	2.637951257	0.6594885084	2.637954034	2.777×10^{-6}
0.6	0.9892317210	2.308207350	0.9892327626	2.308209779	2.429×10^{-6}
0.8	1.318975628	1.978463443	1.318977017	1.978465525	2.082×10^{-6}
1	1.648719535	1.648719536	1.648721271	1.648721271	1.735×10^{-6}

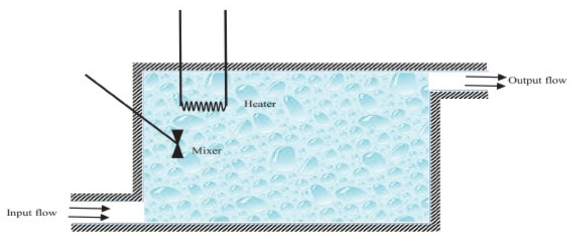

Figure 6: A tank with a heating system in Example (4.2).

Example 4.2. [15] A tank containing a heating system is represented in Figure 6, where $\tilde{\rho} = 0.5$, the thermal capacitance is $\tilde{c} = 2$, and the temperature is ψ. The model is formulated as follows:

$$\begin{cases} (\psi^+)'(t) = -\frac{1}{\tilde{\rho}\tilde{c}}\psi^+(t), \quad 0 \leq t \leq T, \\ \psi^+(0) = (\psi_B(0), \psi_N(0)), \end{cases} \tag{27}$$

Where $\psi_B(0) = (\alpha, 2-\alpha), \psi_N(0) = \tau(0,1,2,\beta)$.
Using the Modified Euler method, $\psi_B(t,r) = (\underline{\psi}_B(t,\alpha), \overline{\psi}_B(t,\alpha))$, the solution of the problem (27) at $t = 1$ and its hidden extended triangular pdf for different values of $\nu, (\nu = 0.9, 0.92, 0.94, 0.96, 0.98, 1)$ are shown in Figure 7.

5 Conclusion

Since fuzzy differential equations (FDEs) cannot assist decision-makers in determining the reliability of output information, we were motivated to address this gap. To achieve this, we investigated z^+-differential equations, which are governed by the concept of z^+-numbers. Z^+-number is

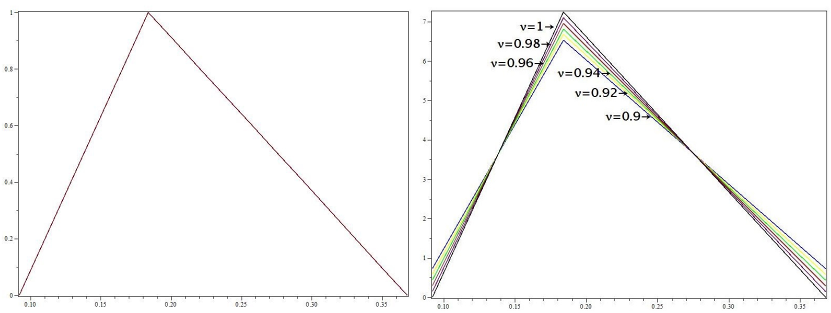

Figure 7: The hidden extended triangular pdf for $\nu = 0.9, 0.92, 0.94, 0.94, 0.96, 0.98, 1$, at $t = 1$, in Example 4.2

Table 2: The global truncation errors for $t = 1$, in Example 4.2

α	$\underline{\psi}_B(t,\alpha)$	$\overline{\psi}_B(t,\alpha)$	Lower bound of Real Value	Upper bound of Real Value	Max Error
0	0	0.7357619558	0	0.7357588824	0.30734×10^{-5}
0.2	0.07357619592	0.6621857599	0.07357588824	0.6621829942	0.27657×10^{-5}
0.4	0.1471523918	0.58860956407	0.1471517765	0.5886071059	0.24581×10^{-5}
0.6	0.2207285878	0.5150333680	0.2207276647	0.5150312177	0.21503×10^{-5}
0.8	0.2943047837	0.4414571721	0.2943035530	0.4414553294	0.18427×10^{-5}
1	0.3678809796	0.3678809762	0.3678794412	0.3678794412	0.15350×10^{-5}

a combination of possibilities and probabilities. Due to the high computational complexity involved in working with continuous distribution functions, this paper focuses on studying z^+-numbers with a triangular distribution as initial values in uncertain differential equations. We then extended the Modified Euler method to solve the z^+-initial value problem, and we also proved the convergence and stability of the method. Finally, the accuracy and efficiency of the method were demonstrated through several examples.

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