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Original Research Paper

On Tricomplex Ernst-type Sequence

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Abstract. In this article, we present two new sequences of Ernst-type that mark a significant step in the theoretical exploration of such mathematical constructions. Specifically, we develop the concept of Ernst and Ernst-Lucas tricomplex numbers, offering a detailed analysis of their properties and their links to classical Ernst numbers. We establish several fundamental identities that clarify the properties and structure of these sequences. In addition, we derive the generating function for this new class of sequences and propose a Binet-type formula. Our research also validates several classical identities related to Ernst tricomplex numbers, including those attributed to Tagiuri-Vajda, d'Ocagne, Catalan, and Cassini. This research not only improves our understanding of these distinctive sequences, but also provides a solid foundation for future research in this emerging domain.

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1 Introduction

The Ernst sequence $\{E_n\}_{n\geq 0}$ is defined by the third-order recurrence relation:

$$E_n = 2E_{n-1} + E_{n-2} - 2E_{n-3}, \tag{1}$$

for all integers $n \geq 3$, with initial terms $E_0 = 0$, $E_1 = 1$, and $E_2 = 2$. The first elements of the Ernst sequence are $0, 1, 2, 5, 10, 21, 42, 85, 170, 341, 682$ and so on. The Ernst-Lucas numbers $\{H_n\}_{n\geq 0}$, are defined by the same recurrence relation:

$$H_n = 2H_{n-1} + H_{n-2} - 2H_{n-3}, \tag{2}$$

for all integers $n \geq 3$, and have the initial values $H_0 = 3$, $H_1 = 2$, and $H_2 = 6$. The first few terms of the Ernst-Lucas sequence are 3, 2, 6, 8, 18, 32, 66, 128, 258, 512, 1026, ..., and so on. These sequences will serve as the basis for the study that we will present in this article.

According to [5], the Ernst numbers belong to the family of Leonardo-Alwyn numbers and are also referred to as the Purkiss sequence. The Ernst sequence has been the focus of extensive research, with numerous studies dedicated to its properties and applications. According [6] these numbers exhibit interesting properties, connections, or relations to the puzzle of the Chinese rings. Further details can be found in the rich domain of literature, such as [5], [6], and [13]. In [14], the authors explore generalized Ernst sequences, focusing in detail on two specific cases: the Ernst sequence and the Ernst-Lucas sequence. Additionally, in [1], the Gaussian generalized Ernst numbers are introduced as a novel complex recursive number sequence, effectively extending the traditional Ernst numbers to their Gaussian counterparts. Refs. [15, 16] investigate some third-order linear polynomials called generalized Horadam-Leonardo polynomials, which generalize various third-order sequences, for example the Ernst sequence.

In this paper, we present two novel Ernst-types sequences given below in the ring of Tricomplex $\mathbb{T} \subset \mathbb{R}^3$ (see [8] and [9]).

Definition 1.1. For all integers $n \geq 0$, in $\mathbb{T}$ we define the Tricomplex Ernst sequence $\{\mathbb{T}E_n\}_{n\geq 0}$ as $\mathbb{T}E_n = (E_n, E_{n+1}, E_{n+2})$, and the Tricomplex Ernst–Lucas sequence $\{\mathbb{T}H_n\}_{n\geq 0}$ as $\mathbb{T}H_n = (H_n, H_{n+1}, H_{n+2})$ for all $n \geq 0$.

To illustrate, we may consider the following examples: $\mathbb{T}E_0 = (0, 1, 2)$, $\mathbb{T}E_1 = (1, 2, 5)$ and $\mathbb{T}E_2 = (2, 5, 10)$. Furthermore, the value of $\mathbb{T}E_3$ is given by $(5, 10, 21)$. Note that,

$$\begin{aligned} 2\mathbb{T}E_2 + \mathbb{T}E_1 - 2\mathbb{T}E_0 &= 2 \cdot (2, 5, 10) + (1, 2, 5) - 2 \cdot (0, 1, 2) \\ &= (5, 10, 21) = \mathbb{T}E_3 \ . \end{aligned} \tag{3}$$

In [2], the authors introduced a novel family of Fibonacci-type numbers known as the Tricomplex Fibonacci sequence, extending the classical Fibonacci sequence. Their study explored various properties of this sequence, including its recurrence relation, summation formula, generating function, and several classical identities. Similarly, in [3], the authors presented the Tricomplex Repunit sequence, establishing symmetrical properties analogous to those of the ordinary repunit sequence. These works served as the primary motivation for our research.

The structure of the present work, which is divided into five further sections, is as follows. In Section 2, we introduce the sequences of Ernst, Ernst-Lucas, and Jacobsthal, highlighting key relationships between them. Additionally, we present the generating functions for the Ernst and Ernst-Lucas sequences, along with several important identities involving these sequences. Section 3 focuses on the Tricomplex rings associated with Ernst and Ernst-Lucas sequences ($\{\mathbb{T}E_n\}_{n\geq 0}$ and $\{\mathbb{T}H_n\}_{n\geq 0}$), detailing their properties, including addition, multiplication, and the derivation of a Binet-type formula. Generating functions for these sequences are also provided. In Section 4, we explore a collection of identities specific to the Tricomplex rings $\{\mathbb{T}E_n\}_{n\geq 0}$ and $\{\mathbb{T}H_n\}_{n\geq 0}$. Finally, Section 5 looks at the properties of summation involving the $\{\mathbb{T}E_n\}_{n\geq 0}$ and $\{\mathbb{T}H_n\}_{n\geq 0}$ rings, further expanding the discussion of these structures. In Section 6, we recall that research into sequences in the tricomplex ring is relatively recent, which could still increase our knowledge of this algebraic structure.

This study extends the Ernst sequence in the tricomplex ring T, and examines the similarity of the Tricomplex Ernst-type sequence with the ordinary Ernst sequence, $\{E_n\}_{n\geq 0}$. So, in the ring of $\mathbb{T} \subset \mathbb{R}^3$, we designate Tricomplex Ernst-type sequence as an element in the three-dimensional space. By emphasizing these results, we believe to inspire further exploration of this class of tricomplex numbers.

2 Background and Preliminary Results

This section is dedicated to presenting some concepts and results about Ernst and Ernst-Lucas numbers that we will use to develop the Tricomplex Ernst and Tricomplex Ernst-Lucas number, in addition, we state and prove the Tagiuri-Vajda identity for Ernst numbers. Again, let us revisit some essential concepts about the tricomplex ring $\mathbb{T}$ for the proper development of this article.

2.1 The Ernst and Jacobsthal sequences

As we have mentioned in the previous section, the ordinary Ernst sequence is defined by the third order recurrence relation (1), and the sequence has the id A000975 in [12]. Also $\{H_n\}_{n\geq 0}$ is the sequence of Ernst-Lucas numbers that are given with the recurrence relation (2), and the sequence is cataloged by A001047 in [12].

According [1], a generalization of the Ernst and Ernst-Lucas sequences is given by

$$G_n = 2G_{n-1} + G_{n-2} - 2G_{n-3}\,, \tag{4}$$

with $G_0 = a$, $G_1 = b$, and $G_2 = c$. Where a, b and c are fixed constants (typically integers). In this case, the Binet formula for the sequence $\{G_n\}_{n\geq 0}$ is presented in the next result.

Lemma 2.1. *([1], Equation 1.6)) Let n be a non-negative integer. The Binet formula for the sequence $G_n = 2G_{n-1} + G_{n-2} - 2G_{n-3}$, with $G_0 = a$, $G_1 = b$ and $G_2 = c$ is given by*

$$G_n = \frac{c-a}{3}\alpha^n + \frac{c+2a-3b}{6}\beta^n + \left(a + \frac{b-c}{2}\right)\gamma^n,$$

where a, b and c are arbitrary fixed integers, with $\alpha = 2$, $\beta = -1$ and $\gamma = 1$ are the roots of the characteristic equation associated with Equation (4).

As a consequence of Lemma 2.1, we can express the respective Binet formulas for Ernst and Ernst-Lucas sequences as

$$E_n = \frac{2}{3}\alpha^n - \frac{1}{6}\beta^n - \frac{1}{2}\,, \tag{5}$$

and

$$H_n = \alpha^n + \beta^n + 1 \ , \tag{6}$$

respectively, where $\alpha = 2$, $\beta = -1$ and 1 be the distinct roots of the characteristic equation

$$r^3 - 2r^2 - r + 2 = 0\,. \tag{7}$$

Note that

$$\begin{aligned} \alpha + \beta + \gamma &= 2, \\ \alpha\beta + \alpha\gamma + \beta\gamma &= -1, \\ \alpha\beta\gamma &= -2, \\ \alpha - \beta &= 3. \end{aligned}$$

Let us still consider the sequence $\{I_n\}_{n\geq 0}$ given by $I_n = I_{n-1} + kI_{n-2}$, where k is a natural number, $I_0 = d$ and $I_1 = e$, with d, e integer numbers. For the sequence $\{I_n\}_{n\geq 0}$, the Binet formula is given by,

$$I_n = \frac{dr_2 - e}{r_2 - r_1} r_1^n + \frac{e - dr_1}{r_2 - r_1} r_2^n,$$

where r_1 and r_2 are the roots of the characteristic equation $r^2 - r - k$=0. If $k = 2$, then we have the generalizations of Jacobsthal sequence $\{J_n\}_{n\geq 0}$ (OEIS: A001045, [12]) and the Jacobsthal-Lucas sequence $\{j_n\}_{n\geq 0}$ (OEIS: A014551, [12]), both defined by the same second-order recurrence relation:

$$J_n = J_{n-1} + 2J_{n-2}. \tag{8}$$

However, they differ in their initial terms $J_0 = d = 0$, $J_1 = e = 1$ for the Jacobsthal sequence, and $j_0 = d = 2$, $j_1 = e = 1$ for the Jacobsthal-Lucas sequence. As we shall see, there is a connection between Jacobsthal numbers and Ernst numbers. Note that, when $k = 2$ it follows that $r_1 = \alpha$ and $r_2 = \beta$. The Binet formula for the sequences J_n and j_n are, respectively, given by

$$J_n = \frac{\alpha^n - \beta^n}{3}. \tag{9}$$

$$j_n = \alpha^n + \beta^n. \tag{10}$$

Motivated by the previous definitions, we will introduce the Tricomplex Ernst numbers and study some properties of this new Ernst-type sequence of numbers in the paper. Moreover, this article will explore the connection between the Tricomplex Ernst numbers and the Ernst numbers by considering the following identities for non-negative integers n,

[14, Equation 2.5]	$2E_n$	$= J_{n+2} - 1$,	(11)
[14, Equation 2.6]	H_n	$= j_n + 1$,	(12)
[14, Corollary 4.3]	$18E_n$	$= 5H_{n+4} - 3H_{n+3} - 11H_{n+2},$	(13)
[14, Corollary 4.3]	$18E_n$	$= 7H_{n+3} - 6H_{n+2} - 10H_{n+1},$	(14)
[14, Corollary 4.3]	$18E_n$	$= 8H_{n+2} - 3H_{n+1} - 14H_n,$	(15)
[14, Corollary 4.3]	$18E_n$	$= 13H_{n+1} - 6H_n - 16H_{n-1},$	(16)
[14, Corollary 4.3]	$18E_n$	$= 20H_n - 3H_{n-1} - 26H_{n-2},$	(17)
[14, Corollary 4.3]	$8H_n$	$= E_{n+4} + 16E_{n+3} - 33E_{n+2},$	(18)
[14, Corollary 4.3]	$4H_n$	$= 9E_{n+3} - 16E_{n+2} - E_{n+1},$	(19)
[14, Corollary 4.3]	$2H_n$	$= E_{n+2} + 4E_{n+1} - 9E_n,$	(20)
[14, Corollary 4.3]	H_n	$= 3E_{n+1} - 4E_n - E_{n-1},$	(21)
[14, Corollary 4.3]	H_n	$= 2E_n + 2E_{n-1} - 6E_{n-2}.$	(22)

where, respectively, J_n and j_n are the n-th Jacobsthal and Jacobsthal-Lucas numbers.

Complementing the Equations (11) and (12), and according [14] there are close relations between Ernst, Ernst-Lucas, and Jacobsthal, Jacobsthal-Lucas numbers.

Lemma 2.2. *[14, p. 137] For all non-negative integers n, the Ernst, Ernst-Lucas, and Jacobsthal, Jacobsthal-Lucas numbers satisfy the following interrelations:*

$$18E_n = 5j_{n+1} + 2j_n - 9\,, \tag{23}$$

$$2H_n = 4J_{n+1} - 2J_n + 2\,. \tag{24}$$

2.2 Generating functions

In the study of recurrence sequences, generating functions provide an efficient way to analyze their asymptotic behavior and also help in understanding the growth of related sequences.

The ordinary generating function of Ernst and Ernst-Lucas numbers are displayed on the next result.

Lemma 2.3. *[14, Corollary 2.2] For all $n \geq 0$ the ordinary generating function for the Ernst sequence $\{E_n\}_{n\geq 0}$ is*

$$G_{E_n}(x) = \frac{x}{1-2x-x^2+2x^3}\ , \tag{25}$$

and for the Ernst-Lucas sequence $\{H_n\}_{n\geq 0}$ is

$$G_{H_n}(x) = \frac{3-4x-x^2}{1-2x-x^2+2x^3}\ . \tag{26}$$

The exponential generating function $E_{a_n}(x)$ for a sequence $\{a_n\}_{n\geq 0}$ is expressed as a power series:

$$E_{a_n}(x) = a_0 + a_1 x + \frac{a_2 x^2}{2!} + \cdots + \frac{a_n x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}.$$

In the next result, we take $a_n = E_n$ and, utilizing Equation (5), which provides the Binet formula for the Ernst sequence, we derive the classical exponential generating function for the Ernst sequence $\{E_n\}_{n\geq 0}$.

Proposition 2.4. *For all $n \geq 0$, the exponential generating function for the Ernst sequence $\{E_n\}_{n\geq 0}$ is*

$$E_{E_n}(x) = \sum_{n=0}^{\infty} \frac{E_n x^n}{n!} = \frac{2}{3}e^{\alpha x} - \frac{1}{6}e^{\beta x} - \frac{1}{2}e^{x}\ , \tag{27}$$

where $\alpha = 2$, $\beta = -1$ and 1 are the roots of the Equation (7) .

Proof. Note that,

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{E_n x^n}{n!} &= \frac{2}{3}\sum_{n=0}^{\infty} \frac{\alpha^n}{n!}x^n - \frac{1}{6}\sum_{n=0}^{\infty} \frac{\beta^n}{n!}x^n - \frac{1}{2}\sum_{n=0}^{\infty} \frac{1^n}{n!}x^n \\
&= \frac{2}{3}\sum_{n=0}^{\infty} \frac{(\alpha x)^n}{n!} - \frac{1}{6}\sum_{n=0}^{\infty} \frac{(\beta x)^n}{n!} - \frac{1}{2}\sum_{n=0}^{\infty} \frac{(1x)^n}{n!} \\
&= \frac{2}{3}e^{\alpha x} - \frac{1}{6}e^{\beta x} - \frac{1}{2}e^{1x}
\end{aligned}$$

as required. □

In a similar way to the Proposition 2.4, we have the next result

Proposition 2.5. *For all $n \geq 0$ the exponential generating function for the Ernst-Lucas sequence $\{H_n\}_{n\geq 0}$ is*

$$E_{H_n}(x) = e^{\alpha x} + e^{\beta x} + e^{x} \ , \tag{28}$$

where $\alpha = 2$, $\beta = -1$ and 1 are the roots of the Equation (7) .

2.3 Identities

To aid in understanding the upcoming result, it is essential to first establish two preliminary auxiliary findings.

Lemma 2.6. *Let n, s and k be non-negative integers and a and b be real numbers. Then the following identities are valid:*

$$\begin{array}{lll} (a) & a^{n+s+k} - a^{n+s} - a^{n+k} + a^{n} & = a^{n}(a^{s}-1)(a^{k}-1)\,, \\ (b) & a^{k+s} - a^{k}b^{s} + b^{k+s} - a^{s}b^{k} & = (a^{k}-b^{k})(a^{s}-b^{s})\,. \end{array}$$

The proof of Lemma 2.6 is just algebraic manipulation, which is why we have omitted the proof.

As a consequence of the Binet formula we find next the Tagiuri-Vajda identity for the Ernst sequence $\{E_n\}_{n\geq 0}$.

Theorem 2.7 (Tagiuri-Vajda)**.** *Let m, s, k be any non-negative integers, and $\{E_n\}_{n\geq 0}$ is the Ernst sequence. We have the following identity:*

$$E_{m+s}E_{m+k} - E_m E_{m+s+k} = \frac{1}{3}\alpha^{m}A(s,k) - \frac{1}{12}\beta^{m}B(s,k) + (-2)^{m}J_sJ_k\,. \tag{29}$$

where $A(s,k) = (\alpha^{s}-1)(\alpha^{k}-1)$, $B(s,k) = (\beta^{s}-1)(\beta^{k}-1)$, and $\{J_n\}_{n\geq 0}$ is the Jacobsthal sequence.

Proof. Using Equation (5), we can calculate the products: $E_{m+s}E_{m+k}$ and E_mE_{m+s+k}:

$$\begin{aligned} & E_{m+s}E_{m+k} \\ = \ & \left[\frac{2}{3}\alpha^{m+s} - \frac{1}{6}\beta^{m+s} - \frac{1}{2}\right]\left[\frac{2}{3}\alpha^{m+k} - \frac{1}{6}\beta^{m+k} - \frac{1}{2}\right] \\ = \ & \frac{4}{9}\alpha^{2m+s+k} - \frac{1}{9}\alpha^{m+s}\beta^{m+k} - \frac{1}{3}\alpha^{m+s} - \frac{1}{9}\alpha^{m+k}\beta^{m+s} + \frac{1}{36}\beta^{2m+s+k} \\ & + \frac{1}{12}\beta^{m+s} - \frac{1}{3}\alpha^{m+k} + \frac{1}{12}\beta^{m+k} + \frac{1}{4}\,, \end{aligned}$$

and

$$\begin{aligned}
& E_m E_{m+s+k} \\
= & \left[\frac{2}{3}\alpha^m - \frac{1}{6}\beta^m - \frac{1}{2}\right]\left[\frac{2}{3}\alpha^{m+s+k} - \frac{1}{6}\beta^{m+s+k} - \frac{1}{2}\right] \\
= & \frac{4}{9}\alpha^{2m+s+k} - \frac{1}{9}\alpha^{m+k+s}\beta^m - \frac{1}{3}\alpha^{m+s+k} - \frac{1}{9}\alpha^m\beta^{m+s+k} + \frac{1}{36}\beta^{2m+s+k} \\
& + \frac{1}{12}\beta^{m+s+k} - \frac{1}{3}\alpha^m + \frac{1}{12}\beta^m + \frac{1}{4},
\end{aligned}$$

The subtraction $E_{m+s}E_{m+k} - E_m E_{m+s+k}$ yields:

$$\begin{aligned}
& E_{m+s}E_{m+k} - E_m E_{m+s+k} \\
= & \frac{1}{3}(\alpha^{m+s+k} - \alpha^{m+s} - \alpha^{m+k} + \alpha^m) \\
& - \frac{1}{12}(\beta^{m+s+k} - \beta^{m+s} - \beta^{m+k} + \beta^m) \\
& + \frac{1}{9}(\alpha\beta)^m(\alpha^{s+k} - \alpha^s\beta^k - \alpha^k\beta^s + \beta^{s+k}).
\end{aligned}$$

Using Lemma 2.6, after simplification, the final result is:

$$\begin{aligned}
& E_{m+s}E_{m+k} - E_m E_{m+s+k} \\
= & \frac{1}{3}\alpha^m(\alpha^s - 1)(\alpha^k - 1) - \frac{1}{12}\beta^m(\beta^s - 1)(\beta^k - 1) \\
& + \frac{1}{9}(\alpha\beta)^m(\alpha^s - \beta^s)(\alpha^k - \beta^k), \\
= & \frac{1}{3}\alpha^m A(s,k) - \frac{1}{12}\beta^m B(s,k) + (-2)^m J_s J_k,
\end{aligned}$$

where we make $A(s,k) = (\alpha^s - 1)(\alpha^k - 1)$, $B(s,k) = (\beta^s - 1)(\beta^k - 1)$, and $J_n = \frac{\alpha^n - \beta^n}{3}$ is the n-th Jacobsthal number. This completes the proof. □

The next auxiliary result will be used in the Tricomplex version of the Tagiuri-Vajda identity.

Lemma 2.8. *Let s and k be non-negative integers and $A(s,k) = (\alpha^s - 1)(\alpha^k - 1)$ and $B(s,k) = (\beta^s - 1)(\beta^k - 1)$. Then the following identities* *are valid:*

$$
\begin{array}{lll}
(a) & A(s,k)+\alpha A(s,k+1)+\alpha^2 A(s,k-1) & = 7A(s,k)\,,\\
(b) & B(s,k)+\beta B(s,k+1)+\beta^2 B(s,k-1) & = B(s,k)\,,\\
(c) & (-2)^m J_s\big(J_k+(-2)J_{k+1}+(-2)^2 J_{k-1}\big) & = (-2)^{m+1} j_s j_k\,,\\
(d) & (-2)^m J_s\big(J_{k+1}+(-2)J_{k-1}+(-2)^2 J_k\big) & = (-2)^m 5 J_s J_k\,,\\
(e) & (-2)^m J_s\big(J_{k+2}+(-2)J_k+(-2)^2 J_{k-2}\big) & = (-2)^m 3 J_s J_k\,,
\end{array}
$$

where $\alpha = 2$, $\beta = -1$ and 1 are the roots of the Equation (7), and J_n and j_n are, respectively, the n-th Jacobsthal and n-th Lucas-Jacobsthal number.

Proof. (a) Note that

$$
\begin{aligned}
& A(s,k)+\alpha A(s,k+1)+\alpha^2 A(s,k-1)\\
= \;& (\alpha^s-1)(\alpha^k-1)+\alpha(\alpha^s-1)(\alpha^{k+1}-1)+\alpha^2(\alpha^s-1)(\alpha^{k-1}-1)\\
= \;& (\alpha^s-1)\big((\alpha^k-1)+\alpha(\alpha^{k+1}-1)+\alpha^2(\alpha^{k-1}-1)\big)\\
= \;& (\alpha^s-1)\big(\alpha^k-1+\alpha^{k+2}-\alpha+\alpha^{k+1}-\alpha^2\big)\\
= \;& (\alpha^s-1)\big(\alpha^k(\alpha^2+\alpha+1)-(\alpha^2+\alpha+1)\big)\\
= \;& (\alpha^s-1)(\alpha^k-1)(\alpha^2+\alpha+1)\,.
\end{aligned}
$$

Since $\alpha^2+\alpha+1 = 7$ for $\alpha = 2$, this completes the proof.

(b) The proof follows a similar approach to the one used in item (a). So

$$
\begin{aligned}
& B(s,k)+\beta B(s,k+1)+\beta^2 B(s,k-1)\\
= \;& (\beta^s-1)(\beta^k-1)+\beta(\beta^s-1)(\beta^{k+1}-1)+\beta^2(\beta^s-1)(\beta^{k-1}-1)\\
= \;& (\beta^s-1)(\beta^k-1)(\beta^2+\beta+1)\,.
\end{aligned}
$$

Since $\beta^2+\beta+1 = 1$ for $\beta = -1$, this establishes the result.

(c) According to Equation (8) we have

$$
\begin{aligned}
& (-2)^m J_s\big(J_k+(-2)J_{k+1}+(-2)^2 J_{k-1}\big)\\
= \;& (-2)^m J_s(J_k-2J_{k+1}+4J_{k-1})\\
= \;& (-2)^m J_s\big(J_k-2(J_k+2J_{k-1})+4J_{k-1}\big)\\
= \;& -(-2)^m J_s J_k\,,
\end{aligned}
$$

as required.

(d) Using Equation (8), we have

$$\begin{aligned} & (-2)^m J_s\big(J_{k+1} + (-2)J_{k-1} + (-2)^2 J_k\big) \\ = & (-2)^m J_s\big(J_k + 2J_{k-1} + (-2)J_{k-1} + (-2)^2 J_k\big) \\ = & (-2)^m 5 J_s J_k\,. \end{aligned}$$

.

(e) By the use of Equation (8)

$$\begin{aligned} & (-2)^m J_s\big(J_{k+2} + (-2)J_k + (-2)^2 J_{k-2}\big) \\ = & (-2)^m J_s\big(J_{k+1} + 2J_k + (-2)J_k + (-2)^2 J_{k-2}\big) \\ = & (-2)^m J_s\big(J_{k+1} + (-2)^2 J_{k-2}\big) \\ = & (-2)^m J_s\big(J_k + 2J_{k-1} + (-2)^2 J_{k-2}\big) \\ = & (-2)^m J_s\big(J_{k-1} + 2J_{k-2} + 2j_{k-1} + (-2)^2 J_{k-2}\big) \\ = & (-2)^m J_s 3\big(J_{k-1} + 2J_{k-2}\big) \\ = & (-2)^m 3 J_s J_k\ , \end{aligned}$$

as required. □

To finalize this section, the two following results present a partial sum formulas for Ernst and Ernst-Lucas numbers.

Lemma 2.9. *[14, Corollary 7.2.1] For $n \geq 0$, the Ernst numbers satisfy the following properties:*

$$\begin{aligned} &(a) \quad \sum_{k=0}^{n} E_k &&= \frac{1}{4}\big(5J_{n+1} + 6J_n - 2n - 5\big)\,, \\ &(b) \quad \sum_{k=0}^{n} E_{2k} &&= \frac{1}{6}\big((2-n)J_{2n+1} + 2(n+6)J_{2n} - 3n - 2\big)\,, \\ &(c) \quad \sum_{k=0}^{n} E_{2k+1} &&= \frac{1}{12}\big((2n+29)J_{2n+1} - 2(2n-3)J_{2n} - 6n - 17\big)\,, \end{aligned}$$

where J_n is the n-th Jacobsthal numbers.

Lemma 2.10. *[14, Corollary 7.2.2] For $n \geq 0$, the Ernst–Lucas numbers satisfy the following properties:*

$$\text{(a)} \quad \sum_{k=0}^{n} H_k = \frac{1}{2}\big(j_{n+1} + 2j_n + 2n + 1\big)\,,$$

$$\text{(b)} \quad \sum_{k=0}^{n} H_{2k} = \frac{1}{3}\big(2(n+3)j_{2n} - (n+1)j_{2n+1} + 3n - 2\big)\,,$$

$$\text{(c)} \quad \sum_{k=0}^{n} H_{2k+1} = \frac{1}{6}\big((2n+11)j_{2n+1} - 2(2n+3)j_{2n} + 6n + 13\big)\,,$$

where j_n is the n-th Jacobsthal–Lucas numbers.

3 The Tricomplex Ring

In mathematics, a *tricomplex number* is an element of a number system that extends the complex numbers. While complex numbers have a real part and an imaginary part, tricomplex numbers have one real part and two imaginary parts. The concept of tricomplex numbers was introduced by Olariu [8], and a general tricomplex number is expressed in the form:

$$(a, b, c) = a + b\mathbf{i} + c\mathbf{j}\,,$$

where a, b, and c are real numbers, and $\mathbf{i}$ and $\mathbf{j}$ are distinct imaginary units.

Let $\mathbb{T} = (\mathbb{T}, +, \times)$ denote the *ring of tricomplex numbers*, which consists of ordered triples of real numbers (x, y, z). The operations of addition and multiplication are defined as follows:

- Addition: for two tricomplex numbers (x_1, y_1, z_1) and (x_2, y_2, z_2), their sum is given by:

  $$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2), \tag{30}$$

  this operation simply adds the corresponding components of the two triples.

- Scalar multiplication: the operation of scalar multiplication for all $x \in \mathbb{R}$ is given by

$$x(x_1, y_1, z_1) = (xx_1, xy_1, xz_1), \tag{31}$$

this operation simply multiplies the scalar by each component of the triple.

- Multiplication: the product of two tricomplex numbers (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by the following formula:

$$\begin{aligned} & (x_1, y_1, z_1) \times (x_2, y_2, z_2) \\ = \; & (x_1x_2 + y_1z_2 + z_1y_2, z_1z_2 + x_1y_2 + y_1x_2, y_1y_2 + x_1z_2 + z_1x_2), \end{aligned} \tag{32}$$

the multiplication rule is defined in such a way that it extends the behavior of complex numbers to accommodate the two imaginary units, **i** and **j**.

Note that the multiplication rules for the imaginary units given by $\mathbf{ij} = \mathbf{ji} = 1$, $\mathbf{i}^2 = \mathbf{j}$, and $\mathbf{j}^2 = \mathbf{i}$ (see, for instance, [4, 7, 8, 9, 10]). Thus, the ring $\mathbb{T}$ of tricomplex numbers provides a rich structure that generalizes the complex numbers by incorporating additional imaginary components, which opens up interesting possibilities for algebraic and geometric applications. Tricomplex numbers can be a powerful tool in various mathematical contexts, especially in the study of three-dimensional systems. They also find applications in physics and engineering, where three-dimensional systems are intrinsic to many phenomena. Additionally, in [11], the author explore a four-dimensional complex algebraic structure, which has applications in the construction of directional probability distributions within four-dimensional spaces.

3.1 Binet's formula and generating function for tricomplex Ernst-type sequence

According to Definition 1.1, the Tricomplex Ernst sequence $\{\mathbb{T}E_n\}_{n\geq 0}$ is defined as:

$$\mathbb{T}E_n = (E_n, E_{n+1}, E_{n+2}),$$

while the Tricomplex Ernst-Lucas sequence $\{\mathcal{T}H_n\}_{n\geq 0}$ is defined as:

$$\mathcal{T}H_n = (H_n, H_{n+1}, H_{n+2}),$$

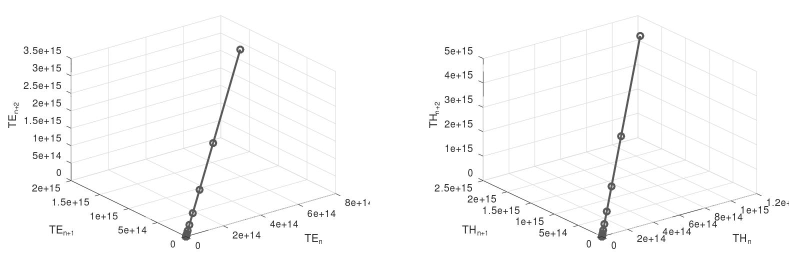
for all non-negative integers $n \geq 0$.

The table below presents some terms of the Tricomplex Ernst and Tricomplex Ernst-Lucas numbers for specific cases.

Table 1: terms for n smaller than 9

n	$\mathcal{T}E_n$	$\mathcal{T}H_n$
0	(0,1,2)	(3,2,6)
1	(1,2,5)	(2,6,8)
2	(2,5,10)	(6,8,18)
3	(5,10,21)	(8,18,32)
4	(10,21,42)	(18,32,66)
5	(21,42,85)	(32, 66, 128)
6	(42,85,170)	(66, 128, 258)
7	(85,170,341)	(128, 258, 512)
8	(170,341,682)	(258, 512, 1026)
9	(341,682,1365)	(512, 1026, 2048)

In addition, the Figures 1(a) and 1(b) illustrate the terms of the Tricomplex sequences $\{\mathcal{T}E_n\}_{n\geq 0}$ and $\{\mathcal{T}H_n\}_{n\geq 0}$ in tridimensional space, for n from 0 to 50.



(a) Ernst. (b) Ernst-Lucas.

Figure 1: Tricomplex Sequences $0 \leq n \leq 50$ - Ernst 1(a), Ernst-Lucas 1(b).

In Equation (3) we saw that $2\mathbb{T}E_2 + \mathbb{T}E_1 - 2\mathbb{T}E_0 = \mathbb{T}E_3$, let us show that the Ernst recurrence given by Equation (1), is again valid to Tricomplex Ernst for all $n \geq 3$, or rather, the Tricomplex Ernst sequence is also an Ernst-type sequence.

Proposition 3.1. *The Tricomplex Ernst sequence satisfies the recurrence relation*

$$\mathbb{T}E_n = 2\mathbb{T}E_{n-1} + \mathbb{T}E_{n-2} - 2\mathbb{T}E_{n-3},$$

for all integer $n \geq 3$, and with initial terms $\mathbb{T}E_0 = (0, 1, 2)$, $\mathbb{T}E_1 = (1, 2, 5)$ and $\mathbb{T}E_2 = (2, 5, 10)$.

Proof. Using the Definition 1.1, the vector addition given in Equation (30), scalar multiplication given in Equation (31) and Equation (1), it follows that

$$\begin{aligned}
& 2\mathbb{T}E_{n-1} + \mathbb{T}E_{n-2} - 2\mathbb{T}E_{n-3} \\
= \; & 2(E_{n-1}, E_n, E_{n+1}) + (E_{n-2}, E_{n-1}, E_n) - 2(E_{n-3}, E_{n-2}, E_{n-1}) \\
= \; & (2E_{n-1} - E_{n-2} - 2E_{n-31},\ 2E_n - E_{n-1} - 2E_{n-2},\ 2E_{n+1} - E_n - 2E_{n-1}) \\
= \; & (E_n, E_{n+1}, E_{n+2}) \ ,
\end{aligned}$$

and we obtain the result. □

Next, by applying the Binet formula for the n-th Ernst number given by Equation (5), to each coordinate, we have derived the Binet formula for the n-th Tricomplex Ernst number.

Proposition 3.2. *For all non-negative integers n, the Binet formula for the Tricomplex Ernst sequence $\{\mathbb{T}E_n\}_{n\geq 0}$ is*

$$\mathbb{T}E_n = \frac{2}{3}(\alpha^n, \alpha^{n+1}, \alpha^{n+2}) - \frac{1}{6}(\beta^n, \beta^{n+1}, \beta^{n+2}) - \frac{1}{2}(1, 1, 1) \ ,$$

where $\alpha = 2$, $\beta = -1$ and 1 are the distinct roots of the Equation (7).

Proof. Combining Definition 1.1 and Equation (5) we obtain that

$$\begin{aligned}
\mathbb{T}E_n &= (E_n, E_{n+1}, E_{n+2}) \\
&= (\frac{2}{3}\alpha^n - \frac{1}{6}\beta^n - \frac{1}{2},\ \frac{2}{3}\alpha^{n+1} - \frac{1}{6}\beta^{n+1} - \frac{1}{2},\ \frac{2}{3}\alpha^{n+2} - \frac{1}{6}\beta^{n+2} - \frac{1}{2}) \\
&= \frac{2}{3}(\alpha^n, \alpha^{n+1}, \alpha^{n+2}) - \frac{1}{6}(\beta^n, \beta^{n+1}, \beta^{n+2}) - \frac{1}{2}(1, 1, 1),
\end{aligned}$$

as required. □

In a similar way to Proposition 3.2, we have

Proposition 3.3. *For all non-negative integers n, the Binet formula for the Tricomplex Ernst-Lucas sequence $\{\mathbb{T}H_n\}_{n\geq 0}$ is*

$$\mathbb{T}H_n = (\alpha^n, \alpha^{n+1}, \alpha^{n+2}) + (\beta^n, \beta^{n+1}, \beta^{n+2}) + I \ ,$$

where $\alpha = 2$, $\beta = -1$ and 1 are the distinct roots of the Equation (7), and $I = (1, 1, 1)$ is a tricomplex number.

Proof. Combining Definition 1.1 and Equation (6). □

3.2 Generating functions of tricomplex Ernst-type sequence

Let $\mathcal{E}_{\mathbb{T}E}(x) = \left(E_{E_n}(x), E_{E_{n+1}}(x), E_{E_{n+2}}(x)\right)$ represent the exponential generating function for the Tricomplex Ernst sequence $\{\mathbb{T}E_n\}_{n\geq 0}$. By combining Definition 1.1 and Equation (27), we arrive at the following result.

Proposition 3.4. *For all $n \geq 0$, the exponential generating function for the Tricomplex Ernst sequence $\{\mathbb{T}E_n\}_{n\geq 0}$ is*

$$\mathcal{E}_{\mathbb{T}E}(x) = \frac{2}{3}e^{\alpha x}(1, 2, 4) - \frac{1}{6}e^{\beta x}(1, -1, 1) - \frac{1}{2}e^{x}(1, 1, 1) \ ,$$

where $\alpha = 2$, $\beta = -1$ and 1 are the distinct roots of the Equation (7).

Proof. According Equation (27) we have

$$\begin{aligned} & \left(E_{E_n}(x), E_{E_{n+1}}(x), E_{E_{n+2}}(x)\right) \\ = & \left(\frac{2}{3}e^{\alpha x} - \frac{1}{6}e^{\beta x} - \frac{1}{2}e^{t}, \frac{2}{3}\alpha e^{\alpha x} - \frac{1}{6}\beta e^{\beta x} - \frac{1}{2}e^{t}, \frac{2}{3}\alpha^2 e^{\alpha x} - \frac{1}{6}\beta^2 e^{\beta x} - \frac{1}{2}e^{t}\right) \\ = & \frac{2}{3}e^{\alpha x}(1, 2, 4) - \frac{1}{6}e^{\beta x}(1, -1, 1) - \frac{1}{2}e^{t}(1, 1, 1) \ . \end{aligned}$$

Since $\alpha = 2$ and $\beta = -1$, we get the result required. □

As well as with Proposition 3.4, we have

Proposition 3.5. *For all $n \geq 0$ the exponential generating function for the Tricomplex Ernst-Lucas sequence $\{\mathbb{T}H_n\}_{n\geq 0}$ is*

$$\mathcal{E}_{\mathbb{T}H}(x) = e^{\alpha x}(1,2,4) + e^{\beta x}(1,-1,1) + e^{x}(1,1,1)\ ,$$

where $\alpha = 2$, $\beta = -1$ and 1 are the distinct roots of the Equation (7).

As $\mathbb{T}E_n = (E_n, E_{n+1}, E_{n+2})$, the ordinary generating function for the Tricomplex sequence $\{\mathbb{T}E_n\}_{n\geq 0}$ is a triple of generating functions, that is, we write:

$$\begin{aligned} G_{\mathbb{T}E}(x) &= \sum_{n=0}^{\infty} \mathbb{T}E_n x^n = \sum_{n=0}^{\infty}(E_n, E_{n+1}, E_{n+2})x^n \\ &= \left(\sum_{n=0}^{\infty} E_n x^n, \sum_{n=0}^{\infty} E_{n+1}x^n, \sum_{n=0}^{\infty} E_{n+2}x^n\right). \end{aligned}$$

Therefore,

$$G_{\mathbb{T}E}(x) = \Bigg(G_{E_n}(x), xG_{E_{n+1}}(x), x^2 G_{E_{n+2}}(x)\Bigg),$$

where $G_{E_n}(x)$ is given by Equation (25).

This preliminary discussion helps to ensure the following result:

Proposition 3.6. *The ordinary generating function for the Tricomplex Ernst sequence $\{\mathbb{T}E_n\}_{n\geq 0}$, denoted by $G_{\mathbb{T}E}(x)$, is given by*

$$\left(\frac{x}{1-2x-x^2+2x^3}\ , \frac{1}{1-2x-x^2+2x^3}\ , \frac{2+x-2x^2}{1-2x-x^2+2x^3}\right)\ . \quad (33)$$

Proof. The first coordinate of the vector defined in Equation (33) is a direct application of Equation (25).

We have $G_{E_{n+1}}(x) = \sum_{n=0}^{\infty} E_{n+1}x^n$. Then, by expanding equation $xG_{E_{n+1}}(x)$, we obtain

$$\begin{aligned} xG_{E_{n+1}}(x) &= E_1 x + E_2 x^2 + E_3 x^3 + \ldots \\ &= G_{E_n}(x) - E_0 \\ &= \frac{x}{1-2x-x^2+2x^3} - E_0\ . \end{aligned}$$

Since $E_0 = 0$, this proves the second coordinate of the vector defined in Equation (33). The proof of the third coordinate of the vector defined in Equation (33) is performed in a similar way. □

Similarly to Proposition 3.6, we have

Proposition 3.7. *The ordinary generating function for the Tricomplex Ernst-Lucas sequence* $\{\mathbb{T}H_n\}_{n\geq 0}$, *denoted by* $G_{\mathbb{T}H}(x)$, *is given by*

$$\left(\frac{3-4x-x^2}{1-2x-x^2+2x^3}, \frac{2+2x-6x^2}{1-2x-x^2+2x^3}, \frac{6x-4x-4x^2}{1-2x-x^2+2x^3}\right) .$$

4 Identities for Tricomplex Ernst Sequence

In this section, we establish some identities for the Tricomplex Ernst sequence. We will established some classical identities for the Tricomplex Ernst sequence in this section, for example the Tagiuri-Vajda identities and their Catalan, Cassini and d'Ocganes derivations.

First, using Theorem 2.7 the next result establishes Tagiuri-Vajda's identity for the Tricomplex Ernst sequence $\{\mathbb{T}E_n\}_{n\geq 0}$.

Theorem 4.1. *Let* m, s, k *be any non-negative integers and* $\{\mathbb{T}E_n\}_{n\in\mathbb{N}_0}$ *the Tricomplex Ernst sequence. We have*

$$\mathbb{T}E_{m+s} \times \mathbb{T}E_{m+k} - \mathbb{T}E_m \times \mathbb{T}E_{m+s+k} = (X, Y, Z) \ . \tag{34}$$

where

$$X = \frac{7}{3}\alpha^m A(s,k) - \frac{1}{12}\beta^m B(s,k) - (-2)^m J_s J_k \ ,$$

$$Y = \frac{7}{3}\alpha^m A(s,k) - \frac{1}{12}\beta^m B(s,k) + (-2)^m 5 J_s J_k \ ,$$

$$Z = \frac{7}{3}\alpha^m A(s,k) - \frac{1}{12}\beta^m B(s,k) + (-2)^m 3 J_s J_k \ ,$$

$A(s,k) = (\alpha^s - 1)(\alpha^k - 1)$, $B(s,k) = (\beta^s - 1)(\beta^k - 1)$, $J_n = \frac{\alpha^n - \beta^n}{3}$ *is the* n*-th Jacobsthal number, and* $\alpha = 2$, $\beta = -1$ *and* 1 *are the distinct roots of the Equation* (7).

Proof. By Equation (31), we have

$$\begin{aligned} & TE_{m+s} \times TE_{m+k} \\ = & (E_{m+s}, E_{m+s+1}, E_{m+s+2}) \times (E_{m+k}, E_{m+k+1}, E_{m+k+2}) \\ = & (\ E_{m+s}E_{m+k} + E_{m+s+1}E_{m+k+2} + E_{m+s+2}E_{m+k+1}, \\ & E_{m+s+2}E_{m+k+2} + E_{m+s}E_{m+k+1} + E_{m+s+1}E_{m+k}, \\ & E_{m+s+1}E_{m+k+1} + E_{m+s}E_{m+k+2} + E_{m+s+2}E_{m+k}\ ), \end{aligned} \tag{35}$$

and again by Equation (31),

$$\begin{aligned} & TE_{m} \times TE_{m+s+k} \\ = & (E_{m}, E_{m+1}, E_{m+2}) \times (E_{m+s+k}, E_{m+s+k+1}, E_{m+s+k+2}) \\ = & (\ E_{m}E_{m+s+k} + E_{m+1}E_{m+s+k+2} + E_{m+2}E_{m+s+k+1}, \\ & E_{m+2}E_{m+s+k+2} + E_{m}E_{m+s+k+1} + E_{m+1}E_{m+s+k}, \\ & E_{m+1}E_{m+s+k+1} + E_{m}E_{m+s+k+2} + E_{m+2}E_{m+s+k}\ ). \end{aligned} \tag{36}$$

To obtain the first coordinate of the vector defined in Equation (34), we subtract Equation (35) from Equations (36), that is,

$$\begin{aligned} & (E_{m+s}E_{m+k} - E_{m}E_{m+s+k}) + (E_{m+s+1}E_{m+k+2} - E_{m+1}E_{m+s+k+2}) \\ & +(E_{m+s+2}E_{m+k+1} - E_{m+2}E_{m+s+k+1}) \\ = & \big(E_{m+s}E_{m+k} - E_{m}E_{m+s+k}\big) \\ & +\big(E_{(m+1)+s}E_{(m+1)+(k+1)} - E_{(m+1)}E_{(m+1)+s+(k+1)}\big) \\ & +\big(E_{(m+2)+s}E_{(m+2)+(k-1)} - E_{(m+2)}E_{(m+2)+s+(k-1)}\big). \end{aligned}$$

By Theorem 2.7, we have

$$\begin{aligned} & \big(E_{m+s}E_{m+k} - E_{m}E_{m+s+k}\big) \\ & +\big(E_{(m+1)+s}E_{(m+1)+(k+1)} - E_{(m+1)}E_{(m+1)+s+(k+1)}\big) \\ & +\big(E_{(m+2)+s}E_{(m+2)+(k-1)} - E_{(m+2)}E_{(m+2)+s+(k-1)}\big) \\ = & \frac{1}{3}\alpha^{m}\Big(A(s,k) + \alpha A(s,k+1) + \alpha^{2}A(s,k-1)\Big) \\ & -\frac{1}{12}\beta^{m}\Big(B(s,k) + \beta B(s,k+1) + \beta^{2}B(s,k-1)\Big) \\ & +(-2)^{m}J_{s}\Big(J_{k} + (-2)J_{k+1} + (-2)^{2}J_{k-1}\Big). \end{aligned}$$

According to Lemma 2.8 we obtain that

$$\begin{aligned}
&\big(E_{m+s}E_{m+k} - E_m E_{m+s+k}\big)\\
&+\big(E_{(m+1)+s}E_{(m+1)+(k+1)} - E_{(m+1)}E_{(m+1)+s+(k+1)}\big)\\
&+\big(E_{(m+2)+s}E_{(m+2)+(k-1)} - E_{(m+2)}E_{(m+2)+s+(k-1)}\big)\\
=\ & \frac{7}{3}\alpha^m A(s,k) - \frac{1}{12}\beta^m B(s,k) - (-2)^m J_s J_k\ .
\end{aligned}$$

This establishes the proof for the first coordinate of the vector defined in Equation (34).

The proofs for the second and third coordinates follow a similar approach, using Theorem 2.7, the Tagiuri-Vajda identity for the Ernst sequence. □

As consequences of Tagiuri-Vajda's identity (Theorem 4.1), the next results of this section establish d'Ocagne's identity, Catalan's identity, and Cassini's identity for the Tricomplex Ernst sequence $\{\mathbb{T}E_n\}_{n\geq 0}$.

Proposition 4.2. *(d'Ocagne's identity) Let m, n be any non-negative integers and $n \geq m$. For the Tricomplex Ernst sequence $\{\mathbb{T}E_n\}_{n\in\mathbb{N}_0}$ the following identity holds*

$$\mathbb{T}E_{m+1} \times \mathbb{T}E_n - \mathbb{T}E_m \times \mathbb{T}E_{n+1} = (X_1, Y_1, Z_1),$$

where $X_1 = \frac{7}{3}(\alpha^n - \alpha^m) + \frac{1}{6}(\beta^n - \beta^m) - (-2)^m J_{n-m}$, $Y_1 = \frac{7}{3}(\alpha^n - \alpha^m) + \frac{1}{6}(\beta^n - \beta^m) + (-2)^m 5J_{n-m}$, $Z_1 = \frac{7}{3}(\alpha^n - \alpha^m) + \frac{1}{6}(\beta^n - \beta^m) + (-2)^m 3J_{n-m}$, $J_n = \frac{\alpha^n - \beta^n}{3}$ is the n-th Jacobsthal number, and $\alpha = 2$, $\beta = -1$ and 1 are the distinct roots of the Equation (7).

Proof. Taking $k = n - m$ and $s = 1$ in Equation (34) gives

$$\begin{aligned}
&\mathbb{T}E_{m+1} \times \mathbb{T}E_n - \mathbb{T}E_m \times \mathbb{T}E_{n+1}\\
=\ &\Big(\frac{7}{3}\alpha^m A(1, n-m) - \frac{1}{12}\beta^m B(1, n-m) - (-2)^m J_1 J_{n-m},\\
&\frac{7}{3}\alpha^m A(1, n-m) - \frac{1}{12}\beta^m B(1, n-m) + (-2)^m 5J_1 J_{n-m},\\
&\frac{7}{3}\alpha^m A(1, n-m) - \frac{1}{12}\beta^m B(1, n-m) + (-2)^m 3J_1 J_{n-m}\Big).
\end{aligned}$$

Since $A(s,k) = (\alpha^s - 1)(\alpha^k - 1)$, $B(s,k) = (\beta^s - 1)(\beta^k - 1)$, $J_1 = 1$, $\alpha = 2$ and $\beta = -1$, we obtain the result. □

In a similar way to Proposition 4.2 we have the Catalan identity.

Proposition 4.3. *(Catalan's identity) Let n, s be non-negative integers with $n \geq s$. For the Tricomplex Ernst sequence $\{\mathit{TE}_n\}_{n\in\mathbb{N}_0}$, the following identity is satisfied:*

$$(\mathit{TE}_n)^2 - \mathit{TE}_{n-s} \times \mathit{TE}_{n+s} = (X_2, Y_2, Z_2) , \tag{37}$$

where $X_2 = \frac{7}{3}\alpha^{n-s}A(s,s) - \frac{1}{12}\beta^{n-s}B(s,s) - (-2)^{n-s}{J_s}^2$, $Y_2 = \frac{7}{3}\alpha^{n-s}A(s,s) - \frac{1}{12}\beta^{n-s}B(s,s) + (-2)^{n-s}5{J_s}^2$, $A(s,s) = (\alpha^s - 1)^2$, $B(s,s) = (\beta^s - 1)^2$, $Z_2 = \frac{7}{3}\alpha^{n-s}A(s,s) - \frac{1}{12}\beta^{n-s}B(s,s) + (-2)^{n-s}3{J_s}^2$, $J_s = \frac{\alpha^s - \beta^s}{3}$ *is the n-th Jacobsthal number, and $\alpha = 2$, $\beta = -1$ and 1 are the distinct roots of the Equation (7).*

Proof. Using $k = s$ and $m + s = n$ in the equation (34), we have the result.

□

Now, we obtain the Cassini identity.

Proposition 4.4. *(Cassini's identity) For all non-negative integers n we have*

$$(\mathit{TE}_n)^2 - \mathit{TE}_{n-1} \times \mathit{TE}_{n+1} = \Big(\frac{7}{3}\alpha^{n-1} - \frac{1}{3}\beta^{n-1} - (-2)^{n-1},$$
$$\frac{7}{3}\alpha^{n-1} - \frac{1}{3}\beta^{n-1} + (-2)^{n-1}5,\ \frac{7}{3}\alpha^{n-1} - \frac{1}{3}\beta^{n-1} + (-2)^{n-1}3\Big) ,$$

where $\{\mathit{TE}_n\}_{n\in\mathbb{N}_0}$ is the Tricomplex Ernst sequence, $\alpha = 2$, $\beta = -1$ and 1 are the distinct roots of the Equation (7).

Proof. It suffices to consider $s = 1$ in Equation (37). □

4.1 Other interesting identities

Similarly, there are significant relationships between the Tricomplex Ernst and Tricomplex Ernst-Lucas numbers. We present some of these relationships in what follows.

Proposition 4.5. *For all non-negative integers n, the following identities hold:*

$$18\mathbb{T}E_n = 5\mathbb{T}H_{n+4} - 3\mathbb{T}H_{n+3} - 11\mathbb{T}H_{n+2}\,; \tag{38}$$

$$18\mathbb{T}E_n = 7\mathbb{T}H_{n+3} - 6\mathbb{T}H_{n+2} - 10\mathbb{T}H_{n+1}\,; \tag{39}$$

$$18\mathbb{T}E_n = 8\mathbb{T}H_{n+2} - 3\mathbb{T}H_{n+1} - 14\mathbb{T}H_{n}\,; \tag{40}$$

$$18\mathbb{T}E_n = 13\mathbb{T}H_{n+1} - 6\mathbb{T}H_{n} - 16\mathbb{T}H_{n-1}\,; \tag{41}$$

$$18\mathbb{T}E_n = 20\mathbb{T}H_{n} - 3\mathbb{T}H_{n-1} - 26\mathbb{T}H_{n-2}\,; \tag{42}$$

$$8\mathbb{T}H_n = \mathbb{T}E_{n+4} + 16\mathbb{T}E_{n+3} - 33\mathbb{T}E_{n+2}\,; \tag{43}$$

$$4\mathbb{T}H_n = 9\mathbb{T}E_{n+3} - 16\mathbb{T}E_{n+2} - \mathbb{T}E_{n+1}\,; \tag{44}$$

$$2\mathbb{T}H_n = \mathbb{T}E_{n+2} + 4\mathbb{T}E_{n+1} - 9\mathbb{T}E_{n}\,; \tag{45}$$

$$\mathbb{T}H_n = 3\mathbb{T}E_{n+1} - 4\mathbb{T}E_{n} - \mathbb{T}E_{n-1}\,; \tag{46}$$

$$\mathbb{T}H_n = 2\mathbb{T}E_{n} + 2\mathbb{T}E_{n-1} - 6\mathbb{T}E_{n-2}\,; \tag{47}$$

where $\mathbb{T}E_n$ is the n-th Tricomplex Ernst numbers and $\mathbb{T}H_n$ is the n-th Tricomplex Ernst–Lucas numbers.

Proof. By combining Definition 1.1 and Identity (13) we obtain

$$\begin{aligned} & 5\mathbb{T}H_{n+4} - 3\mathbb{T}H_{n+3} - 11\mathbb{T}H_{n+2} \\ = \; & 5(H_{n+4}, H_{n+5}, H_{n+6}) - 3(H_{n+3}, H_{n+4}, H_{n+5}) - 11(H_{n+2}, H_{n+3}, H_{n+4}) \\ = \; & (18E_n, 18E_{n+1}, 18E_{n+2}) \ , \end{aligned}$$

which verifies the Identity 38.

Similarly, by using Definition 1.1 and Identity (14), we obtain Equation (39), as well as, by using Definition 1.1 and Identity (15) we obtain (40), as well as, by using Definition 1.1 and Identity (16), we obtain Equation (41), as well as, by using Definition 1.1 and Identity (17) we obtain Equation (42), as well as, by using Definition 1.1 and Identity (18) we obtain Equation (43), as well as, by using Definition 1.1 and Identity (19), we obtain Equation (44), as well as, by using Definition 1.1 and Identity (20) we obtain Equation (45), as well as, by using Definition 1.1 and Identity (21) we obtain Equation (46), as well as, by using Definition 1.1 and Identity (22), we obtain Equation (47). □

5 Partial Sum of Terms Involving the Tricomplex Ernst-type Numbers

In this section, we present results on partial sums of the first $n+1$ terms of the Tricomplex Ernst and Tricomplex Ernst–Lucas numbers.

First, we consider the sequence of partial sums $\sum_{k=0}^{n} \mathbb{T}E_k = \mathbb{T}E_0 + \mathbb{T}E_1 + \mathbb{T}E_2 + \cdots + \mathbb{T}E_n$, for $n \geq 0$, where $\{\mathbb{T}E_n\}_{n\geq 0}$ is the Tricomplex Ernst sequence.

Proposition 5.1. *Let $\{\mathbb{T}E_n\}_{n\geq 0}$ be the Tricomplex Ernst sequence, we have the following formulas:*

(a)
$$\sum_{k=0}^{n} \mathbb{T}E_k = \Bigg(\frac{1}{4}\big(5J_{n+1} + 6J_n - 2n - 5\big), \frac{1}{4}\big(5J_{n+2} + 6J_{n+1} - 2(n+1) - 5\big), \frac{1}{4}\big(5J_{n+3} + 6J_{n+2} - 2(n+2) - 5 - 1\big)\Bigg),$$

(b)
$$\sum_{k=0}^{n} \mathbb{T}E_{2k} = \Bigg(\frac{1}{6}\big((2-n)J_{2n+1} + 2(n+6)J_{2n} - 3n - 2\big), \tfrac{1}{12}\big((2n+29)J_{2n+1} - 2(2n-3)J_{2n} - 6n - 17\big), \tfrac{1}{6}\big((1-n)J_{2n+3} + 2(n+7)J_{2n+2} - 3(n+1) - 2\big)\Bigg),$$

(c)
$$\sum_{k=0}^{n} \mathbb{T}E_{2k+1} = \Bigg(\frac{1}{12}\big((2n+29)J_{2n+1} - 2(2n-3)J_{2n} - 6n - 17\big), \big((2-n)J_{2n+1} + 2(n+6)J_{2n} - 3n - 2\big), \tfrac{1}{12}\big((2n+31)J_{2n+3} - 2(2n-1)J_{2n+2} - 6n - 24\big)\Bigg),$$

where, J_n is the n-th Jacobsthal numbers.

Proof. (a) Follows from the definition of sum of terms of the Tricomplex

Ernst numbers that

$$\begin{aligned}\sum_{k=0}^{n} \mathbb{T}E_k &= \mathbb{T}E_0 + \mathbb{T}E_1 + \cdots + \mathbb{T}E_n \\ &= (E_0, E_1, E_2) + (E_1, E_2, E_3) + \cdots + (E_n, E_{n+1}, E_{n+2}) \\ &= (E_0 + E_1 + \cdots + E_n,\ E_1 + E_2 + \cdots + E_{n+1},\ E_2 + E_3 + \cdots + E_{n+2}) \\ &= \left(\sum_{k=0}^{n} E_k,\ \sum_{k=1}^{n+1} E_k,\ \sum_{k=2}^{n+2} E_k\right) \\ &= \left(\sum_{k=0}^{n} E_k,\ \left(\sum_{k=0}^{n+1} E_k\right) - E_0,\ \left(\sum_{k=0}^{n+2} E_k\right) - (E_0 + E_1)\right) .\end{aligned}$$

According Lemma 2.9, item (a), we have that

$$\begin{aligned}\sum_{k=0}^{n} \mathbb{T}E_k &= \Big(\frac{1}{4}\big(5J_{n+1} + 6J_n - 2n - 5\big),\ \frac{1}{4}\big(5J_{n+2} + 6J_{n+1} - 2(n+1) - 5\big), \\ &\quad \frac{1}{4}\big(5J_{n+3} + 6J_{n+2} - 2(n+2) - 5 - 1\big)\Big) .\end{aligned}$$

Since $E_0 = 0$ and $E_1 = 1$, we get the result.

(b) Note that

$$\begin{aligned}\sum_{k=0}^{n} \mathbb{T}E_{2k} &= \mathbb{T}E_0 + \mathbb{T}E_2 + \cdots + \mathbb{T}E_{2n} \\ &= \left(\sum_{k=0}^{n} E_{2k},\ \sum_{k=0}^{n} E_{2k+1},\ \sum_{k=0}^{n} E_{2(k+1)}\right) = \left(\sum_{k=0}^{n} E_k,\ \sum_{k=0}^{n} E_{2k+1},\ \sum_{k=0}^{n+1} E_{2k}\right) .\end{aligned}$$

We get the result making use of the Lemma 2.9, item (b) and (c).

(c) Similarly, we have

$$\begin{aligned}\sum_{k=0}^{n} \mathbb{T}E_{2k+1} &= \mathbb{T}E_1 + \mathbb{T}E_3 + \cdots + \mathbb{T}E_{2n+1} \\ &= \left(\sum_{k=0}^{n} E_{2k+1},\ \sum_{k=0}^{n+1} E_{2k},\ \left(\sum_{k=0}^{n+1} E_{2k+1}\right) - 1\right) .\end{aligned}$$

We get the result making use of the Lemma 2.9, item (b) and (c). □

Now, we consider the sequence of partial sums $\sum_{k=0}^{n} T\!H_k = T\!H_0 + T\!H_1 + T\!H_2 + \cdots + T\!H_n$, for $n \geq 0$, where $\{T\!H_n\}_{n\geq 0}$ is the Tricomplex Ernst-Lucas sequence.

Proposition 5.2. *Let $\{T\!H_n\}_{n\geq 0}$ be the Tricomplex Ernst–Lucas sequence, we have the following formulas:*

(a) $$\sum_{k=0}^{n} T\!H_k = \left(\frac{1}{2}(j_{n+1} + 2j_n + 2n + 1), \frac{1}{2}(j_{n+1} + 2j_n + 2n - 5), \frac{1}{2}(j_{n+1} + 2j_n + 2n - 9)\right),$$

(b) $$\sum_{k=0}^{n} T\!H_{2k} = \left(\frac{1}{3}(2(n+3)j_{2n} - (n+1)j_{2n+1} + 3n - 2), \frac{1}{6}((2n+11)j_{2n+1} - 2(2n+3)j_{2n} + 6n + 13), (\frac{1}{3}(2(n+3)j_{2n} - (n+1)j_{2n+1} + 3n - 2)) - 2 + j_{2n+2}\right),$$

(c) $$\sum_{k=0}^{n} T\!H_{2k+1} = \left(\frac{1}{6}((2n+11)j_{2n+1} - 2(2n+3)j_{2n} + 6n + 13), (\frac{1}{3}(2(n+3)j_{2n} - (n+1)j_{2n+1} + 3n - 2)) - 2 + j_{2n+2}, \frac{1}{6}((2n+11)j_{2n+1} - 2(2n+3)j_{2n} + 6n + 13)) - 1 + j_{2n+3}\right).$$

where, j_n is the n-th Jacobsthal-Lucas.

Using the same technique employed in Proposition 5.1, now making use of Lemma 2.10, we omit the proof of this result in the interest of brevity.

6 Conclusions

In this paper, we examined the properties of two newly sequences whose coefficients are elements of the Ernst or Ernst-Lucas sequences within the tricomplex ring, both of which qualify as Ernst-type sequences. In addition to this, we have derived the generating function for this novel class of sequences and formulated a Binet-type formula. Other objectives were to derive various identities for these sequences, particularly classical ones such as the Tagiuri-Vajda, D'Ocagne, Catalan, and Cassini identities. In addition, aspects of our investigation appear to be innovative and may provide original insights that may deepen the understanding of this domain, especially considering that research on the tricomplex ring is still relatively new. By clarifying these identities and providing a structured framework, we hope to inspire further scientific exploration and progress in related areas of mathematics. Soykan [15] points out that linear recurrence relations have many applications. In particular, list some applications of second, third and fourth order sequences are given. In future work, we will investigate some applications of this type of sequence in three-dimensional space in connection with ordinary sequences.

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