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An Improved Bernoulli Collocation Method for Solving Volterra Integral Equations

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Abstract. In this work, an improved collocation method based on Bernoulli polynomials is proposed for solving the Volterra integral equation of the second kind. The main idea of the proposed method is to enhance the performance of the classical Bernoulli collocation method by dividing the interval into several subintervals and applying collocation points in each of them. The collocation nodes are chosen as the zeros of shifted Chebyshev polynomials. The method is implemented sequentially from the first subinterval to the last, resulting in a system of algebraic equations for each subinterval that can be efficiently solved. The convergence of the method is analyzed. To demonstrate the validity and efficiency of the proposed scheme, several numerical examples are presented. The results demonstrate that the improved method achieves higher accuracy than the classical Bernoulli collocation method.

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1 Introduction

Many challenges in science and technology can be effectively addressed using models based on ordinary and partial differential equations. However, integral equations often provide more powerful and flexible methods for modeling and solving complex phenomena. Historically, integral equations have attracted significant attention from mathematicians, especially when the unknown function appears within the integral. Such equations constitute a fundamental class within the broader category of functional equations.

The theory of integral equations is regarded as one of the most robust tools in both pure and applied mathematics, as well as in areas such as mathematical physics, mechanical vibrations, and engineering. Its foundational development dates back to the seminal contributions of Vito Volterra and Gösta Fredholm. Volterra played a crucial role in highlighting the importance of integral equations and systematically studying their analytical properties. In 1896, he introduced a general framework for solving linear integral equations with variable upper limits of integration.

Recently, there has been growing interest in Volterra integral equations (VIEs), mainly due to their wide range of practical applications in physics, engineering, and computational science [3–5, 10, 14, 19, 22, 24, 32, 35, 39, 43]. In this work, we develop a numerical scheme based on Bernoulli polynomials to efficiently approximate the solution of VIEs of the second kind

Bernoulli polynomials play a significant role in many areas of mathematical analysis, such as the theory of distributions in p-adic analysis [28], the theory of modular forms [45], and the polynomial expansions of analytic functions [8]. Applications of Bernoulli polynomials in mathematical physics include their connection to the theory of the KdV equation [13], the solution of the Lamé equation [17], and studies in the field of vertex algebra [12].

The collocation method is a widely used numerical technique for solving various types of functional and applied problems. It has consistently attracted the attention of researchers [1, 2, 6, 7, 9, 11, 15, 16, 18, 20, 21, 23, 25–27, 29, 33, 34, 36–38, 40–42]. Over recent decades, numerous authors have introduced different variants of collocation-based methods. For instance, Doha et al. [19] proposed a Jacobi–Gauss–Lobatto collocation scheme combined with a fourth-order implicit Runge–Kutta method for approximating the solutions to nonlinear Schrödinger equations. Nemati [29] addressed Volterra–Fredholm integral equations using a shifted Legendre polynomial-based collocation method. By employing shifted Gauss–Legendre nodes as collocation points, one can reduce the integral equations to a solvable matrix system.

Mirzaee and Hoseini [27] developed a matrix-based approach using Fibonacci polynomials and collocation points for the numerical solution of Volterra–Fredholm integral equations. Ren and Tian [34] presented a scheme for solving boundary value problems for Kirchhoff-type nonlinear integro-differential equations. Gouyandeh et al. [16] utilized the Tau collocation method to approximate the solutions to nonlinear Volterra–Fredholm–Hammerstein integral equations. Aziz et al. [2] introduced a collocation method based on Haar wavelets for the numerical solution of three-dimensional elliptic partial differential equations with Dirichlet boundary conditions. Çelik [9] applied the Chebyshev wavelet collocation method to study free vibration in non-uniform Euler–Bernoulli beams under various boundary conditions.

Samadyar and Mirzaee [37] developed an orthonormal Bernoulli collocation method to approximate linear singular stochastic Itô type Volterra integral equations. Biçer and Yalçinbas [7] obtained approximate solutions to the telegraph equation by employing a Bernoulli collocation approach. Alijani et al. [1] investigated systems of fuzzy fractional differential equations using a spline collocation technique involving a generalized Hukuhara derivative. Wang et al. [41] proposed a localized Chebyshev collocation method for solving two-dimensional elliptic partial differential equations. Singh [38] employed a Jacobi collocation method to solve fractional advection–dispersion equations arising in porous media.

Kumbinarasaiah et al. [20] developed an operational integration matrix based on Bernoulli wavelets and introduced the Bernoulli wavelet collocation method. In [42], a space–time Sinc-collocation method was proposed for handling a fourth-order nonlocal heat model in viscoelasticity. Laib et al. [21] designed an algorithm based on Taylor polynomials to construct collocation solutions for two-dimensional Volterra integral equations. Wang et al. [40] presented a new collocation method for second-kind Volterra integral equations by using the roots of Chebyshev polynomials as collocation nodes. Furthermore, in [18], the authors introduced a computationally efficient numerical scheme for integro-differential equations based on an operational matrix of Bernstein polynomials. Their approach transforms the governing equations into algebraic systems, enabling the derivation of higher-order approximate solutions along with convergence and error analysis. They further demonstrated the effectiveness of their method through several numerical examples, comparing the results with exact solutions and other well established techniques, including the Bernoulli collocation method (BCM).

In recent years, significant progress has been achieved in the development of advanced numerical methods for solving VIEs and related integro-differential models. Various hybrid, spectral, and collocation-based techniques have been proposed, demonstrating high accuracy and enhanced computational efficiency. For instance, a hybrid approach combining Legendre polynomials with Newton–Cotes quadrature was introduced in [44] for nonlinear VIEs. A spectral collocation method employing shifted Gegenbauer polynomials for fractional Volterra equations was presented in [31]. Moreover, a Chebyshev–Bernstein hybrid scheme incorporating adaptive mesh refinement to address strongly nonlinear and weakly singular kernels was proposed in [30]. These developments underscore the ongoing advancement of numerical techniques and motivate the design of more efficient and adaptable frameworks, such as the method introduced in the present work.

The primary novelty of this work resides in the integration of a piecewise collocation strategy with the classical Bernoulli collocation framework. By partitioning the domain into sub-intervals and utilizing the zeros of shifted Chebyshev polynomials (SCPs) as collocation nodes, the proposed method substantially improves the accuracy and adaptability of the classical scheme. Furthermore, the improved Bernoulli collocation method (IBCM) incorporates a stepwise structure that facilitates enhanced control over local errors. This framework is not confined to the Bernoulli basis and can be extended to refine other collocation-based numerical methods.

The proposed method possesses several notable features. It is straightforward to implement, highly accurate, and computationally efficient, even when employing relatively low-degree polynomial bases. The incorporation of a piecewise structure enhances local adaptivity and facilitates effective error control across sub-intervals. Furthermore, the selection of collocation points from the SCPs contributes to improved numerical stability and accuracy. However, a key limitation of the method is the need for careful selection of the number of sub-intervals to balance computational cost against the desired accuracy. Additionally, although the method performs well for smooth kernels and solutions, its efficiency may deteriorate in the presence of singularities or steep gradients.

The remainder of the paper is organized as follows. Section 2 introduces the Bernoulli polynomials and outlines their main properties. In Section 3, the proposed scheme is described in detail. Section 4 is devoted to the convergence analysis. Numerical results are presented in Section 5. Finally, the conclusions are drawn in Section 6.

2 Bernoulli Polynomials

The classical Bernoulli polynomials $B_n(t)$ are often characterized by the following exponential generating function [6]:

$$\frac{se^{sx}}{e^s-1}=\sum_{z=0}^{\infty} B_z(x)\frac{s^k}{z!}. \tag{1}$$

The Bernoulli polynomials of degree n are defined over the interval $[0,1]$ as follows [33]:

$$B_N(x) = \sum_{z=0}^{N} \binom{N}{z} B_z x^{N-\mathrm{z}}, \tag{2}$$

where B_z=$B_z(0)$ denotes the Bernoulli number for each z= $0, 1, \ldots N$. The first few Bernoulli polynomials are listed as follows:

$$\begin{aligned}
&B_0(x) = 1,\\
&B_1(x) = x - \frac{1}{2},\\
&B_2(x) = x^2 - x + \frac{1}{6},\\
&B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x,\\
&B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30},\\
&B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x,\\
&B_6(x) = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42},\\
&B_7(x) = x^7 - \frac{7}{2}x^6 + \frac{7}{2}x^5 - \frac{7}{6}x^3 + \frac{1}{6}x.
\end{aligned}$$

Bernoulli polynomials have following important properties [36]:

- $\frac{d}{dx}B_N(x) = NB_{N-1}(x), N \geq 1$.
- $B_N(x+1) - B_N(x) = Nx^{N-1}$.
- $\int_0^1 B_N(x)dx = 0, N \geq 1$.
- $\int_0^1 B_N(x)B_M(x)dx = (-1)^{N-1}\frac{M!N!}{(M+N)!}B_{N+M}$.
- $\int_a^x B_N(t)dt = \frac{B_{N+1}(x) - B_{N+1}(a)}{(N+1)}$.
- $B_N(1-x) = (-1)^N B_N(x)$.

2.1 Function approximation

A square-integrable function $u(t)$ can be expressed in terms of Bernoulli polynomials as:

$$u(t) = \sum_{i=0}^{\infty} U_i B_i(t).$$

The truncated form of this series is given by:

$$\tilde{u}(x) \simeq \sum_{i=0}^{M} U_i B_i(x) = U^T B(x), \tag{3}$$

where

$$U = \begin{bmatrix} U_0 & U_1 & U_2 & \dots & U_M \end{bmatrix}^T, \tag{4}$$

is the vector of unknown coefficients and

$$B(x) = \begin{bmatrix} B_0(x) & B_1(x) & B_2(x) & \dots & B_M(x) \end{bmatrix}^T, \tag{5}$$

is the vector of Bernoulli polynomials.

3 Description of the Proposed Scheme

In this section, the proposed scheme for solving second-kind VIEs is presented. The primary objective of the suggested method is to improve the accuracy of the classical collocation approach by partitioning the domain into multiple subintervals and applying the collocation method within each subinterval.

3.1 Solving VIEs of the second kind by BCM

Consider the following VIE of the second kind.

$$u(x) = f(x) + \int_0^x k(x,t) N(u(t)) \, dt, \qquad x \in [0,1], \tag{6}$$

where u is the unknown function, f and the kernel k are known functions, and N is a given continuous function, which may be nonlinear with respect to u.

Substituting the approximation of $u(t)$ from Eq. (3) into Eq. (6) gives:

$$\sum_{i=1}^{M} U_i B_i(x) = f(x) + \int_0^x k(x,t) N\left(\sum_{i=1}^{M} U_i B_i(t) \right) dt. \tag{7}$$

Next, we apply the collocation method to determine the unknown coefficients U_i, for $i = 1, ..., M$. Let c_k, $k = 1, ..., M$, denote the collocation points. Here, the collocation points are chosen as the zeros of the SCPs of degree M on the interval $[0, 1]$. For example, for $M = 3$, the collocation points are: $c_1 = 0.0670$, $c_2 = 0.5$, and $c_3 = 0.9330$.

This leads to a system of nonlinear algebraic equations, which can be easily solved using computational software.

In the special case where $N(u(t)) = u(t)$, the equation becomes linear. By rearranging the equation in terms of the coefficients U_i, we obtain:

$$\sum_{i=1}^{M} \left(B_i\left(x\right) - \int_0^x k\left(x,t\right) B_i\left(t\right) dt \right) U_i = f\left(x\right). \tag{8}$$

By substituting collocation points $c_k,\ k = 1, \ldots, M$ into Eq. (8) we get

$$\sum_{i=1}^{M} \left(B_i\left(c_k\right) - \int_0^{c_k} k\left(c_k,t\right) B_i\left(t\right) dt \right) U_i = f\left(c_k\right), \quad k = 1, \ldots, M. \tag{9}$$

The final equation represents a system of M algebraic equations with unknown coefficients U_i, $i = 1, \ldots, M$ which can be expressed in the following matrix form:

$$AU = F,$$

in which

$$A = \begin{bmatrix} B_1\left(c_1\right) - \int_0^{c_1} k\left(c_1,t\right) B_1\left(t\right) dt & \cdots & B_M\left(c_1\right) - \int_0^{c_1} k\left(c_1,t\right) B_M\left(t\right) dt \\ \vdots & \ddots & \vdots \\ B_1\left(c_M\right) - \int_0^{c_M} k\left(c_M,t\right) B_1\left(t\right) dt & \cdots & B_M\left(c_M\right) - \int_0^{c_M} k\left(c_M,t\right) B_M\left(t\right) dt \end{bmatrix},$$

vector U is defined as Eq. (4) and

$$F_1 = \left[f\left(c_1\right), f\left(c_2\right), \ldots, f\left(c_M\right)\right]^T.$$

3.2 Solving VIEs of the second kind by IBCM

In order to apply the idea of the suggested scheme, first we divide the interval $[0, 1]$ into N sub-intervals as $I_j = [(j-1)\,h, jh]$ where $h = \frac{1}{N}$.

Then, we consider the approximation of u by Bernoulli polynomials of degree M in each sub-interval as follows.

$$u(x) \simeq \sum_{i=1}^{M} U_{i,j} B_i(x) = B(x) U_j, \quad x \in I_j, \quad j = 1, \dots, N, \tag{10}$$

where $U_{i,j}$, $i = 1, \dots, M$, $j = 1, \dots, N$, are unknown coefficients to be determined and $B(x)$ is defined as Eq. (4) and

$$U_j = [U_{1,j}, U_{2,j}, \dots, U_{M,j}]^T, \qquad j = 1, \dots, N. \tag{11}$$

According to Eq. (10), the approximate solution is considered as a piece-wise function in the proposed method.

In general, there are MN unknowns, $U_{1,1}, U_{2,1}, \dots, U_{M,1}, \dots, U_{1,N}$, $U_{2,1}, \dots, U_{M,N}$, to be determined. To find these unknowns, we proceed as follows.

For finding the unknowns $U_{1,1}, U_{2,1}, \dots, U_{M,1}$, suppose that $x \in I_1$. Then, according to Eq. (10) the approximate solution in the interval $I_1 = [0, h]$ is

$$u(x) \simeq \sum_{i=1}^{M} U_{i,1} B_i(x). \tag{12}$$

Substituting Eq. (12) into Eq. (6) leads to

$$\sum_{i=1}^{M} U_{i,1} B_i(x) = f(x) + \int_0^x k(x,t) N\left(\sum_{i=1}^{M} U_{i,1} B_i(t)\right) dt. \tag{13}$$

We use the collocation method to determine unknowns $U_{i,1}$, $i = 1, \dots, M$. Let $c_{k,1}$, $k = 1, \dots, M$ be the zeros of SCPs of degree M in the interval $[0, h]$ as collocation points. For example for $M = 3$, and $N = 4$ the collocation points are $c_{1,1} = 0167$, $c_{2,1} = 0.1250$, $c_{3,1} = 0.2333$, $c_{1,2} = 0.2667$, $c_{2,2} = 0.3750$, $c_{3,2} = 0.4833$, $c_{1,3} = 0.5167$, $c_{2,3} = 0.6250$, $c_{3,3} = 0.7333$, $c_{1,4} = 0.7667$, $c_{2,4} = 0.8750$, $c_{3,4} = 0.9833$. Then, a system of nonlinear algebraic equation is produced that could be easily solved.

In the case of $N(u(t)) = u(t)$, which means the equation is linear, equation (13) could be stated as follows.

$$\sum_{i=1}^{M} \left(B_i(x) - \int_0^x k(x,t) B_i(t)\, dt\right) U_{i,1} = f(x). \tag{14}$$

By substituting collocation points $c_{k,1},\ k = 1, \ldots, M$ into Eq. (13) we will have

$$\sum_{i=1}^{M} \left(B_i(c_{k,1}) - \int_0^{c_{k,1}} k(c_{k,1}, t) B_i(t)\, dt \right) U_{i,1} = f(c_{k,1}). \tag{15}$$

This equation is a system of M algebraic equations with unknown coefficients $U_{i,1},\ i = 1, \ldots, M$ which can be written in the following matrix form.

$$A_1 U_1 = F_1,$$

in which

$$A_1 = \begin{bmatrix} B_1(c_{1,1}) - \int_0^{c_{1,1}} k(c_{1,1}, t) B_1(t)\, dt & \cdots & B_M(c_{1,1}) - \int_0^{c_{1,1}} k(c_{1,1}, t) B_M(t)\, dt \\ \vdots & \ddots & \vdots \\ B_1(c_{M,1}) - \int_0^{c_{M,1}} k(c_{M,1}, t) B_1(t)\, dt & \cdots & B_M(c_{M,1}) - \int_0^{c_{M,1}} k(c_{M,1}, t) B_M(t)\, dt \end{bmatrix},$$

vector U_1 is defined as Eq. (11) for $j = 1$ and

$$F_1 = \left[f(c_{1,1}), f(c_{2,1}), \ldots, f(c_{M,1}) \right]^T.$$

By utilizing the coefficients $U_{i,1},\ i = 1, \ldots, M$, which were determined in the previous stage, the unknown coefficients $U_{i,2},\ i = 1, \ldots, M$, can be found as follows.

Suppose that $x \in I_2$. Then Substituting Eq. (10) for $j = 2$ into Eq. (6) yields

$$\begin{aligned} \sum_{i=1}^{M} B_i(x) U_{i,2} = f(x) &+ \int_0^h k(x, t) \left(\sum_{i=1}^{M} B_i(t) U_{i,1} \right) dt \\ &+ \int_h^x k(x, t) \left(\sum_{i=1}^{N} B_i(t) U_{i,2} \right) dt. \end{aligned} \tag{16}$$

This equation can be rewritten as follows.

$$\sum_{i=1}^{M} \left(B_i(x) - \int_h^x k(x, t) B_i(t)\, dt \right) U_{i,2} = f(x) + \int_0^h k(x, t) \left(\sum_{i=1}^{M} U_{i,1} B_i(t) \right) dt. \tag{17}$$

Let $c_{k,2},\ k=1,\ldots,M$ be the zeros of SCPs of degree M in the interval $[h,2h]$ as collocation points. Substituting the collocation points into Eq. (17) leads to the following relation.

$$\begin{aligned}\sum_{i=1}^{M}\left(B_i(c_{k,2})-\int_{h}^{c_{k,2}}k(c_{k,2},t)B_i(t)\,dt\right)U_{i,2}\\ =f(c_{k,2})+\int_0^h k(c_{k,2},t)\left(\sum_{i=1}^{M}U_{i,1}B_i(t)\right)\,dt.\end{aligned}\tag{18}$$

This equation represents a system of M algebraic equations with unknown coefficients $U_{i,2},\ i=1,\ldots,M$. This system can be expressed in the following matrix form:

$$A_2U_2=F_2,$$

where

$$A_2=\begin{bmatrix} B_1(c_{1,2})-\int_h^{c_{1,2}}k(c_{1,2},t)\,B_1(t)\,dt & \cdots & B_M(c_{1,2})-\int_h^{c_{1,2}}k(c_{1,2},t)\,B_M(t)\,dt \\ \vdots & \ddots & \vdots \\ B_1(c_{M,2})-\int_h^{c_{M,2}}k(c_{M,2},t)\,B_1(t)\,dt & \cdots & B_M(c_{M,2})-\int_h^{c_{M,2}}k(c_{M,2},t)\,B_M(t)\,dt \end{bmatrix},$$

vector U_2 is defined as Eq. (11) for $j=2$ and

$$F_2=\begin{pmatrix} f(c_{1,2})+\int_0^h k(c_{1,2},t)\left(\sum_{i=1}^{M}U_{i,2}B_i(t)\right)dt \\ \vdots \\ f(c_{M,2})+\int_0^h k(c_{M,2},t)\left(\sum_{i=1}^{M}U_{i,2}B_i(t)\right)dt \end{pmatrix}.$$

Now, we can present a general formula to find the unknowns in the interval $I_j=[(j-1)\,h,jh]$.

Suppose that $x\in I_j$. Then, according to Eq. (12) the approximate solution in the interval I_j is

$$u(x)\simeq\sum_{i=1}^{M}U_{i,j}B_i(x).\tag{19}$$

By substituting Eq. (19) into Eq. (6) we will have

$$\sum_{i=1}^{M} B_i\left(x\right)U_{i,j} = f\left(x\right) + \int_0^h k\left(x,t\right)\left(\sum_{i=1}^{M} U_{i,1}B_i\left(t\right)\right)dt + \dots \\ + \int_{(j-2)h}^{(j-1)h} k\left(x,t\right)\left(\sum_{i=1}^{M} U_{i,j-1}B_i\left(t\right)\right)dt \\ + \int_{(j-1)h}^{x} k\left(x,t\right)\left(\sum_{i=1}^{M} U_{i,j}B_i\left(t\right)\right)dt.$$

This equation would be stated as follows.

$$\sum_{i=1}^{M}\left(B_i(x) - \int_{(j-1)h}^{x} k(x,t)B_i(t)\,dt\right)U_{i,j} \\ = f(x) + \int_0^h k(x,t)\left(\sum_{i=1}^{M} U_{i,1}B_i(t)\right)\,dt + \dots \\ + \int_{(j-2)h}^{(j-1)h} k(x,t)\left(\sum_{i=1}^{M} U_{i,j-1}B_i(t)\right)\,dt. \tag{20}$$

Let $c_{k,j},\ k = 1, \dots, M$ be the zeros of SCPs of degree M in the interval I_j as collocation points. Substituting the collocation points into Eq. (20) yields

$$\sum_{i=1}^{M}\left(B_i\left(c_{k,j}\right) - \int_{(j-1)h}^{c_{k,j}} k\left(c_{k,j},t\right)B_i\left(t\right)dt\right)U_{i,j} \\ = f\left(c_{k,j}\right) + \int_0^h k\left(c_{k,j},t\right)\left(\sum_{i=1}^{M} U_{i,1}B_i\left(t\right)\right)dt + \dots \\ + \int_{(j-2)h}^{(j-1)h} k\left(c_{k,j},t\right)\left(\sum_{i=1}^{M} U_{i,j-1}B_i\left(t\right)\right)dt.$$

This equation is a system of M algebraic equations with unknown coefficients $U_{i,j}$, $i = 1, \dots, M$. It has the following matrix form.

$$A_jU_j = F_j,$$

where

$$A_j = \begin{bmatrix} B_1(c_{1,j}) - \int_{(j-1)h}^{c_{1,j}} k(c_{1,j},t) B_1(t)\,dt & \cdots & B_M(c_{1,j}) - \int_{(j-1)h}^{c_{1,j}} k(c_{1,j},t) B_M(t)\,dt \\ \vdots & \ddots & \vdots \\ B_1(c_{M,j}) - \int_{(j-1)h}^{c_{M,j}} k(c_{M,j},t) B_1(t)\,dt & \cdots & B_M(c_{M,j}) - \int_{(j-1)h}^{c_{M,j}} k(c_{M,j},t) B_M(t)\,dt \end{bmatrix},$$

vector U_j is defined as Eq. (11) and

$$F_j = \begin{pmatrix} f(c_{1,j}) + \sum_{r=1}^{j-1} \int_{(r-1)h}^{rh} k(c_{1,j},t) \left(\sum_{i=1}^{M} U_{i,r} B_i(t)\right) dt \\ \vdots \\ f(c_{M,j}) + \sum_{r=1}^{j-1} \int_{(r-1)h}^{rh} k(c_{M,j},t) \left(\sum_{i=1}^{M} U_{i,r} B_i(t)\right) dt \end{pmatrix}.$$

Therefore, by continuing this process step by step and computing the unknowns in the subsequent subintervals using the coefficients obtained in the previous steps, all unknowns can eventually be determined.

To compute the unknowns in the final subinterval, suppose that $x \in I_N$. Then, according to Eq. (10), the approximate solution in this interval is given by

$$u(x) \simeq \sum_{i=1}^{M} U_{i,N} B_i(x). \tag{21}$$

Substituting Eq. (21) into Eq. (6) gives

$$\begin{aligned} \sum_{i=1}^{M} B_i(x) U_{i,N} = f(x) &+ \int_0^h k(x,t) \left(\sum_{i=1}^{M} U_{i,1} B_i(t)\right) dt + \dots \\ &+ \int_{(N-2)h}^{(N-1)h} k(x,t) \left(\sum_{i=1}^{M} U_{i,N-1} B_i(t)\right) dt \\ &+ \int_{(N-1)h}^{x} k(x,t) \left(\sum_{i=1}^{M} U_{i,N} B_i(t)\right) dt. \end{aligned}$$

This equation can be rewritten in terms of $U_{i,N}$ as follows.

$$\sum_{i=1}^{M}\left(B_i(x)-\int_{(N-1)h}^{x}k(x,t)B_i(t)\,dt\right)U_{i,N} = f(x)+\int_0^h k(x,t)\left(\sum_{i=1}^{M}U_{i,1}B_i(t)\right)dt + \cdots + \int_{(N-2)h}^{(N-1)h}k(x,t)\left(\sum_{i=1}^{M}U_{i,N-1}B_i(t)\right)dt. \tag{22}$$

Let $c_{k,N},\ k=1,\dots,M$ be the zeros of SCPs of degree M in the interval I_N as collocation points. By substituting these points into Eq. (22) we will have

$$\sum_{i=1}^{M}\left(B_i\left(c_{k,N}\right)-\int_{(N-1)h}^{c_{k,N}}k\left(c_{k,N},t\right)B_i\left(t\right)dt\right)U_{i,N} = f\left(c_{k,N}\right)+\int_0^h k\left(c_{k,N},t\right)\left(\sum_{i=1}^{M}U_{i,1}B_i\left(t\right)\right)dt+\dots + \int_{(N-2)h}^{(N-1)h}k\left(c_{k,N},t\right)\left(\sum_{i=1}^{M}U_{i,N-1}B_i\left(t\right)\right)dt.$$

This equation is a system of M algebraic equations with unknown coefficients $U_{i,N},\ i=1,\dots,M$ which can be stated in a matrix form as follows.

$$A_N U_N = F_N,$$

in which

$$A_N=\begin{bmatrix} B_1\left(c_{1,N}\right)-\int_{(N-1)h}^{c_{1,N}}k\left(c_{1,N},t\right)B_1\left(t\right)dt & \cdots & B_M\left(c_{M,N}\right)-\int_{(N-1)h}^{c_{1,N}}k\left(c_{1,N},t\right)B_M\left(t\right)dt \\ \vdots & \ddots & \vdots \\ B_1\left(c_{M,N}\right)-\int_{(N-1)h}^{c_{M,N}}k\left(c_{M,N},t\right)B_1\left(t\right)dt & \cdots & B_M\left(c_{M,N}\right)-\int_{(N-1)h}^{c_{M,N}}k\left(c_{M,N},t\right)B_M\left(t\right)dt \end{bmatrix},$$

vector U_N is defined as Eq. (11) for $j=N$ and

$$F_N=\begin{pmatrix} f\left(c_{1,N}\right)+\sum_{r=1}^{N-1}\int_{(r-1)h}^{rh}k\left(c_{1,N},t\right)\left(\sum_{i=1}^{M}U_{i,r}B_i\left(t\right)\right)dt \\ \vdots \\ f\left(c_{M,N}\right)+\sum_{r=1}^{N-1}\int_{(r-1)h}^{rh}k\left(c_{M,N},t\right)\left(\sum_{i=1}^{M}U_{i,r}B_i\left(t\right)\right)dt \end{pmatrix}.$$

Eventually, we can calculate the solution by the following piecewise function.

$$\widetilde{u} = \begin{cases} \sum_{i=1}^{M} U_{i,1} B_i(x), & x \in I_1, \\ \sum_{i=1}^{M} U_{i,2} B_i(x), & x \in I_2, \\ \vdots \\ \sum_{i=1}^{M} U_{i,N} B_i(x), & x \in I_N, \end{cases} \tag{23}$$

where $\tilde{u}$ is the approximation of the exact solution u.

4 Convergence Analysis

In this section, we analyze the convergence of the proposed scheme. To this end, we begin by recalling a key theorem regarding the residual interpolation error associated with Chebyshev nodes.

Theorem 4.1. *Let u be a sufficiently smooth function on $I = [0,\ 1]$ and Π_M be the space of polynomials of order M. Also, let $P_M \in \Pi_M$ be the interpolating polynomials of u at points $c_1, \ldots, c_{M+1}$ which are the zeros of the SCP of degree $M{+}1$ on I. Then, the following relation is established.*

$$u(t) - P_M(t) = \frac{\partial^{M+1} u(\xi)}{\partial x^{M+1} (M+1)!} \prod_{i=0}^{M} (t - c_i)\,, \tag{24}$$

where $\xi{\in}I$.

Proof. [15]. □

According to the last theorem, we can write

$$|u(t) - P_M(t)| \leq \max_{x \in I} \left| \frac{\partial^{M+1} u(t)}{\partial x^{M+1}} \right| \frac{\prod_{i=0}^{M} |t - c_i|}{(M+1)!}. \tag{25}$$

Now, Assume that

$$\max_{x \in I} \left| \frac{\partial^{M+1} u(t)}{\partial x^{M+1}} \right| \leq \eta. \tag{26}$$

Applying this upper bound to Eq. (25) and considering the approximations for Chebyshev interpolation nodes [25] leads to

$$|u(t) - P_M(t)| \leq \frac{\eta}{(M+1)!\, 2^{2M+1}}. \tag{27}$$

Theorem 4.2. *Suppose that $\tilde{u}$ defined in Eq. (3), be the best approximation of real sufficiently smooth function u by Bernoulli polynomials. Then a real constant η exists such that*

$$\|u(t) - \tilde{u}(t)\|_2 \leq \frac{\eta}{(M+1)!\, 2^{2M+1}}. \tag{28}$$

Proof. According to the definition, $\tilde{u}$ is the best approximation of u provided that

$$\forall v(t) \in \Pi_N; \quad \|u(t) - \tilde{u}(t)\|_2 \leq \|u(t) - v(t)\|_2. \tag{29}$$

Particularly, if $v(t) = P_M(t)$ then according to Eq. (27), we get

$$\begin{aligned} \|u(t) - \tilde{u}(t)\|_2^2 \leq \|u(t) - P_M(t)\|_2^2 &= \int_0^1 |u(t) - P_M(t)|^2 \, dt \\ &\leq \int_0^1 \left[\frac{\eta}{(M+1)!\, 2^{2M+1}}\right]^2 dt = \left[\frac{\eta}{(M+1)!\, 2^{2M+1}}\right]^2. \end{aligned} \tag{30}$$

Hence, Eq. (28) is proved. □

According to Eq. (28), it can be written

$$\|u(t) - \tilde{u}(t)\|_2 = \mathcal{O}\left(\frac{1}{(M+1)!\, 2^{2M+1}}\right). \tag{31}$$

So, $\frac{1}{(M+1)!\, 2^{2M+1}} \to 0$ when $M \to \infty$ which implies that $\tilde{u} \to u$. Therefore, the collocation method based on the Bernoulli polynomials is convergent.

Theorem 4.3. *Assume that $u_{M,j}$ be the approximate solution of Eq. (6) in the interval $I_j = [(j-1)\,h, jh]$, and*

$$(1 - L_1\lambda_1)(1 - L_2\lambda_2)\ldots(1 - L_j\lambda_j) > 0,$$

where $j = 1, \ldots, N$. Also, the nonlinear term satisfies the Lipschitz condition as follows:

$$\left\| N\left(u\left(x\right)\right) - N\left(u_{M,j}\left(x\right)\right)\right\| \leq L_j \left\| u\left(x\right) - u_{M,j}\right\|. \tag{32}$$

Then, there is an upper error bound as follows:

$$u(t) - u_{M,j}(t) \leq \frac{\lambda_1}{\left(1-\lambda_1\right)\left(1-\lambda_2\right)\ldots\left(1-\lambda_j\right)}, \tag{33}$$

where

$$\max\left|k(x,t)\right| = \lambda_j, \quad x \in I_j. \tag{34}$$

Proof. The approximate solution of Eq. (6) in the interval I_1 could be stated as

$$u_{M,1}(x) = f(x) + \int_0^x k(x,t) N\left(u_{M,1}(t)\right) dt. \tag{35}$$

From Eqs. (6) and (35), we get

$$u\left(x\right) - u_{M,1}\left(x\right) = \int_0^x \; k\left(x,t\right)\left(N\left(u\left(t\right)\right) - N\left(u_{M,1}\left(t\right)\right)\right) dy.$$

Then, we have

$$\left\| u(x) - u_{M,1}(x)\right\| \leq L_1 \left\| k\left(x,t\right)\right\| \left\| u\left(t\right) - u_{M,1}\left(t\right)\right\|.$$

Using Eq. (34), for i= 1, we have

$$\left\| u(x) - u_{M,1}(x)\right\| \leq L_1 \lambda_1 \left\| u\left(x\right) - u_{M,1}\left(x\right)\right\|.$$

So,

$$\left\| u(x) - u_{M,1}(x)\right\| \leq \frac{1}{1 - L_1 \lambda_1}. \tag{36}$$

Now, consider the approximate solution of Eq. (6) in the interval I_2 as follows:

$$u_{M,2}(x) = f(x) + \int_0^h k(x,t) N\left(u_{M,1}(t)\right) dt + \int_h^x k(x,t) N\left(u_{M,2}(t)\right) dt. \tag{37}$$

From Eqs. (6) and (36), we get

$$u\left(x\right)-u_{M,2}\left(x\right)=\int_{0}^{h} k\left(x,t\right)\ \left(N\left(u\left(t\right)\right)-N\left(u_{M,1}\left(t\right)\right)\right)dt \\ +\int_{h}^{2h} k\left(x,t\right)\ \left(N\left(u\left(t\right)\right)-N\left(u_{M,2}\left(t\right)\right)\right)dt.$$

Then, we have

$$\|u(x)-u_{M,2}(x)\| \leq \|k\left(x,t\right)\| \left\|u\left(x\right)-u_{M,1}\left(x\right)\right\|+k\left(x,t\right)\| \left\|u\left(x\right)-u_{M,2}\left(x\right)\right\|.$$

Using Eq. (34), for $i=2$ and also Eq. (36) we have

$$\|u(x)-u_{M,2}(x)\| \leq L_1\lambda_1\left(\frac{1}{1-L_1\lambda_1}\right)+L_2\lambda_2\left\|u\left(x\right)-u_{M,2}\left(x\right)\right\|.$$

Therefore,

$$|u(x)-u_{M,2}(x)| \leq \frac{L_1\lambda_1}{\left(1-L_1\lambda_1\right)\left(1-L_1\lambda_2\right)}. \tag{38}$$

For the approximate solution of Eq. (6) in the interval I_3 we can write

$$u_{M,3}(x)=f(x)+\int_{0}^{h} k(x,t)N\left(u_{M,1}(t)\right)\,dt+\int_{h}^{2h} k(x,t)N\left(u_{M,2}(t)\right)\,dt \\ +\int_{2h}^{x} k(x,t)N\left(u_{M,3}(t)\right)\,dt. \tag{39}$$

From Eqs. (6) and (36), we can write

$$u\left(x\right)-u_{M,3}\left(x\right)=\int_{0}^{h} k\left(x,t\right)\left(N\left(u\left(t\right)\right)-N\left(u_{M,1}\left(t\right)\right)\right)dt \\ +\int_{h}^{2h} k\left(x,t\right)\left(N\left(u\left(t\right)\right)-N\left(u_{M,2}\left(t\right)\right)\right)dt \\ +\int_{2h}^{x} k\left(x,t\right)\left(N\left(u\left(t\right)\right)-N\left(u_{M,3}\left(t\right)\right)\right)dt.$$

Then, we have

$$\|u(x) - u_{M,3}(x)\| \leq L_1 \|k(x,t)\| \|u(t) - u_{M,1}(t)\| + L_2 \|k(x,t)\| \|u(t) - u_{M,2}(t)\| + L_3 \|k(x,t)\| \|u(t) - u_{M,3}(t)\|.$$

Using Eq. (34), for $i = 3$ and also Eq. (38) we have

$$\begin{aligned} \|u(x) - u_{M,3}(x)\| &\leq \frac{L_1\lambda_1}{1 - L_1\lambda_1} + \frac{L_1\lambda_2\lambda_1}{(1-\lambda_1)(1-\lambda_2)} + \lambda_3 \|u(t) - u_{M,3}(t)\| \\ &= \frac{L_1\lambda_1}{(1-\lambda_1)(1-\lambda_2)} + \lambda_3 \|u(x) - u_{M,3}(x)\|. \end{aligned}$$

Finally, we have

$$\|u(x) - u_{M,3}(x)\| \leq \frac{L_1\lambda_1}{(1 - L_1\lambda_1)(1 - L_2\lambda_2)(1 - L_3\lambda_3)}. \tag{40}$$

By comparing the upper error bounds obtained in previous steps, it can be concluded that an upper error bound for $u_{M,j}$ is as follows:

$$\|u(x) - u_{M,j}(x)\| \leq \frac{L_1\lambda_1}{(1-L_1\lambda_1)(1-L_2\lambda_2)\dots(1-L_j\lambda_{\mathrm{j}})}.$$

Therefore, Eq. (33) is established. □

5 Numerical Results

In this section, we present several numerical examples to demonstrate the validity, efficiency, and accuracy of the proposed IBCM in comparison with the classical BCM and other existing methods. All numerical computations were performed using MATLAB R2018b on a laptop equipped with an Intel Pentium B960 @ 2.20 GHz processor and 4 GB of RAM.

Example 5.1. Consider the following nonlinear Volterra integral equation of the second kind [23].

$$u(x) = \cos x - e^x \sin x + \int_0^x e^x u(t)\, dt. \tag{41}$$

with the exact solution $u(x) = \cos x$.

The numerical results for this example are reported in Tables 1–3. Table 1 compares the absolute errors of the IBCM and BCM for $M = 5$ and two values of N ($N = 3$ and $N = 5$), evaluated at five points over the interval $[0, 1]$. Tables 2 and 3 present similar comparisons with increased values of M. Specifically, Table 2 corresponds to $M = 10$, while in Table 3, M is doubled.

An examination of Tables 1–3 shows that, in both methods, increasing the polynomial degree N improves accuracy. However, the IBCM consistently yields lower errors than the BCM. For instance, for $N = 5$, the error in the IBCM with $M = 20$ (Table 3) is on the order of 10^{-12}, while for the BCM (Table 1), it remains around 10^{-3}. Similarly, for $N = 7$, the IBCM achieves an error of approximately 10^{-16}, whereas the BCM reaches only 10^{-8}, as reported in Table 3.

The precision of the IBCM can be improved by increasing the number of subintervals M, while keeping N fixed. For example, for $N = 5$, the order of error in the IBCM with $M = 5$ (Table 1) is 10^{-9}, whereas it reduces to 10^{-10} for $M = 10$ (Table 2) and further to 10^{-12} for $M = 20$ (Table 3). The absolute error of the IBCM for $M = 20$ and $N = 7$ is illustrated in Figure 1.

This equation was previously solved in [23] using Bernstein's approximation. The authors reported the absolute errors for $n = 2, \ldots, 9$, where n denotes the polynomial degree. According to their results (Table 4), the error for $n = 9$ was on the order of 10^{-10}, while our proposed method achieves an error of 10^{-16} for $n = 7$.

As shown in Figure 2, the approximate solution obtained by the IBCM almost perfectly overlaps the exact solution, demonstrating the extremely high accuracy of the method.

The CPU time for this example was approximately 0.019 seconds for the classical BCM ($N = 7$), and 0.044 seconds for the proposed IBCM ($N = 7$, $M = 20$).

Example 5.2. Consider the following nonlinear Volterra integral equation of the second kind [16].

$$u(x) = e^x - \frac{1}{3}\left(e^{3x} + 1\right) + \int_0^x u^3(t)\,dt, \tag{42}$$

Table 1: The absolute error of the IBCM with $M = 5$ and BCM for example 1.

x	$N = 3$		$N = 5$	
	IBCM	BCM	IBCM	BCM
0.15	4.5550×10^{-6}	1.7512×10^{-3}	2.1507×10^{-10}	6.0265×10^{-6}
0.35	1.2274×10^{-5}	2.7788×10^{-3}	8.2472×10^{-10}	7.7318×10^{-6}
0.55	1.9155×10^{-5}	7.9400×10^{-4}	1.3400×10^{-9}	3.1127×10^{-6}
0.75	2.4561×10^{-5}	7.4072×10^{-4}	1.8279×10^{-9}	6.0874×10^{-6}
0.95	2.7405×10^{-5}	2.9919×10^{-3}	2.2953×10^{-9}	3.2830×10^{-6}

Table 2: The absolute error of the IBCM with $M = 10$ and BCM for example 1.

x	$N = 3$		$N = 5$	
	IBCM	BCM	IBCM	BCM
0.1	3.2037×10^{-7}	7.5868×10^{-4}	9.4060×10^{-11}	8.0179×10^{-6}
0.2	8.2926×10^{-7}	2.4247×10^{-3}	2.5412×10^{-10}	1.6861×10^{-6}
0.3	1.3229×10^{-6}	2.9075×10^{-4}	4.1141×10^{-11}	6.2700×10^{-6}
0.4	1.8168×10^{-6}	2.4544×10^{-3}	5.6437×10^{-11}	7.0803×10^{-6}
0.5	2.2875×10^{-6}	1.4077×10^{-4}	7.1154×10^{-11}	9.1337×10^{-7}
0.6	2.7390×10^{-6}	2.0181×10^{-4}	8.5158×10^{-11}	6.5449×10^{-6}
0.7	3.1703×10^{-6}	6.4116×10^{-4}	9.8331×10^{-11}	8.4324×10^{-6}
0.8	3.5826×10^{-6}	5.1679×10^{-4}	1.1058×10^{-10}	1.9262×10^{-6}
0.9	3.9812×10^{-6}	1.2557×10^{-3}	1.2185×10^{-10}	9.6383×10^{-6}

Table 3: The absolute error of the IBCM with $M = 20$ and BCM for example 1.

x	IBCM		BCM
	$N = 5$	$N = 7$	$N = 7$
0.1	4.0059×10^{-13}	2.2043×10^{-16}	3.0790×10^{-9}
0.2	9.0170×10^{-13}	1.0987×10^{-16}	1.1425×10^{-8}
0.3	1.3914×10^{-12}	2.4797×10^{-16}	2.2238×10^{-9}
0.4	1.8717×10^{-12}	1.8339×10^{-16}	1.2202×10^{-8}
0.5	2.3303×10^{-12}	1.9098×10^{-16}	1.8206×10^{-9}
0.6	2.7679×10^{-12}	2.8180×10^{-16}	1.0911×10^{-8}
0.7	3.1737×10^{-12}	1.2333×10^{-15}	3.8603×10^{-9}
0.8	3.5513×10^{-12}	2.2374×10^{-15}	1.1863×10^{-9}
0.9	3.8946×10^{-12}	3.2271×10^{-15}	6.6650×10^{-11}

Table 4: The computed errors $\|e_n\|$ for Example 1 in [23].

n	$\|e_n\|$
2	1.95950×10^{-3}
3	2.27437×10^{-4}
4	6.17039×10^{-6}
5	5.39633×10^{-7}
6	1.09152×10^{-8}
7	1.11546×10^{-9}
8	3.37632×10^{-10}
9	3.32584×10^{-10}
10	3.40973×10^{-10}

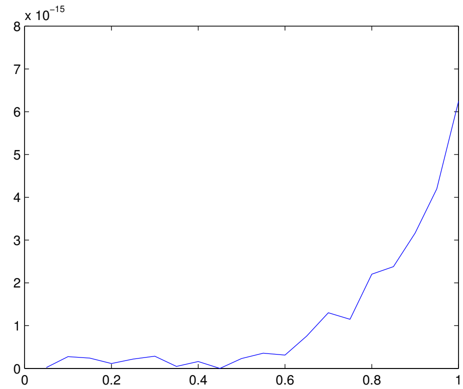

Figure 1: The absolute error of the IBCM with $M = 20$ and $N = 7$ for example 1.

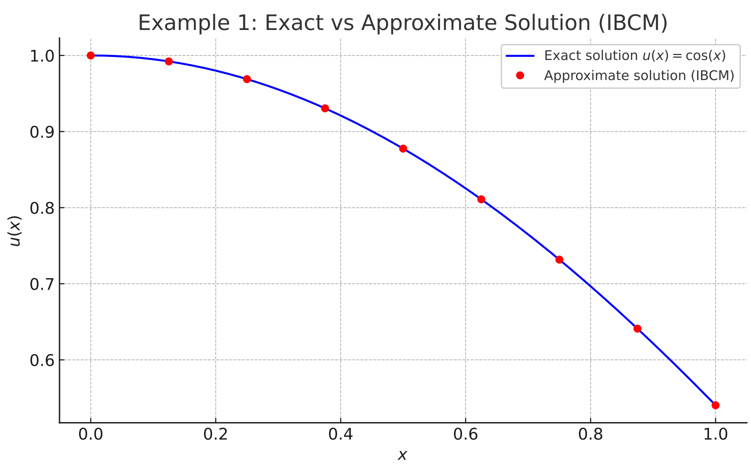

Figure 2: The exact and approximate solutions obtained by the IBCM for Example 1 with $M = 20$ and $N = 7$.

with the exact solution $u(x) = e^x$.

The numerical results for this example are presented in Tables 5 and 6. Table 5 compares the absolute errors of the IBCM with $M = 10$ and the BCM for two values of N ($N = 4$ and $N = 6$). An examination of this table reveals that, for both methods, increasing the polynomial degree N improves the accuracy. Moreover, the IBCM consistently outperforms the BCM in terms of precision.

Table 6 presents a comparison among the IBCM, BCM, and the Tau-collocation method [16] for $N = 8$, demonstrating the superior accuracy of the IBCM. According to this table, the accuracy of the proposed method exceeds that of the approach in [16]. The absolute error of the IBCM for $M = 5$ and $N = 8$ is illustrated in Figure 3.

As shown in Figure 4, the approximate solution nearly overlaps the exact solution, confirming the high accuracy of the IBCM.

The CPU time for this example was approximately 0.024 seconds for the BCM ($N = 8$) and 0.026 seconds for the IBCM ($N = 8$, $M = 5$).

Table 5: The absolute error of the IBCM with $M = 10$ and BCM for example 2.

x	$N = 4$		$N = 6$	
	IBCM	BCM	IBCM	BCM
0.1	3.1892×10^{-8}	2.8966×10^{-4}	6.6974×10^{-13}	5.4474×10^{-7}
0.2	3.1164×10^{-8}	3.2696×10^{-4}	6.5893×10^{-13}	4.4242×10^{-7}
0.3	2.9990×10^{-8}	1.0663×10^{-4}	6.4054×10^{-13}	5.2662×10^{-7}
0.4	2.8363×10^{-8}	1.3136×10^{-4}	6.1384×10^{-13}	1.7092×10^{-7}
0.5	2.6204×10^{-8}	2.4239×10^{-4}	5.7933×10^{-13}	6.3315×10^{-7}
0.6	2.3438×10^{-8}	1.7382×10^{-4}	5.3553×10^{-13}	3.2899×10^{-7}
0.7	1.9935×10^{-8}	3.9408×10^{-5}	4.8214×10^{-13}	3.2364×10^{-7}
0.8	1.5477×10^{-8}	2.7993×10^{-4}	4.1705×10^{-13}	3.7016×10^{-7}
0.9	9.7004×10^{-9}	3.5403×10^{-4}	3.3804×10^{-13}	3.4501×10^{-7}

Example 5.3. Consider the following nonlinear Volterra integral equa-

Table 6: Numerical results for example 2.

x	$N = 8$		Tau-collocation method [16]
	IBCM ($M = 5$)	BCM	for ($N = 15$)
0.0	1.9404×10^{-15}	7.3181×10^{-10}	2.2046×10^{-11}
0.2	1.8559×10^{-15}	3.3474×10^{-10}	1.8409×10^{-11}
0.4	1.7551×10^{-15}	1.0685×10^{-10}	8.5021×10^{-12}
0.6	1.6675×10^{-15}	2.2912×10^{-11}	1.8216×10^{-13}
0.8	1.3226×10^{-15}	1.5064×10^{-10}	2.8661×10^{-13}
1.0	1.3194×10^{-15}	4.6726×10^{-10}	4.8397×10^{-12}

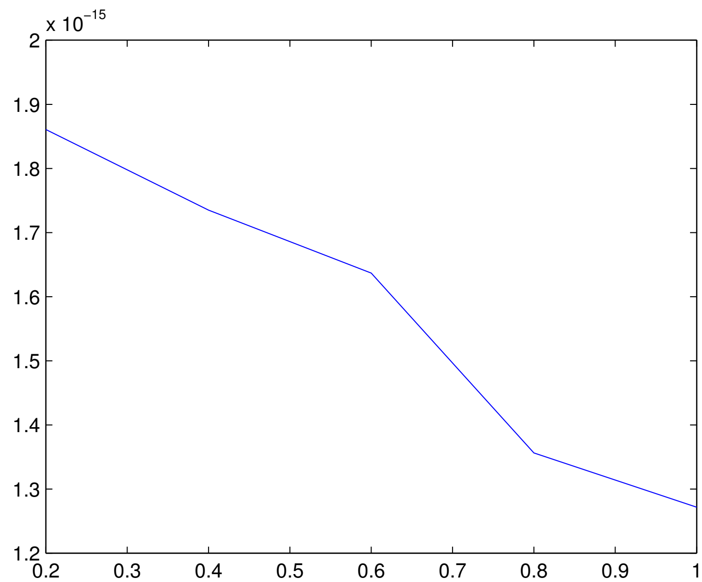

Figure 3: The absolute error of the IBCM with $M = 5$ and $N = 8$ for example 2.

tion of the second kind [16].

$$u(x) = -\frac{1}{10}x^4 + \frac{5}{6}x^2 + \frac{3}{8} + \int_0^x \frac{1}{2x} u(t)^2 \, dt, \tag{43}$$

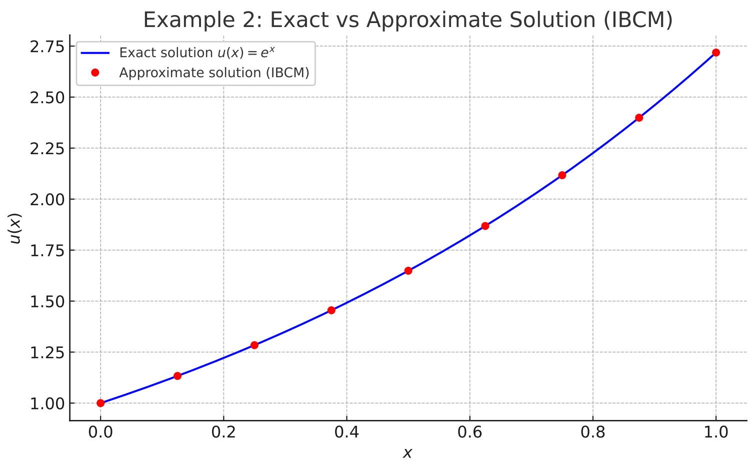

Figure 4: The exact and approximate solutions obtained by the IBCM for Example 2 with $M = 5$ and $N = 8$.

with the exact solution $u(x) = x^2 + \frac{1}{2}$.

The numerical results for this example are presented in Table 7. This table compares the numerical results obtained by the BCM and the IBCM (for $N = 11$) with those reported in [16] using the Tau-collocation method. The results clearly demonstrate that the proposed IBCM achieves significantly higher accuracy compared to the Tau-collocation method in [16].

The absolute error of the IBCM for $M = 5$ and $N = 11$ is illustrated in Figure 5. As shown in Figure 6, the approximate solution is almost indistinguishable from the exact solution, confirming the excellent performance of the IBCM.

The CPU time for this example was approximately 0.016 seconds for the BCM ($N = 11$) and 0.040 seconds for the IBCM ($N = 11, M = 5$).

Example 5.4. Consider the following nonlinear Volterra integral equation of the second kind [26].

$$u(x) = \sin(\pi x) + \int_0^x \sin(\pi t) \cos(\pi x)\, u(t)^3 \, dt, \tag{44}$$

Table 7: Numerical results for example 3.

x	$N=11$		Tau-collocation method [16]
	IBCM ($M=5$)	BCM	for ($N=11$)
0	2.8147×10^{-15}	1.3193×10^{-14}	2.2046×10^{-11}
0.2	1.0679×10^{-16}	9.5633×10^{-15}	1.8409×10^{-11}
0.4	1.7370×10^{-16}	1.0877×10^{-14}	8.5021×10^{-12}
0.6	1.7400×10^{-16}	1.0815×10^{-14}	1.8216×10^{-13}
0.8	9.5338×10^{-17}	9.7200×10^{-15}	2.8661×10^{-13}
1	1.1892×10^{-16}	1.3953×10^{-14}	4.8397×10^{-12}

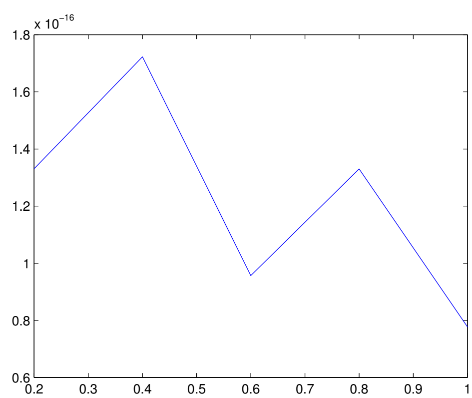

Figure 5: The absolute error of the IBCM with $M=5$ and $N=11$ for example 3.

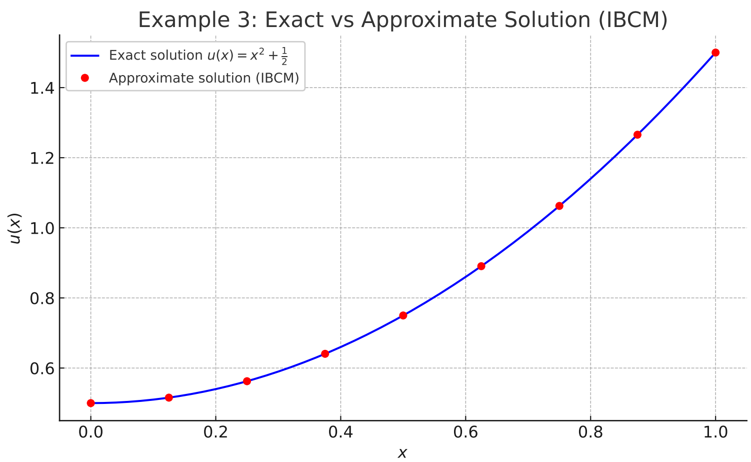

Figure 6: The exact and approximate solutions obtained by the IBCM for Example 3 with $M = 5$ and $N = 11$.

with the exact solution $u(x) = \sin(\pi x) + \frac{20-\sqrt{391}}{3}\cos(\pi x)$.

The numerical results for this example are presented in Tables 8 and 9. Table 8 compares the exact solution with the approximate solutions obtained by the IBCM and the modification of hat functions method [26]. The comparison indicates that the IBCM achieves higher accuracy.

Table 9 compares the absolute errors of the IBCM (with $M = 10$) and the classical BCM (with $N = 5$). As expected, increasing the polynomial degree improves accuracy in both methods. Furthermore, the IBCM consistently outperforms the BCM in terms of precision.

The absolute error of the IBCM for $M = 10$ and $N = 5$ is illustrated in Figure 7. As shown in Figure 8, the approximate solution closely follows the exact solution, confirming the effectiveness of the IBCM.

The CPU time for this example was approximately 0.020 seconds using the BCM ($N = 5$) and 0.049 seconds using the IBCM ($M = 10, N = 5$).

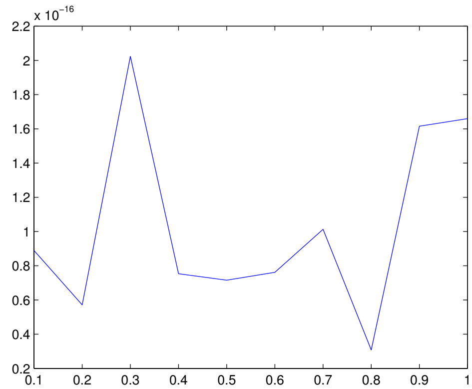

Figure 7: The absolute error of the IBCM with $M = 10$ and $N = 5$ for example 4.

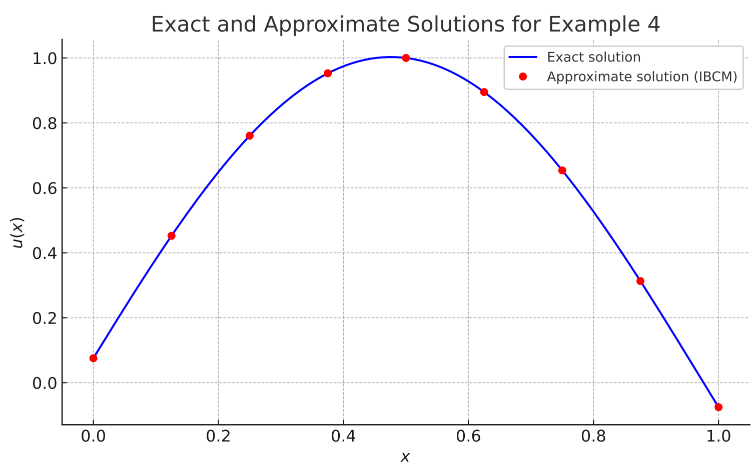

Figure 8: The exact and approximate solutions obtained by the IBCM for Example 4 with $M = 10$ and $N = 5$.

Table 8: Comparison of exact solution and approximate solution of example 4.

x	Exact solution	IBCM ($M = 10, N = 5$)	Modification of hat functions [26]
0.1	0.3807520	0.3807520	0.3807489
0.2	0.6488067	0.6488067	0.6488007
0.3	0.8533517	0.8533517	0.8533529
0.4	0.9743646	0.9743646	0.9743612
0.5	1.0000000	1.0000000	1.0000000
0.6	0.9277484	0.9277484	0.9277518
0.7	0.7646823	0.7646823	0.7646811
0.8	0.5267638	0.5267638	0.5267698
0.9	0.2372820	0.2372820	0.2372851

Table 9: The absolute error of the IBCM and BCM with $N = 5$ for example 4.

x	IBCM (M=10)	BCM
0.1	3.1892×10^{-8}	1.1337×10^{-5}
0.2	3.1164×10^{-8}	1.2295×10^{-6}
0.3	2.9999×10^{-8}	9.4557×10^{-6}
0.4	2.8363×10^{-8}	9.5335×10^{-6}
0.5	2.6204×10^{-8}	1.0443×10^{-6}
0.6	2.3438×10^{-8}	7.8075×10^{-6}
0.7	1.9935×10^{-8}	9.0435×10^{-6}
0.8	1.5477×10^{-8}	6.9274×10^{-7}
0.9	9.7004×10^{-9}	7.9773×10^{-6}

6 Conclusions

In this work, a novel numerical scheme, the IBCM, was introduced for solving VIEs of the second kind. This approach enhances the classical BCM by partitioning the domain into multiple subintervals and applying the collocation process locally within each subinterval. Collocation points are selected as the zeros of the SCPs, resulting in increased flexibility and computational efficiency. Unlike traditional methods that treat the entire domain simultaneously, the IBCM employs a piecewise strategy that significantly improves numerical stability and accuracy. A system of algebraic equations is formulated and solved on each subinterval, yielding a global approximate solution expressed as a piecewise Bernoulli expansion. Theoretical convergence of the method was established, and extensive numerical experiments were conducted to assess its performance. Results demonstrate that the IBCM consistently outperforms the classical BCM in terms of accuracy. Furthermore, increasing the number of subintervals M substantially enhances accuracy, even when the polynomial degree N is fixed. The proposed method shows excellent agreement with exact solutions and surpasses several existing methods in the literature. Given its robustness and adaptability, applying the piecewise collocation approach to other basis functions or collocation frameworks is recommended for future research.

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