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Some Remarks on the Weak Integral Closure of a Filtration Relative to a Module

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Abstract. In this paper, we will see some new results about the weak integral closure and the asymptotic prime divisors of a filtration relative to a module. Especially, we obtain some new results for the weak integral closure of a filtration relative to an injective module. For example, if $f = \{I_n\}_{n\geq 0}$ is a Noetherian filtration on a Noetherian ring R and E is an injective $R-$module, then it is shown that the asymptotic prime divisors of the filtration f relative to E can be characterized by the asymptotic prime divisors of the filtration f and also it is shown that the sequence $(Ass_R(R/Clos_R(f^{(n)}, E)))_{n\in\mathbb{N}}$ is ultimately constant.

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1 Introduction

Throughout this paper, R is a commutative ring with a non-zero identity and M is an $R-$module. Also, $\mathbb{N}$ denotes the set of all positive integers.

Let R be a Noetherian ring and I be an ideal of R. We denote the integral closure of I by I^-. Also, we know from [11, 2.7], the set

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$\{P : P \in Ass_R(R/(I^n)^-)\ for\ some\ n \geq 1\}$ is a finite set. This set is denoted by $\hat{A}^*(I)$ and every element of $\hat{A}^*(I)$ is called the asymptotic prime divisor of I.

We recall that for every Noetherian $R-$module M, $I^{-(M)}$ denotes the integral closure of an ideal I relative to M. Now, let R be a Noetherian ring and E be an injective $R-$module. We recall that $I^{*(E)}$ denotes the integral closure of an ideal I relative to E. For more information about them we can see [13] and [3].

A filtration $f = \{I_n\}_{n\geq 0}$ on a commutative ring R is a sequence of ideals of R such that $I_0 = R$, $I_{n+1} \subseteq I_n$, and $I_nI_m \subseteq I_{n+m}$ for all non-negative integers m and n. Let $f = \{I_n\}_{n\geq 0}$ and $g = \{J_n\}_{n\geq 0}$ be two filtrations on R. We know $f \leq g$, if $I_n \subseteq J_n$ for all n. Also, if $f = \{I_n\}_{n\geq 0}$ is a filtration on R and k is a positive integer, then we know $\{I_{nk}\}_{n\geq 0}$ is a filtration on R. This filtration is denoted by $f^{(k)}$. Further for every $n \geq 0$, $I_{n0} = R$ and this shows $f^{(0)}$ is also a filtration on R.

In this paper, we encounter instances where the filtration is required to be a Noetherian filtration. We know from [8, 3.2.1] and [9, 2.2.1], a filtration $f = \{I_n\}_{n\geq 0}$ on a Noetherian ring R is a Noetherian filtration if and only if there exists a positive integer e such that $I_{e+i} = I_eI_i$ for all $i \geq e$. For example, if R is a Noetherian ring and I is an ideal of R, then the $I-$adic filtration $f_I = \{I^n\}_{n\geq 0}$ on R is a Noetherian filtration with $e = 1$.

The weak integral closure of a filtration $f = \{I_n\}_{n\geq 0}$ on a commutative ring R is defined in [8]. Let $(I_k)_w$ is the set of all $x \in R$ such that x satisfies an equation of the form $x^m + a_1x^{m-1} + \cdots + a_m = 0$, where $a_i \in I_{ki}$ for every $1 \leq i \leq m$. We know from [8, 2.2], the sequence $\{(I_k)_w\}_{k\geq 0}$ of ideals of R is a filtration on R. This filtration is called the weak integral closure of the filtration $f = \{I_n\}_{n\geq 0}$ and is denoted by f_w. According to our notations in this paper, we prefer to denote the weak integral closure of the filtration f by f^-. Also, for every filtration $f = \{I_n\}_{n\geq 0}$ on a Noetherian ring R, every element of

$$A^-(f) = \{P : P \in Ass_R(R/(I_n)_w)\ for\ some\ n \geq 1\}$$

is called the asymptotic prime divisor of f. If $f = \{I_n\}_{n\geq 0}$ is a Noetherian filtration on a Noetherian ring R, then there exists a positive integer e such that $I_{e+i} = I_e I_i$ for all $i \geq e$. In [8, 3.3], it is proved that $A^{-}(f) = \hat{A}^{*}(I_e)$.

In [4], for a filtration $f = \{I_n\}_{n\geq 0}$ of ideals on R, the ideal $(I_1)_w$ is denoted by $Clos_R(f)$. Also, we have

$$(I_k)_w = Clos_R(f^{(k)}) \quad for\ every\ k \geq 0.$$

In particular, if R is a Noetherian ring and I is an ideal of R, then for the $I-$adic filtration $f_I = \{I^n\}_{n\geq 0}$ on R, we have $Clos_R(f_I^{(k)}) = (I^k)^{-}$ for every $k \geq 0$.

We now recall a useful notation introduced in [5]. An element $x \in R$ is said to be $M-$integral over a filtration $f = \{I_n\}_{n\geq 0}$, if there exists a positive integer m such that

$$x^m + a_1 x^{m-1} + \cdots + a_m \in (0 :_R M),$$

where $a_i \in I_i$ for every $1 \leq i \leq m$. In [5], it is shown that the set of all elements of R which are $M-$integral over a filtration $f = \{I_n\}_{n\geq 0}$ is an ideal. This ideal is denoted by $Clos_R(f, M)$.

Let I be an ideal of R. By [5, 2.6], for the $I-$adic filtration $f_I = \{I^n\}_{n\geq 0}$ on R and for every Noetherian $R-$module M, we have

$$Clos_R(f_I^{(k)}, M) = (I^k)^{-(M)}$$

for every $k \geq 0$.

Let $f = \{I_n\}_{n\geq 0}$ be a filtration of ideals on R and M be an $R-$module. In [7], it is proved that $\{Clos_R(f^{(k)}, M)\}_{k\geq 0}$ is a filtration on R. This filtration is called the weak integral closure of the filtration $f = \{I_n\}_{n\geq 0}$ relative to M and is denoted by $f^{-(M)}$. Also, if R is a Noetherian ring, then every element of

$$A^{-}(f, M) = \{P : P \in Ass_R(R/Clos_R(f^{(k)}, M))\ for\ some\ k \geq 1\}$$

is called the asymptotic prime divisor of f relative to M.

In this paper, we will obtain some results concerning the weak integral closure of a filtration relative to a module. In particular, we will obtain new results concerning the weak integral closure of a filtration relative to an injective module. For instance, we will see that if R is a Noetherian ring and I is an ideal of R, then for the $I-$adic filtration $f_I = \{I^n\}_{n\geq 0}$ on R, and for every injective $R-$module E, we have

$Clos_R(f_I^{(k)}, E) = (I^k)^{*(E)}$ for every $k \geq 0$. This implies that, the integral closure of an ideal I relative to a Noetherian module M and the integral closure of an ideal I relative to an injective module E on Noetherian rings are defined differently in [13] and [3], they fundamentally originate from the same concept.

2 The Asymptotic Prime Divisors of a Filtration Relative to a Module

In this section, we obtain some new facts about both the weak integral closure of a filtration relative to a module and the asymptotic prime divisors of a filtration relative to a module. In the remainder of this paper, for every $R-$module N, the symbol $E(N)$ denotes the injective envelope of N.

The following lemma is proven for special rings (for example, a Noetherian ring or a local ring), while the same proof can be stated without those special conditions (see [12, 2.1]).

Lemma 2.1. *Let R be a commutative (not necessarily Noetherian) ring and M be an $R-$module. Suppose $E = \bigoplus_{m \in Max(R)} E(R/m)$, where $Max(R)$ is the set of all maximal ideals of R. If $D(M) = Hom_R(M, E)$, then*

$$(0 :_R M) = (0 :_R D(M)).$$

Proof. This is clear $(0 :_R M) \subseteq (0 :_R D(M))$. So it is enough to show that $(0 :_R D(M)) \subseteq (0 :_R M)$. Let $t \in (0 :_R D(M))$. The inclusion map $\iota : tM \to M$ induces the $R-$homomorphism $\iota^* : Hom_R(M, E) \to Hom_R(tM, E)$ defined by $\iota^*(f) = f\iota$ for every $f \in Hom_R(M, E)$. Since $t \in (0 :_R D(M))$, we have $\iota^*(f) = 0$ for every $f \in Hom_R(M, E)$. We know from [1, 18-16], $E = \bigoplus_{m \in Max(R)} E(R/m)$ is a cogenerator. Then by [1, 18-14], we have $\iota = 0$ and so $tM = 0$. Therefore $t \in (0 :_R M)$ and so $(0 :_R D(M)) \subseteq (0 :_R M)$. □

Proposition 2.2. *Let $f = \{I_n\}_{n \geq 0}$ be a filtration on R and M be an $R-$module. Let $E = \bigoplus_{m \in Max(R)} E(R/m)$ and as above $D(M) =$*

$Hom_R(M, E)$. Then $f^{-(M)} = f^{-(D(M))}$. Further, if R is a Noetherian ring, then

$$A^{-}(f, M) = A^{-}(f, D(M)).$$

Proof. This is clear by 2.1. $\square$

Remark 2.3. Let $f = \{I_n\}_{n\geq 0}$ be a filtration on R and S be a multiplicatively closed subset of R. It is easy to see that, $\{S^{-1}I_n\}_{n\geq 0}$ is a filtration on $S^{-1}R$. We will show this filtration on $S^{-1}R$ by $S^{-1}f$.

Proposition 2.4. *Let $f = \{I_n\}_{n\geq 0}$ be a filtration on R and M be a finitely generated R-module. Then*

$$S^{-1}Clos_R(f, M) = Clos_{S^{-1}R}(S^{-1}f, S^{-1}M).$$

Proof. Since M is a finitely generated R-module, it is easy to see that

$$S^{-1}Clos_R(f, M) \subseteq Clos_{S^{-1}R}(S^{-1}f, S^{-1}M)$$

by [5, 2.6].

For converse inclusion, Let $\frac{x}{s} \in Clos_{S^{-1}R}(S^{-1}f, S^{-1}M)$. Then there exists a positive integer m such that

$$(\tfrac{x}{s})^m + \tfrac{a_1}{s_1}(\tfrac{x}{s})^{m-1} + \cdots + \tfrac{a_m}{s_m} \in (0 :_{S^{-1}R} S^{-1}M),$$

where $\frac{a_i}{s_i} \in S^{-1}I_i$ for every $1 \leq i \leq m$. Without loss of generality, we can assume that $a_i \in I_i$ for every $1 \leq i \leq m$. Since M is a finitely generated R-module, we can choose an element $t \in S$ such that $t(s_1 \cdots s_m)x \in Clos_R(f, M)$. Then

$$\tfrac{x}{s} = \tfrac{t(s_1\cdots s_m)x}{t(s_1\cdots s_m)s} \in S^{-1}Clos_R(f, M).$$

Thus $Clos_{S^{-1}R}(S^{-1}f, S^{-1}M) \subseteq S^{-1}Clos_R(f, M)$ and so the proof is completed. $\square$

Corollary 2.5. *Let $f = \{I_n\}_{n\geq 0}$ be a filtration on R and M be a finitely generated R-module. Then*

$$S^{-1}f^{-(M)} = (S^{-1}f)^{-(S^{-1}M)}.$$

Proof. This is clear by 2.4. $\square$

Theorem 2.6. *Let $f = \{I_n\}_{n\geq 0}$ be a filtration on a Noetherian ring R and M be a finitely generated R-module. Then*

$$A^-(S^{-1}f, S^{-1}M) = \{S^{-1}P : P \in A^-(f, M), P \cap S = \varnothing\}.$$

Proof. Let $k \geq 1$. By 2.4, we have $Clos_{S^{-1}R}(S^{-1}f^{(k)}, S^{-1}M) = S^{-1}Clos_R(f^{(k)}, M)$. Since

$$\frac{S^{-1}R}{S^{-1}Clos_R(f^{(k)}, M)} \simeq S^{-1}(\frac{R}{Clos_R(f^{(k)}, M)}),$$

we have

$$Ass_{S^{-1}R}(\frac{S^{-1}R}{Clos_{S^{-1}R}(S^{-1}f^{(k)}, S^{-1}M)}) = Ass_{S^{-1}R}(S^{-1}(\frac{R}{Clos_R(f^{(k)}, M)})).$$

We know

$$Ass_{S^{-1}R}(S^{-1}(\frac{R}{Clos_R(f^{(k)}, M)})) = \{S^{-1}P : P \in Ass_R(\frac{R}{Clos_R(f^{(k)}, M)}), P \cap S = \varnothing\}$$

and so the proof is completed. □

Theorem 2.7. *Let $f = \{I_n\}_{n\geq 0}$ be a filtration on R and M an $R-$module. Then*

$$(f^{-(M)})^- = (f^-)^{-(M)}.$$

So $Clos_R((f^{-(M)})^{(k)}) = Clos_R((f^-)^{(k)}, M)$ for every $k \geq 0$.

Proof. This immediately follows from [7, 2.6]. □

Remark 2.8. (See [15, 1.5].) Let I be an ideal of a commutative Noetherian ring R and M be a finitely generated $R-$module. We consider the $I-$adic filtration $f_I = \{I^n\}_{n\geq 0}$ on R. Then by 2.7, we have

$$(I^{-(M)})^- = Clos_R(((f_I)^{-(M)})^{(1)}) = Clos_R(((f_I)^-)^{(1)}, M) = (I^-)^{-(M)}.$$

Theorem 2.9. *Let $f = \{I_n\}_{n\geq 0}$ be a filtration on a Noetherian ring R and M be an $R-$module. Then*

$$A^-(f^{-(M)}) = A^-(f^-, M).$$

Proof. By 2.7, we have

$$Clos_R((f^{-(M)})^{(k)}) = Clos_R((f^-)^{(k)}, M)$$

for every $k \geq 1$. So

$$Ass_R(\frac{R}{Clos_R((f^{-(M)})^{(k)})}) = Ass_R(\frac{R}{Clos_R((f^-)^{(k)},M)})$$

for every $k \geq 1$. Now the proof is clear. □

Remark 2.10. Let M be an $R-$module. The ring $R/(0 :_R M)$ is a commutative ring. This ring is denoted by $\widetilde{R}$. Also for every ideal I of R, the ideal $(I + (0 :_R M))/(0 :_R M)$ of $\widetilde{R}$ is denoted by $\widetilde{I}$. It is useful for us to remember that if $f = \{I_n\}_{n\geq 0}$ is a filtration of ideals on R, then $\{\widetilde{I_n}\}_{n\geq 0}$ is a filtration of ideals on $\widetilde{R}$. This filtration is denoted by $\widetilde{f}$.

Theorem 2.11. *Let $f = \{I_n\}_{n\geq 0}$ be a Noetherian filtration of ideals on a Noetherian ring R and M be an $R-$module. Then there exists a positive integer e such that*

$$A^-(f, M) = \{(\widetilde{P})^c : \widetilde{P} \in \hat{A}^*(\widetilde{I}_e)\}$$

which $(\widetilde{P})^c$ is the contraction of the ideal $\widetilde{P}$ under the natural epimorphism $R \to R/(0 :_R M)$.

Proof. At first, we note that $P \in A^-(f, M)$ if and only if $\widetilde{P} \in A^-(\widetilde{f})$.

Now, since $(0 :_R M) \subseteq Clos_R(f^{(k)}, M)$ for every $k \geq 1$, we have $(0 :_R M) \subseteq P$ for every $P \in A^-(f, M)$ and so $(\widetilde{P})^c = P$ for every $P \in A^-(f, M)$. Thus we have

$$A^-(f, M) = \{(\widetilde{P})^c : \widetilde{P} \in A^-(\widetilde{f})\}.$$

Since, $f = \{I_n\}_{n\geq 0}$ is a Noetherian filtration of ideals on a Noetherian ring R, then $\widetilde{f} = \{\widetilde{I_n}\}_{n\geq 0}$ is a Noetherian filtration of ideals on the Noetherian ring $\widetilde{R}$. Now the proof is completed by [8, 3.3.3]. □

3 The Asymptotic Prime Divisors of a Filtration Relative to an Injective Module

In this section, we use the notation $I(\mathcal{P})$ for an ideal I and a subset $\mathcal{P}$ of $Spec(R)$, where $Spec(R)$ is the set of all prime ideals of R. To recall the concept of $I(\mathcal{P})$, we refer to [3]. In [6], it is shown that if $f=\{I_n\}_{n\geq 0}$ is a filtration on R and $\mathcal{P}$ is a subset of $Spec(R)$, then $\{I_n(\mathcal{P})\}_{n\geq 0}$ is also a filtration on R. This filtration is denoted by $f(\mathcal{P})$.

Theorem 3.1. *Let $f=\{I_n\}_{n\geq 0}$ be a filtration on a Noetherian ring R and E be an injective $R-$module. Then*

$$Clos_R(f^{(k)},E)=(Clos_R(f^{(k)}))(Ass_R(E)),$$

for every $k\geq 0$.

Proof. For $k=0$, it is clear that

$$Clos_R(f^{(0)},E)=R=(Clos_R(f^{(0)}))(Ass_R(E)).$$

So, let $k>0$. Let $E=\bigoplus_{\lambda\in\Lambda} E(R/P_\lambda)$. As we know, the set $\{P_\lambda:\lambda\in\Lambda\}$ is $Ass_R(E)$. Let $x\in Clos_R(f^{(k)},E)$. Then there exists a positive integer m such that

$$x^m+a_1x^{m-1}+\cdots+a_m\in(0:_R E),$$

where $a_i\in I_{ki}$ for every $1\leq i\leq m$. But this is valid if and only if

$$x^m+a_1x^{m-1}+\cdots+a_m\in(0:_R E(R/P_\lambda))$$

for every $\lambda\in\Lambda$. But by [14, 2.26] and [10, 18.4], $x^m+a_1x^{m-1}+\cdots+a_m\in(0:_R E(R/P_\lambda))$ if and only if there exists an element $s\in R-P_\lambda$ such that $s(x^m+a_1x^{m-1}+\cdots+a_m)=0$. But this means that $(\frac{x}{1})^m+\frac{a_1}{1}(\frac{x}{1})^{m-1}+\cdots+\frac{a_m}{1}=\frac{0}{1}$ where $\frac{a_i}{1}\in I_{ki}R_{P_\lambda}$ for every $1\leq i\leq m$. Then $x\in Clos_R(f^{(k)},E)$ if and only if $\frac{x}{1}\in(I_kR_{P_\lambda})_w$ for every $\lambda\in\Lambda$. By [6, 2.7], we have $(I_kR_{P_\lambda})_w=(I_k)_wR_{P_\lambda}$. Since $(I_k)_w=Clos_R(f^{(k)})$, we have $x\in Clos_R(f^{(k)},E)$ if and only if $\frac{x}{1}\in Clos_R(f^{(k)})R_{P_\lambda}$ for every $\lambda\in\Lambda$. So $x\in Clos_R(f^{(k)},E)$ if and only if $x\in(Clos_R(f^{(k)}))(P_\lambda)$ for every $P_\lambda\in Ass_R(E)$. This implies that

$$\begin{aligned}Clos_R(f^{(k)},E)&=\bigcap_{P_\lambda\in Ass_R(E)}(Clos_R(f^{(k)}))(P_\lambda)\\&=(Clos_R(f^{(k)}))(Ass_R(E)).\end{aligned}$$

□

Corollary 3.2. *Let $f = \{I_n\}_{n\geq 0}$ be a filtration on a Noetherian ring R and E be an injective $R-$module. Then*

$$f^{-(E)} = f^{-}(Ass_R(E)).$$

Proof. This is clear by 3.1. □

Remark 3.3. Let $f = \{I_n\}_{n\geq 0}$ be a filtration on a Noetherian ring R and E be an injective $R-$module. For every $n \geq 0$, let U_n contains all $x \in R$ such that

$$(0 :_E \sum_{i=1}^{t} x^{t-i} I_{ni}) \subseteq (0 :_E x^t)$$

for a positive integer t. Then $\{U_n\}_{n\geq 0}$ is a filtration on R by [6, 2.8]. This filtration is called the integral closure of the filtration $f = \{I_n\}_{n\geq 0}$ relative to E and is denoted by $f^{*(E)}$. But we know from [6, 3.1], $f^{*(E)} = f^{-}(Ass_R(E))$. Thus by 3.2, we have $f^{-(E)} = f^{*(E)}$ and so $Clos_R(f^{(k)}, E) = U_k$ for every $k \geq 0$.

Now, let I be an ideal of R and $f_I = \{I^n\}_{n\geq 0}$ be the $I-$adic filtration on R. Concerning this situation we have

$$Clos_R({f_I}^{(k)}, E) = U_k = (I^k)^{*(E)}$$

for every $k \geq 0$.

Theorem 3.4. *Let $f = \{I_n\}_{n\geq 0}$ be a filtration on a Noetherian ring R and E be an injective $R-$module. Then*

$$A^{-}(f, E) = \{P \in A^{-}(f) : P \subseteq Q \ for\ some\ Q \in Ass_R(E)\}.$$

Proof. We know

$$A^{-}(f, E) = \{P : P \in Ass_R(R/Clos_R(f^{(k)}, E)),\ for\ some\ k \geq 1\}.$$

By 3.1, we have

$$Ass_R(R/Clos_R(f^{(k)}, E)) = ass(Clos_R(f^{(k)}, E))$$

$$= ass((Clos_R(f^{(k)}))(Ass_R(E))),$$

for every $k \geq 1$. This shows that

$$Ass_R(R/Clos_R(f^{(k)}, E)) = \{P \in ass(Clos_R(f^{(k)})) : P \subseteq Q \ for\ some \\ Q \in Ass_R(E)\}$$

for every $k \geq 1$. Thus

$$A^-(f, E) = \{P \in A^-(f) : P \subseteq Q \ for\ some\ Q \in Ass_R(E)\}.$$

□

Corollary 3.5. *Let $f = \{I_n\}_{n\geq 0}$ be a Noetherian filtration of ideals on a Noetherian ring R and E be an injective $R-$module. Let e be a positive integer such that $I_{e+i} = I_e I_i$ for all $i \geq e$. Then*

$$A^-(f, E) = \{P \in \hat{A}^*(I_e) : P \subseteq Q \ for\ some\ Q \in Ass_R(E)\}.$$

Proof. Since $f = \{I_n\}_{n\geq 0}$ is a Noetherian filtration, we have $A^-(f) = \hat{A}^*(I_e)$ by [8, 3.3.3]. Now the proof is clear by 3.4. □

Corollary 3.6. *Let $f = \{I_n\}_{n\geq 0}$ be a Noetherian filtration on a Noetherian ring R and E be an injective $R-$module. Let e be a positive integer such that $I_{e+i} = I_e I_i$ for all $i \geq e$.*

(a) For every $q \geq 1$ and for every $n \geq e$, we have

$$Ass_R(R/Clos_R(f^{(qe)}, E)) \subseteq Ass_R(R/Clos_R(f^{(qe+n)}, E)).$$

(b) For every $q \geq 1$ and for every fixed r that $0 \leq r \leq e-1$

$$Ass_R(R/Clos_R(f^{(qe+r)}, E)) \subseteq Ass_R(R/Clos_R(f^{((q+1)e+r)}, E)).$$

(c) $Ass_R(R/Clos_R(f^{(n)}, E)) = A^-(f, E)$ for all large n. In other words the sequence $(Ass_R(R/Clos_R(f^{(n)}, E)))_{n\in\mathbb{N}}$ is ultimately constant.

Proof. (a) By [8, 3.4.1], we have

$$ass(Clos_R(f^{(qe)})) \subseteq ass(Clos_R(f^{(qe+n)})),$$

for every $q \geq 1$ and for every $n \geq e$. Now (a) is clear by 3.1.

(b) By [8, 3.4.2], we have

$$ass(Clos_R(f^{(qe+r)})) \subseteq ass(Clos_R(f^{((q+1)e+r)})),$$

for every $q \geq 1$ and for every fixed r that $0 \leq r \leq e-1$. Now (b) is clear by 3.1.

(c) By 3.1, we have

$$Ass_R(R/Clos_R(f^{(k)}, E)) = ass((Clos_R(f^{(k)}))(Ass_R(E))),$$

for every $k \geq 0$. By [8, 3.4.3], $ass(Clos_R(f^{(n)})) = \hat{A}^*(I_e)$ for all large n. Then for all large n,

$$ass((Clos_R(f^{(n)}))(Ass_R(E))) = \{P \in \hat{A}^*(I_e) : P \subseteq Q \ for \ some \ Q \in Ass_R(E)\}.$$

Now (c) is clear by 3.5. □

The following remark indicates a well known result for the integral closure of an ideal relative to an injective module.

Remark 3.7. (See [2, 3.2].) Let I be an ideal of a Noetherian ring R and E be an injective $R-$module. Let $f_I = \{I^n\}_{n \geq 0}$ be the I-adic filtration on R. By 3.3, we have

$$Clos_R((f_I)^{(k)}, E)) = (I^k)^{*(E)},$$

for every $k \geq 0$. Using each 3.6(a) or 3.6(b), can imply that

$$Ass_R(R/(I^q)^{*(E)}) \subseteq Ass_R(R/(I^{q+1})^{*(E)}),$$

for every $q \geq 1$. Then the sequence of sets $(Ass_R(R/(I^n)^{*(E)}))_{n \in \mathbb{N}}$ is increasing. Also 3.6(c), shows the sequence of sets $(Ass_R(R/(I^n)^{*(E)}))_{n \in \mathbb{N}}$ is ultimately constant with the ultimately constant value $A^-(f_I, E)$. But

$$A^-(f_I, E) = \{P \in \hat{A}^*(I) : P \subseteq Q \ for \ some \ Q \in Ass_R(E)\},$$

by 3.5.

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