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A Study of Decomposer and Related Functions on Groups

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Abstract. Decomposer functions in algebraic structures are studied in many recent papers. They have close relations to factorization by two subsets. Also, idempotent endomorphisms form a class of (strong) decomposer functions in groups. Now, if the algebraic structure is a group, then by introducing a type of local homomorphisms we obtain several properties and equivalent conditions for many classes of decomposer functions and get a new result regarding to factorization of a group by its two subsets. Moreover, we prove existence of (two-sided) decomposer type functions in non-simple groups.

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1. Introduction

There are many classes of functions from algebraic structures to themselves which have close relations to some aspects of algebraic properties. One of them is the class of decomposer type functions, introduced and studied in [2]. They have many relations to factorization by two subsets, associative , multiplicative symmetric and canceller functions (see [1,4,6]). Indeed, decomposer type functions satisfy some functional equations on algebraic structures (see [5,7]). In the case that the algebraic structure is a group, the properties are so much more and there are many important sub-classes. Fore instance, cyclic decomposer (parter) functions that are periodic, and specially b-parts functions on the additive real numbers group (see [3]). We found that they have also some connections to a

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type of local homomorphisms which we call them $\Delta \times \Omega$-homomorphisms. Let us recall decomposer type functions on groups (from [2,4]) first, and then introduce the new conceptions. By applying them, we obtain many new results regarding to decomposer type functions on groups and prove some related theorems.
Let $(G, .)$ be a group with the identity element e. If f and g are functions from G to G, then define the functions $f.g$ and f^- by

$$f.g(x) = f(x)g(x) \quad , \quad f^-(x) = f(x)^{-1} \; : \qquad \forall x \in G.$$

We denote the identity function on G by ι_G and put $f^* = \iota_G.f^-$, $f_* = f^-.\iota_G$ and call f^* [resp. f_*] *left $*$-conjugate of f* [resp. *right $*$-conjugate of f*]. They are also called *$*$-conjugates of f*. Note that $f^*(e) = f_*(e) = f^-(e) = f(e)^{-1}$.
If $(G, +)$ is additive group, then the notations e, f^-, $f.g$, $f.g^-$ are replaced by 0, $-f$, $f + g$, $f - g$ and we have $f^* = f_* = \iota_G - f$. It is easy to see that

$$f \text{ is idempotent} \Leftrightarrow f^*f = e \Leftrightarrow f_*f = e$$

$$f^{*^2} = f^* \Leftrightarrow ff^* = e \Leftrightarrow f^{**}f^* = e \;\; , \;\; f_*^2 = f_* \Leftrightarrow ff_* = e \Leftrightarrow f_{**}f_* = e.$$

Also, if f is endomorphism then $f^-f = ff^-$, $f^*f = ff^*$ and $f_*f = ff_*$ (i.e. the compositions of f and its $*$-conjugations are commutative).

Now, we recall from [2,4] decomposer type functions as follows.
A function f from (a group) G to G is called:
(a) *right [resp. left] strong decomposer* if

$$f(f^*(x)y) = f(y) \quad [\text{resp. } f(xf_*(y)) = f(x)] \quad : \forall x, y \in G.$$

(b) *right [resp. left] semi-strong decomposer* if

$$f(f^*(x)y) = f(f^*(e)y) \quad [\text{resp. } f(xf_*(y)) = f(xf_*(e))] \quad : \forall x, y \in G.$$

(c) *right [resp. left] decomposer* if

$$f(f^*(x)f(y)) = f(y) \quad [\text{resp. } f(f(x)f_*(y)) = f(x)] \quad : \forall x, y \in G.$$

(d) *right [resp. left] weak decomposer* if

$$f(f^*(e)f(x)) = f(x) \; , \; f(f^*(x)f(e)) = f(e) \quad : \forall x \in G.$$

$$[\text{resp. } f(f(x)f_*(e)) = f(x) \; , \; f(f(e)f_*(x)) = f(e) \quad : \forall x \in G].$$

(e) *right [resp. left] separator* if $f^*(G) \cap f(G) = \{f(e)\}$ [resp. $f(G) \cap f_*(G) = \{f(e)\}$].

In each of the above parts if $f(e)=e$, then we will add the word *standard* to the titles. For example "f is a standard right separator" means $f^*(G)\cap f(G)=\{e\}$.

We call f *a decomposer* or *a two-sided decomposer* [resp. a separator] if it is a left and right decomposer [resp. separator].

Example 1.1. Consider $G=\{1,a,a^2,a^3,b,ba,ba^2,ba^3\}\cong D_4$ ($a^4=b^2=1, bab=a^{-1}=a^3$). Put $\Omega=\{1,ba,ba^2,ba^3\}$ and

$$f(x)=\begin{cases} x & x\in\Omega \\ bx & x\notin\Omega. \end{cases}$$

Considering the relation $x\notin\Omega\Leftrightarrow bx\in\Omega$, it can be seen that f is a (standard) right strong decomposer.
Now, consider the additive group $\mathbb{R}$ and fix $b\in\mathbb{R}\setminus\{0\}$. For a real number a denote by $[a]$ the largest integer not exceeding a and put $(a)=a-[a]$ (the decimal part of a). Now, set

$$[a]_b=b[\frac{a}{b}] \ \ , \ \ (a)_b=b(\frac{a}{b}).$$

We call $[a]_b$ *the b-integer part of a* and $(a)_b$ *the b-decimal part of a*. Also $[\ ]_b$, $(\ )_b$ are called *b-decimal part function and b-integer part function*, respectively. The b-parts functions are decomposers. Moreover, the b-decimal part function $(\ )_b$ is a strong decomposer (see [3,4]). Also, for every constant real number c, the function $f:=(\ )_b+c$ is a semi-strong decomposer.

It is shown in [2,4] that
(a) f is a right strong decomposer $\Rightarrow$ f is a right semi-strong decomposer $\Rightarrow$ f is a right decomposer $\Rightarrow$ f is a right weak decomposer.
Note that the converses of the above implications are not necessarily true. For example, the real function $f:=(\ )_1+\frac{1}{2}$ is a (right) semi-strong decomposer but it is not a (right) strong decomposer.
(b) f is a standard right strong decomposer $\Leftrightarrow$ f is a standard right semi-strong decomposer $\Rightarrow$ f is a standard right decomposer $\Rightarrow$ f is a standard right weak decomposer $\Rightarrow$ f is a standard right separator.
(c) If f is a right strong decomposer, then f is a right separator, an idempotent, $ff^*=f(e)$ and

$$f^*(e).ff^*=f^*f=e \qquad , \qquad \langle f(e)\rangle\leqslant f^*(G)\leqslant G.$$

Similar properties hold for the left case. Also, a left or right decomposer function is a (two-sided) strong decomposer if and only if $f^*(G)=f_*(G)\trianglelefteq G$.

2. Relations to Local Homomorphisms

Below introduces a type of local (partial) homomorphisms whic is very useful for the topic.

Definition 2.1. *Assume Δ, Ω are nonempty subsets of G. We call a map f from G to G a "$\Delta \times \Omega$-homomorphism" if $f(\delta\omega) = f(\delta)f(\omega)$, for all $\delta \in \Delta, \omega \in \Omega$.*

If f is a $\Delta \times \Omega$-homomorphism and $e \in \Delta \cup \Omega$, then $f(e) = e$. A function f is an endomorphism if and only if it is a $G \times G$-homomorphism.
Now, let g,h be arbitrary functions from G to G. The function f is a $g(G) \times h(G)$-homomorphism if and only if

$$f(g(x)h(y)) = fg(x)fh(y) \ ; \qquad \forall x, y \in G.$$

Hence f is a $f^*(G) \times f(G)$-homomorphism if and only if

$$f(f^*(x)f(y)) = ff^*(x)f^2(y) \ ; \qquad \forall x, y \in G,$$

and it is a $f^*(G) \times G$-homomorphism if and only if

$$f(f^*(x)y) = ff^*(x)f(y) \ ; \qquad \forall x, y \in G.$$

Also, if f is an idempotent $f(G) \times G$-homomorphism then

$$f(f(x)y) = f(x)f(y) \ ; \qquad \forall x, y \in G.$$

It is interesting to know that if $f(e) = e$, then the converse is also valid (notice that without the condition $f(e) = e$, the converse is not true, e.g. if $k \in \mathbb{Z}$, then $f(x) = [x]+k$ satisfies in the additive real numbers group, but $f^2 \neq f$ whenever $k \neq 0$). It is easy to see that if f satisfies, then the following conditions are equivalent:
(i) $f(e) = e$, (ii) f is an idempotent , (iii) f is a $f(G) \times G$-homomorphism.
For if f satisfies and $f(e) = e$ then $f^2(x) = f(f(x)e) = f(x)f(e) = f(x)$, for all $x \in G$. So, $f(f(x)y) = f(x)f(y) = f^2(x)f(y)$, for all $x \in G$, which means (iii) holds. Also, if f is a $f(G) \times G$-homomorphism then putting $x = y = e$ we obtain $f^2(e) = f^2(e)f(e)$ and so $f(e) = e$.
Finally, f satisfies if and only if f_* is a right strong decomposer (see [2; p. 549]). Similar properties hold for:

$$f(xf(y)) = f(x)f(y) \ ; \qquad \forall x, y \in G.$$

If f is is an idempotent $f(G) \times f(G)$-homomorphism then

$$f(f(x)f(y)) = f(x)f(y) \ ; \qquad \forall x, y \in G.$$

If $f^2 = f$, then the converse is also valid (e.g. if $f(e) = e$). For the above functional equation the conditions $f(e) = e$ and $f^2 = f$ are not equivalent, for if $c > 0$ is a fixed real number and $f(x) = \max\{x, c\}$, then f satisfies in the additive real numbers group and $f^2 = f$ but $f(0) = c \neq 0$. It is clear that if f satisfies, then $f(G)$ is a sub-semigroup of G but (in general) $f(G)$ is not a subgroup of G (e.g. $f(x) = \max\{x, c\}$).

Example 2.2. The functions $(\ )_b$ and $[\ ]_b$ are $b\mathbb{Z} \times \mathbb{R}$-homomorphisms. But they are not endomorphisms.

Notation. By $H \dot{\leqslant} G$ we mean H is a sub-semigroup of G. Also, we put $f_e := f^-(e) \cdot f$.
It can be shown that if f is a type of right decomposer (weak, ordinary, semi-strong or strong), then f_e is a standard form of the same type of decomposer.

Lemma 2.3. *Let $f : G \to G$.*
(a) If f is a right decomposer and $f^*(G) \dot{\leqslant} G$, then it is a right strong decomposer.
(b) If f is a right weak decomposer and f_e is a $f_e^*(G) \times f_e(G)$-homomorphism, then f is a right decomposer.
(c) If f is a standard right separator and $ff^* = f^*f$, then it is a standard right weak decomposer.

Proof. (a) Since $f^*(x)f^*(y) \in f^*(G)$, then

$$f(f^*(x)y) = f(f^*(x)f^*(y)f(y)) = f(y) \ : \quad \forall x, y \in G.$$

(b) f_e is a standard right weak decomposer and so $f_e f_e^* = e$, $f_e^2 = f_e$. Hence, we have

$$f_e(f_e^*(x)f_e(y)) = f_e f_e^*(x) f_e^2(y) = f_e(y) \ : \quad \forall x, y \in G.$$

Therefore f_e is a standard right decomposer, so f is a right decomposer.
(c) Fix $x \in G$. If $c = f(f^*(x)) = f^*(f(x))$, then $c \in f^*(G) \cap f(G)$ so $c = e$. Therefore $ff^* = f^*f = e$ so f is a standard right weak decomposer. $\square$

Recall from [2] that if Δ and Ω are subsets of G, then the notation $A = \Delta \cdot \Omega$ means $A = \Delta\Omega$ and if $\delta_1\omega_1 = \delta_2\omega_2$ where $\delta_1, \delta_2 \in \Delta$, $\omega_1, \omega_2 \in \Omega$, then $\delta_1 = \delta_2$ and $\omega_1 = \omega_2$. If $A = \Delta \cdot \Omega$, then we say A is *direct product of (subsets)* Δ *and* Ω. By the notation $A = \Delta : \Omega$ we mean $A = \Delta \cdot \Omega$ and $\Delta \cap \Omega = \{e\}$ and say A is *standard direct product of* Δ *and* Ω. If $A = \Delta.\Omega$, then Δ [resp. Ω] is called *left [resp. right] factor* of A. Note that additive notations are $\Delta \dot{+} \Omega$ (direct sum of subsets) and $\Delta \ddot{+} \Omega$ (standard direct sum of subsets).
Clearly if $\Delta\Omega = \Delta \cdot \Omega$, then $|\Delta\Omega| = |\Delta||\Omega| = |\Omega\Delta|$. Also if Δ and Ω are

nonempty subsets of G, then $\Delta\Omega = \Delta\cdot\Omega$ if and only if $(\Delta^{-1}\Delta)\cap(\Omega\Omega^{-1}) = \{e\}$ (in additive notation $(\Delta-\Delta)\cap(\Omega-\Omega) = \{0\}$). Moreover, if Δ and Ω are finite, then $\Delta\Omega = \Delta\cdot\Omega$ if and only if $|\Delta\Omega| = |\Delta||\Omega|$. If $\Delta\Omega = \Delta\cdot\Omega$ and $\Delta\cap\Omega$ has an element that commutes with every element of $\Delta\cap\Omega$, then $|\Delta\cap\Omega| = 1$. Especially if $\Delta\Omega = \Delta\cdot\Omega$ and $e\in\Delta\cap\Omega$, then $\Delta\Omega = \Delta:\Omega$. If $G = \Delta.\Omega$, then $G = \Delta_e : \Omega_e$ where $\Delta_e = \Delta\delta_0^{-1}$, $\Omega_e = \omega_0^{-1}\Omega$ and $e = \delta_0\omega_0$. One may see more information about factorization of a group by its subsets in [8].

Example 2.4. Consider the additive real numbers group $\mathbb{R}$ and put $\mathbb{R}_b = b[0,1) = \{bd|0\leqslant d<1\}$, $\langle b\rangle = b\mathbb{Z}$ where $b\neq 0$ is a constant real number. We have $\mathbb{R} = \langle b\rangle\ddot{+}\mathbb{R}_b$ (see [2]). But the natural numbers set is not a factor (subset) of $\mathbb{R}$. Also, $S_3 = \langle\sigma\rangle : \langle\tau\rangle$ where σ and τ are the elements of order two and three, respectively, but S_3 is not decomposable by its non-trivial (normal) subgroups.

Projections. Let $G = \Delta\cdot\Omega$. Define the functions $P_\Delta,\ P_\Omega$, from G to G, by $P_\Delta(x) = \delta$, $P_\Omega(x) = \omega$, where $x = \delta\omega, \delta\in\Delta, \omega\in\Omega$. Clearly, they are well-defined and $P_\Delta(G) = \Delta,\ P_\Omega(G) = \Omega,\ P_\Omega^* = P_\Delta$. We call P_Ω, [resp. P_Δ] *right [resp. left] projection.*

Example 2.5. The b-parts functions are projections of the direct decomposition $\mathbb{R} = \langle b\rangle\ddot{+}\mathbb{R}_b$.
Now, we can state many equivalent conditions for a function $f : G\to G$ to be a right weak, an ordinary, a semi-strong and a strong decomposer.

Theorem 2.6. *Every conditions in each part (a),...,(f) are equivalent.*

(a)

i) *f is a right decomposer.*

ii) *$G = f^*(G).f(G)$.*

iii) *f_e is a standard right decomposer.*

iv) *f_e is a standard right weak decomposer and a $f_e^*(G)\times f_e(G)$-homomorphism.*

v)*$f^*(f^*(x)f(y)) = f^*(x),\ \forall x,y\in G$.*

(b)

i)*f is a standard right decomposer.*

ii) *$G = f^*(G) : f(G)$.*

iii)*f is a standard right weak decomposer and $f^*(G)\times f(G)$-homomorphism.*

iv) *f is a standard right weak decomposer and f^* is a $f^*(G)\times f(G)$-homomrphism.*

(c)
i) *f is a right strong decomposer.*

ii) *f is a right decomposer and $f^*(G)\dot{\leqslant}G$.*

iii) *f is a right decomposer and $f^*(G) \leqslant G$.*

iv) *f^*_e is a $f^*_e(G) \times G$-homomorphism and $e \in f^*(G)$.*

v) *f_e is a $f^*_e(G) \times G$-homomorphism and f^*_e is an idempotent, $f(e) \in f^*_e(G)$.*

vi) *f^*_e is a $f^*_e(G) \times G$-homomorphism and an idempotent, $f(e) \in f^*_e(G)$.*
vii) *$f^*(f^*(x)y) = f^*(x)f^*(y)$, $\forall x, y \in G$.*

(d)
i) *f is a standard right strong decomposer.*

ii) *f is a right strong decomposer and f^* is an idempotent.*

iii) *f is a right strong decomposer and a $f^*(G) \times G$-homomorphism.*

iv) *f is a $f^*(G) \times G$-homomorphism and f^* is an idempotent.*

v) *f^* is a $f^*(G) \times G$-homomorphism and an idempotent.*

(e)
i) *f is a standard right weak decomposer.*

ii) *f and f^* are idempotent.*

iii) *f is an idempotent and $ff^* = f^*f$.*

iv) *f^* is an idempotent and $ff^* = f^*f$.*

v) *f is a right separator and $ff^* = f^*f$.*

vi) *$ff^* = f^*f = e$.*

vii) *$ff^* = f^*f = c$, for some $c \in G$.*

(f)
i) *f is a right semi-strong decomposer.*

ii) *$f(f^*(e)f(x)y) = f(xy)$, $\forall x, y \in G$ (i.e. f is a left semi-canceler).*

iii) *$f^*(e) \cdot f$ is a standard right strong decomposer.*

iv)*f is a right decomposer and $f^*(G)f(e)\dot{\leqslant}G$.*

v)*f is a right decomposer and $f^*(G)f(e) \leqslant G$.*

vi) *There exists a standard right strong decomposer g and $c \in G$ such that $f = c \cdot g$.*

vii)*$f^*(f^*(x)y) = f^*(x)f(e)f^*(f^*(e)y)$, $\forall x, y \in G$.*

Moreover, if f is an endomorphism, then f is a standard right strong decomposer $\Leftrightarrow$ f is a right decomposer $\Leftrightarrow$ f is a right weak decomposer $\Leftrightarrow$ f is a right separator $\Leftrightarrow$ f is an idempotent $\Leftrightarrow$ f^* or f_* is an idempotent.

Proof. (a) Let f be a right decomposer. If $\delta_1\omega_1 = \delta_2\omega_2$, where $\delta_i \in f^*(G)$ and $\omega_i \in f(G)$, then

$$\omega_1 = f(\delta_1\omega_1) = f(\delta_2\omega_2) = \omega_2,$$

so $\delta_1 = \delta_2$. Therefore $G = f^*(G).f(G)$. Now let $G = f^*(G).f(G)$. The relation

$$f^*(x)f(y) = f^*(f^*(x)f(y))f(f^*(x)f(y)),$$

implies f is right a decomposer. Also, the above relation shows that (i),(v) are equivalent. The parts (i), (iii) are equivalent, obviously. Now, if f is a right decomposer, then f_e is a standard right decomposer and so $f_ef_e^* = e$, $f_e^2 = f_e$. Thus f_e is $f_e^*(G) \times f_e(G)$-homomorphism. Therefore (i) implies (iv). Now if (iv) holds, then f_e a is right decomposer, so f is a right decomposer.
(b) The part (a) implies (b), clearly.
(c) Lemma 2.3 implies that (i),(ii) and (iii) are equivalent and (i) implies (iv). Now, let (iv) holds and put $c = f(e)^{-1}$ (hence $f_e = c.f$). Since $e \in f^*(G)$, then $f(e) \in f_e^*(G) = f^*(G)f(e)$ so $c \in f_e^*(G)$. So

$$f(f^*(x)y) = c^{-1}cf(f^*(x)c^{-1}cy) = c^{-1}f_e(f_e^*(x)cy) = c^{-1}f_e(y) = f(y),$$

for all $x, y \in G$. Therefore (i), (iv) are equivalent. Also, the parts (i),(vii) are equivalent, clearly.
Since $f(e) \in f_e^*(G)$ if and only if $e \in f^*(G)$, then the other parts are equivalent, similarly.
(d) If f^* is idempotent, then f is a standard right strong decomposer if and only if f is a $f^*(G) \times G$-homomorphism. Considering this fact (c) implies (d).
(e) Considering the relations

$$f^2 = (f^*f)^-.f \ , \ {f^*}^2 = f^*.(ff^*)^-,$$

the parts (i),...,(vi) are equivalent. Now, let (viii) holds. Putting $\alpha = f(e)$, $\beta = f^*(e)$ we have $c = f(\beta) = f^*(\alpha)$. On the other hand $f^*f = c$ implies

$f^2 = c^{-1}.f$ so $f^2(\beta) = c^{-1}f(\beta)$. Hence

$$f(c) = f^2(\beta) = c^{-1}f(\beta) = ff^*(\alpha) = c = f(\beta).$$

Therefore $c = e$.
(f) If f is a right semi-strong decomposer, then

$$f(f^*(e)f(x)y) = f(f^*(x)f(x)y) = f(xy),$$

so it is a left semi-canceler (see the next section). Conversely, if f satisfies (ii), then

$$f(f^*(e)y) = f(f^*(e)f(x)f^-(x)y) = f(xf^-(x)y) = f(f^*(x)y),$$

thus f is a right semi-strong decomposer. Also we have

$$f(f^*(x)y) = f(f^*(e)y) \Leftrightarrow f^*(f^*(x)y)^{-1}f^*(x)y = f^*(f^*(e)y)^{-1}f^*(e)y$$

$$\Leftrightarrow f^*(f^*(x)y) = f^*(x)f(e)f^*(f^*(e)y).$$

Considering the above relations and $f^* = f^*_e.f(e)$ and Lemma 2.3, we can conclude other parts of (f).
Finally, if f is endomorphism, then f is $f^*(G)\times G$ and $f^*(G)\times f(G)$-homomorphism and $ff^* = f^*f$. Therefore the last part of the theorem is concluded from (e),(a) and (d). $\square$

Corollary 2.7. *(i) Let f and g be two right decomposer functions with the same $*$-range (i.e. $f^*(G) = g^*(G)$). Then f is a right strong decomposer if and only if g is a right strong decomposer. Moreover, if f or g is a right strong decomposer, then (they are right strong decomposer and) $fg = f$, $gf = g$ and $|f(G)| = |g(G)|$.*
(ii) If $G = \Delta \cdot \Omega_1 = \Delta \cdot \Omega_2$ and $\Delta \dot{\leqslant} G$ or $0 \neq |\Delta| < \infty$, then $|\Omega_1| = |\Omega_2|$ (for example $\mathbb{R} = \mathbb{Z} \ddot{+} [0,1) = \mathbb{Z} \dot{+} [1,2)$ and we have $[0,1) \sim [1,2)$).

Proof. The first part of (i) is concluded from Theorem 2.6. Now, if f and g are right strong decomposers and $f^*(G) = g^*(G)$, then for every $x \in G$

$$x = f^*(x)f(x) = g^*(x)f^*(g(x))f(g(x)).$$

Since $g^*(x)f^*(g(x)) \in f^*(G)$, then $f(x) = fg(x)$ (by Theorem 2.6). Therefore, $f = fg$ and similarly $g = gf$. Putting $\Omega_f = f(G)$ and $\Omega_g = g(G)$, we have

$$\Omega_f = f(g(G)) = f(\Omega_g) \quad , \quad \Omega_g = g(f(G)) = g(\Omega_f).$$

Therefore $|\Omega_f| \leqslant |\Omega_g|$ and $|\Omega_g| \leqslant |\Omega_f|$ so $|\Omega_f| = |\Omega_g|$.
(ii) If Δ is non-empty and finite, then $|\Delta||\Omega_1| = |\Delta||\Omega_2|$ implies $|\Omega_1| = |\Omega_2|$.

Also if $\Delta \dot{\leqslant} G$, then putting $f = P_{\Omega_1}$ and $g = P_{\Omega_2}$ (where P_{Ω_1} and P_{Ω_2} are projections, see next page) we have $f^*(G) = g^*(G) = \Delta \leqslant G$, by Theorem 2.6, so the part (i) implies $|\Omega_1| = |f(G)| = |g(G)| = |\Omega_2|$. $\square$

Corollary 2.8. *f is a left strong decomposer $\Leftrightarrow$ f_* is a right decomposerand $f_*(G) \leqslant G$.*

Remark 2.9. *The most important property of right decomposer functions is $G = f^*(G).f(G)$. In fact this property connects these functions to "decomposition of a group by its subsets." For example the decomposition $\mathbb{Z} = \langle n \rangle \dot{\dot{+}} \mathbb{Z}_n$, by a subgroup and a subset, is produced by the strong decomposer function $(\ )_n$, for every nonzero integer n (notice that $\mathbb{Z}$ is not decomposable with its two non-trivial subgroups).*

2.1 Existence of decomposer types functions

Up to now we have studied the properties of decomposer type functions. Now, we show that how we can construct them and prove their existence, in arbitrary groups.

Definition 2.1.1. *Let $\Delta \subseteq G$. We say G is left [resp. right] Δ-decomposable if Δ is a left [resp. right] factor of G, which means $G = \Delta \cdot \Omega$ [resp. $G = \Omega \cdot \Delta$], for some $\Omega \subseteq G$.*
Let $\Delta \subseteq G$. If Δ is singletons, then G is left and right Δ-decomposable. Also G is left and right G-decomposable (trivially). Now consider the additive group $G = \mathbb{R}$ and fix $b \in \mathbb{R} \setminus \{0\}$. Then $\mathbb{R}$ is $\mathbb{R}_b$-decomposable and $\mathbb{R}$ is not $\mathbb{N}$-decomposable, i,e, the real numbers group does not have natural part property (although it has the integer part property, see Example 2.4).

Remark 2.1.2. *Considering Theorem 2.6(c), if $\Delta \dot{\leqslant} G$ but $\Delta \nleqslant G$, then G is not left and right Δ-decomposable. Therefore, $\mathbb{R}$ is not (M, ∞)-decomposable for all $M \geqslant 0$. Now, if $\Delta \leqslant G$, then G is (standard) left and right Δ-decomposable. Moreover, if $\Delta \trianglelefteq G$, then G is (standard two-sided) Δ-decomposable (see Remark 2.7 of [2]).*
Now, we are ready to prove the existence of a vast class of decomposer type functions.

Lemma 2.1.3. *If $G \neq 0$ is a group (finite or infinite) for which $|G|$ is not a prime number, then nontrivial (standard) right and left strong decomposer functions exist (and vice versa). In addition if G is not a simple group, then nontrivial (standard) strong decomposer functions exist.*

Proof. The hypothesis implies there exists non-trivial subgroup Δ of G. So

Remark 2.9 gives us a subset Ω of G such that $G = \Delta : \Omega$. Putting $f = P_\Omega$, we have $f^*(G) = P_\Delta(G) = \Delta \leqslant G$, thus Theorem 2.6 implies f is a nontrivial left standard strong decomposer function on G. Moreover, if G is not simple group, then there exists such a nontrivial subgroup Δ which is normal. In this case we claim that the function f defined by $f = P_\Omega$ is a (two-sided) strong decomposer. Considering the first part it is enough to show that it is a left strong decomposer. For every y there exist y', y'' such that $y^{-1}f^*(y) = f^*(y')y^{-1}$ and so

$$\begin{aligned} f(xf_*(y)) &= f(xy^{-1}f^*(y)y) = f(xf^*(y')y^{-1}y) \\ &= f(xf^*(y')) = f(f^*(y^{''})x) = f(x). \end{aligned}$$

Therefore the proof is complete. $\square$

Note that there are so many left and right strong decomposer [strong associative] functions in group [non-simple group] G such that $1 \neq |G| \neq p$, for every prime numbers p. For example the cardinal number of all strong decomposer real functions is $2^c = |2^{\mathbb{R}}|$, i.e. there exists a one to one correspondence between the solutions set of b-decimal part functional equation $f(x + y - f(y)) = f(x)$ and all real functions!.

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