Journal of Mathematical Extension
Journal Pre-proof
ISSN: 1735-8299
URL: http://www.ijmex.com
Original Research Paper

Symmetry Reductions and Exact Solutions for Caputo-Type Fractional Differential Equations and Nonlinear Fractional Wave-Diffusion Systems

F. Bahrami*
University of Tabriz

R. Najafi
Maku Branch, Islamic Azad University

P. Vafadar
University of Tabriz

Abstract. We investigate a class of (2+1)-dimensional nonlinear fractional wave-diffusion equations with Caputo derivatives using Lie symmetry analysis. First, we derive a prolongation formula for the infinitesimal generators of the symmetry group acting on Caputo-type fractional derivatives of arbitrary order α generalizing earlier results for Riemann-Liouville derivatives by Gazizov et al. [13]. By constructing an optimal system of subalgebras, we classify all symmetry reductions for this class of equations. As a key application, we obtain exact invariant solutions and solitary wave solutions for the (2+1)-dimensional fractional Burgers equation. This framework opens avenues for extending Lie analysis to other fractional PDEs in continuum mechanics, plasma physics, and transport processes where Caputo operators play a crucial role.

Received: January 2008; Accepted: February 2009
*Corresponding Author

AMS Subject Classification: 70G65; 35R11; 58J70
Keywords and Phrases: Fractional wave-diffusion equation, Lie symmetry method, Fractional derivative of Caputo type, Exact solutions

1 Introduction

Fractional wave-diffusion equations have emerged as powerful mathematical tools for describing various biological and physical systems that exhibit anomalous diffusion. These equations effectively model systems with long-range interactions and hereditary mechanisms, which are not adequately captured by classical integer-order models. This study focuses on a class of (2+1)-dimensional nonlinear fractional wave-diffusion equations of the Caputo type and their associated Lie symmetry groups [20, 9]. Fractional wave-diffusion equations also play a significant role in understanding diffusion processes in heterogeneous media. For instance, they have been instrumental in analyzing diffusion data from human brain tissue, where the heterogeneous structure necessitates a more sophisticated mathematical model to capture the complex diffusion patterns accurately. Notable contributions to the development and analysis of fractional wave-diffusion equations include the work of Bueno et al. [6], Liu et al. [22], and Magin [26].
While linear fractional wave-diffusion equations have been extensively studied, the same cannot be said for one-dimensional nonlinear fractional wave equations. The complexities associated with nonlinearities have posed significant challenges, resulting in limited research on this subject. However, some notable contributions have been made by researchers such as Beckers, Luchko, and Sakamoto (see [5, 23, 36]). To address these challenges, various methods have been explored and developed, including Chebyshev collocation methods, finite element approaches, CDV wavelet basis, finite difference methods, series expansion methods, and Laplace transform methods (as discussed in [11, 1, 8, 14, 38, 19]). Despite the progress made in studying nonlinear fractional wave equations, there is still much work to be done in this area. Continued research into the theory and applications of these equations is essential for a deeper understanding of the underlying mathematical structure and their potential applications in various fields.

From a practical perspective, the Caputo derivative holds greater significance as it allows for initial conditions similar to those found in integer-order differential equations. This compatibility simplifies the modeling of real-world phenomena and makes it easier to incorporate known initial states into problem formulations. In contrast, the Riemann-Liouville (R-L) derivative requires initial conditions involving limit values of fractional derivatives at $t = 0$. This constraint can complicate the modeling process and makes it more challenging to apply R-L fractional derivatives in practical scenarios. To apply Lie symmetry analysis to systems of differential or integral equations, we need to extend the base space representing the independent and dependent variables to include the derivative or integral operators present in the system. In the context of fractional-order differential equations, Gazizov et al. [13] derived an explicit prolongation formula for the R-L fractional derivative operator for the first time. Most studies in this field that have obtained exact solutions using symmetry methods rely on the R-L fractional derivative. Although the prolongation formula for the Caputo fractional derivative operator has been derived in [13] for $0 < \alpha < 1$, an explicit general formula was previously unavailable. This gap has limited the application of Lie symmetry methods to Caputo-type fractional equations. In this paper, we present such an explicit formula, addressing this long-standing challenge. Additionally, the prolongation formula for integral operators has been established in [2]. This method has been applied to some partial differential equations, as demonstrated by researchers such as [18, 3, 37, 21, 28, 34]. Classic Lie symmetry analysis has been applied to differential equations involving fractional R-L derivatives by authors in [16, 35, 15, 33]. Furthermore, the nonclassical method, which is an extension of the Lie symmetry method, has been applied to R-L fractional differential equations in works such as [32, 29, 4, 30, 31].

Our approach is based on the application of Leibniz's formula and the Taylor's expansion of fractional derivatives, with detailed proofs provided in the Appendix. We also present applicable expressions of this formula for engineering purposes, particularly for cases where $0 < \alpha < 1$ and $1 < \alpha < 2$. By extending the existing literature on Caputo-type fractional derivatives, our findings may contribute to the broader application of Lie symmetry analysis in studying FDEs and facilitate the derivation of exact solutions and conservation laws for these equations. In addition to deriving the prolongation formula for Caputo-type fractional derivatives, this study also compares the invariants of equations involving Caputo-type derivatives with those involving R-L derivatives. We present a specific example where using Caputo-type derivatives leads to the discovery of new invariants, which subsequently yield new solutions for the equation. The results of this study provide a general framework for applying Lie symmetry analysis to fractional differential equations with Caputo-type derivatives. A key contribution of this work is the application of the obtained results to fractional wave-diffusion equations with Caputo-type derivatives. To the best of our knowledge, Lie symmetry analysis has not yet been applied to these equations with Caputo-type derivatives. Our study derives invariants of a class of two-dimensional nonlinear wave-diffusion equations, leading to a class of exact solutions. This approach demonstrates the utility of Lie symmetry methods in studying nonlinear fractional wave-diffusion equations. This study is structured as follows: First, we establish the necessary background by revisiting fundamental concepts on fractional integrals and derivatives. This lays the foundation for the subsequent analysis. Next, we derive a prolongation formula for fractional derivatives of the Caputo type within the context of a one-parameter Lie group of transformations on a specific domain. Building upon these results, we then investigate the symmetry properties of (2+1)-dimensional fractional nonlinear wave-diffusion equations. Finally, we showcase the applicability of our symmetry analysis approach by deriving exact solutions for a particular case of the (2+1)-dimensional Burger's equation, thus demonstrating its potential for solving nonlinear fractional wave-diffusion equations. In conclusion, this study provides valuable insights into the application of Lie symmetry analysis to fractional wave-diffusion equations with Caputo-type derivatives. Our findings contribute to the broader understanding of these complex mathematical models and hold potential for further applications in various fields, such as physics and engineering.

2 An Overview on Fractional Derivatives

This section highlights key aspects of fractional derivatives for a given function. It is important to note that several non-equivalent definitions for fractional derivatives exist, including the Riemann-Liouville, Grunwald-Letnikov, Caputo, Riesz, and Miller and Ross fractional derivatives [27]. Our focus in this paper is on the R-L and Caputo type fractional derivatives. These two fractional derivatives are chosen for their distinct and complementary characteristics. While the R-L derivative extends classical integer-order calculus to non-integer orders in a natural way, the Caputo derivative allows for direct inclusion of initial conditions in the problem formulation. This advantage makes the Caputo derivative particularly appealing for various applications across disciplines such as physics and engineering, [20, 9].

Definition 2.1. [9] If $m \in \mathbb{N}$ and $0 \leq m-1 < \alpha < m$, the R-L and Caputo fractional derivatives of order α are defined as

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} D_t^m \int_0^t (t-s)^{m-\alpha-1} f(s) ds,$$

$${}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} D_s^m f(s) ds,$$

provided the right-hand sides integrals exist. Here Γ is the Gamma function and $D_t^m = \frac{d^m}{dt^m}$.
We state the Leibniz's formula for R-L fractional derivative.

Proposition 2.2. *[9] Let $\alpha > 0$, and assume that f and g are analytic on $(-h, h)$ with some $h > 0$. Then,*

$$D_t^{\alpha}\left[f(t)g(t)\right] = \sum_{n=0}^{\infty} \binom{\alpha}{n} D_t^n f(t) D_t^{\alpha-n} g(t), \binom{\alpha}{n} = \frac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{n!\Gamma(1-\alpha)}, \alpha > 0.$$

Proposition 2.3. *Let f be analytic in $(-h, h)$ for some $h > 0$, and $m-1 < \alpha < m$, $m \in \mathbb{N}$, then*

$${}^{C}D_t^{\alpha} f(t) = \sum_{k=m}^{\infty} \binom{\alpha-m}{k-m} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} D_t^k f(t),$$

for $0 < t < h/2$.

Proof. The proof follows a similar approach to that of the R-L fractional derivative found in [9]. Let recall Definition 2.1

$$^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} D_s^m f(s) \mathrm{d}s,$$

then we employ the Taylor's expansion of $D_s^m f(s)$ with respect to s around $s = t$, and substitute the result into the formula above, and compute the integral. Thus we deduce

$$^{C}D_t^{\alpha} f(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{m+k-\alpha}}{k!(m+k-\alpha)\Gamma(m-\alpha)} D_t^{m+k} f(t).$$

Since $\frac{(-1)^k}{k!(m+k-\alpha)\Gamma(m-\alpha)} = \binom{\alpha-m}{k} \frac{1}{\Gamma(k+m+1-\alpha)}$, then it implies the desired result. $\square$

Proposition 2.4. *[9] Let $m \geq 1$ and $m-1 < \alpha < m$. Assume that f is such that both $D_t^{\alpha} f$ and $^{C}D_t^{\alpha} f$ exist. Then*

$$D_t^{\alpha} f = {}^{C}D_t^{\alpha} f + \sum_{k=0}^{m-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} D_t^k f|_{t=0}.$$

Remark 2.5. When the function f is multivariable, $D_t^{\alpha} f(x, y, t, u)$ represents the partial fractional derivative of f with respect to t, where the other variables x, y and u are held constant. If u is a dependent function of x, y, and t, then $\mathcal{D}_x^{\alpha} f$, $\mathcal{D}_y^{\alpha} f$, and $\mathcal{D}_t^{\alpha} f$ denote the total fractional derivatives of f with respect to x, y, and t, respectively. $\mathcal{D}_x f$, $\mathcal{D}_y f$, and $\mathcal{D}_t f$ represent the total derivatives of f with respect to x, y, and t, respectively.

3 On the Prolongation Formula to the Fractional Operator of the Caputo Type

To apply the Lie symmetry method to differential or integral equations, a crucial step is deriving the prolongation formula for the associated operators. While this has been established for integer-order differential operators and R-L fractional derivatives in [13], we focus here on extending these results to Caputo fractional differential operators of arbitrary order for a given one-parameter Lie group of transformations on a domain. Our primary approach relies on leveraging Leibniz-type formulas for fractional differentiation. We now consider the one-parameter Lie group, which acts on an open subset M within the space $\mathbb{R}^3 \times \mathbb{R}$:

$$\begin{aligned} \bar{x} &= x + \varepsilon\xi(x,y,t,u) + O(\varepsilon^2), \qquad \bar{y} = y + \varepsilon\eta(x,y,t,u) + O(\varepsilon^2), \\ \bar{t} &= t + \varepsilon\tau(x,y,t,u) + O(\varepsilon^2), \qquad \bar{u} = u + \varepsilon\varphi(x,y,t,u) + O(\varepsilon^2), \end{aligned} \tag{1}$$

with the infinitesimal generator

$$V = \xi(x,y,t,u)\frac{\partial}{\partial x} + \eta(x,y,t,u)\frac{\partial}{\partial y} + \tau(x,y,t,u)\frac{\partial}{\partial t} + \varphi(x,y,t,u)\frac{\partial}{\partial u}, \tag{2}$$

where ε is the group parameter. The one-parameter Lie group of transformations is prolonged to the m-th partial derivative with respect to t as follows:

$$D_t^m \bar{u}(\bar{x},\bar{y},\bar{t}) = D_t^m u(x,y,t) + \varepsilon\varphi^{(m,t)} + O(\varepsilon^2), \qquad m = 0,1,2,\cdots$$

where

$$\varphi^{(m,t)} = \mathcal{D}_t^m(\varphi - \xi u_x - \eta u_y - \tau u_t) + \xi D_t^m u_x + \eta D_t^m u_y + \tau D_t^{m+1} u.$$

and $\mathcal{D}_t$ is the total derivative operator with respect to t.
We now discuss the prolongation of a one-parameter group of transformations to the Caputo fractional derivative ${}^C D_t^\alpha u(x,y,t)$, where $\alpha > 0$ and $m-1 < \alpha < m$. It is important to note that in Definition 2.1, the lower limit of the integral is fixed, and therefore should be invariant under the group of transformations (1), i.e.

$$\tau(x,y,0,u(x,y,0)) = 0, \qquad (x,y) \in \mathbb{R}^2.$$

With the previous analysis completed, we are now prepared to construct the extension formula for the fractional derivatives of the Caputo type.

Theorem 3.1. *Let $u(x,y,t)$ be analytic in $t \in (-h,h)$ and $(x,y) \in \mathbb{R}^2$ for some $h > 0$. Let (1) be the one-parameter Lie group of transformations acting on $\mathbb{R}^4$ with the corresponding infinitesimal generator V. Then for $m-1 < \alpha < m$, $m \in \mathbb{N}$, we have*

$$ {}^{C}D_{\bar{t}}^{\alpha}\bar{u}(\bar{x},\bar{y},\bar{t}) = {}^{C}D_{t}^{\alpha}u(x,y,t) + \varepsilon\varphi_{C}^{(\alpha,t)} + O(\varepsilon^2), $$

with the α-th prolongation of V,

$$ Pr_{C}^{(\alpha)}V = V + \varphi_{C}^{(\alpha,t)}\frac{\partial}{\partial\, {}^{C}D_{t}^{\alpha}u}, $$

where

$$ \varphi_{C}^{(\alpha,t)} = {}^{C}\mathcal{D}_{t}^{\alpha}\varphi - {}^{C}\mathcal{D}_{t}^{\alpha}\left(\xi u_x\right) + \xi\, {}^{C}D_{t}^{\alpha}u_x - {}^{C}\mathcal{D}_{t}^{\alpha}\left(\eta u_y\right) + \eta\, {}^{C}D_{t}^{\alpha}u_y - \mathcal{D}_{t}^{\alpha}\left(\tau u_t\right) $$
$$ + \tau D_{t}^{\alpha}u_t + \sum_{k=1}^{m-1}\frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)}\mathcal{D}_{t}^{k}(\tau u_t)\Big|_{t=0} - \tau\sum_{k=0}^{m-2}\frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)}D_{t}^{k+1}u\Big|_{t=0}. $$

Proof. Due to the involved nature of the proof and the need for additional technical considerations, we have decided to present the complete proof in the Appendix. This allows for a more focused presentation of the main ideas in the main text while still providing the necessary details for interested readers to review and understand the underlying mathematics. □

It is worth noting that this expression for the prolongation formula involves $(m-1)$ initial values of the function u, which is a significant difference compared to the R-L prolongation formula [13].

Remark 3.2. In the case $0 < \alpha < 1$, $\varphi_{C}^{(\alpha,t)}$ can be represented as

$$ \begin{aligned} \varphi_{C}^{(\alpha,t)} &= {}^{C}\mathcal{D}_{t}^{\alpha}\varphi - {}^{C}\mathcal{D}_{t}^{\alpha}\left(\xi u_x\right) + \xi\, {}^{C}D_{t}^{\alpha}u_x \\ &\quad - {}^{C}\mathcal{D}_{t}^{\alpha}\left(\eta u_y\right) + \eta\, {}^{C}D_{t}^{\alpha}u_y - \mathcal{D}_{t}^{\alpha}\left(\tau u_t\right) + \tau D_{t}^{\alpha}u_t, \end{aligned} $$

which is an equivalent representation of the formula given by Gazizov et al. [13].

For practical purposes, we provide explicit representations of $\varphi_{C}^{(\alpha,t)}$ separately for the cases $0 < \alpha < 1$ and $1 < \alpha < 2$:

- $0 < \alpha < 1$

$$\varphi_C^{(\alpha,t)} = {}^C D_t^\alpha \varphi + \varphi_u {}^C D_t^\alpha u - u\, {}^C D_t^\alpha \varphi_u + \nu - {}^C \mathcal{D}_t^\alpha (\xi u_x) + \xi\, {}^C D_t^\alpha u_x$$
$$- {}^C \mathcal{D}_t^\alpha (\eta u_y) + \eta\, {}^C D_t^\alpha u_y - \alpha \mathcal{D}_t \tau\, {}^C D_t^\alpha u + \sum_{k=1}^{\infty} \left[\binom{\alpha}{k} D_t^k \varphi_u \right.$$
$$\left. - \binom{\alpha}{k+1} \mathcal{D}_t^{k+1} \tau \right] D_t^{\alpha-k} u - \sum_{k=1}^{\infty} \binom{\alpha}{k+1} \frac{\Theta(x,y) t^{k-\alpha}}{\Gamma(k+1-\alpha)} \mathcal{D}_t^{k+1} \tau$$
$$+ \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \Big[\varphi_u(x,y,t,u) - \varphi_u(x,y,0,u) \Big] u(x,y,0).$$

- $1 < \alpha < 2$

$$\varphi_C^{(\alpha,t)} = {}^C D_t^\alpha \varphi + \varphi_u {}^C D_t^\alpha u - u\, {}^C D_t^\alpha \varphi_u + \nu - {}^C \mathcal{D}_t^\alpha (\xi u_x) + \xi\, {}^C D_t^\alpha u_x$$
$$- {}^C \mathcal{D}_t^\alpha (\eta u_y) + \eta\, {}^C D_t^\alpha u_y - \alpha \mathcal{D}_t \tau\, {}^C D_t^\alpha u + \sum_{k=1}^{\infty} \left[\binom{\alpha}{k} D_t^k \varphi_u \right.$$
$$\left. - \binom{\alpha}{k+1} \mathcal{D}_t^{k+1} \tau \right] D_t^{\alpha-k} u - \sum_{k=2}^{\infty} \binom{\alpha}{k+1} \frac{\Theta(x,y) t^{k-\alpha}}{\Gamma(k+1-\alpha)} \mathcal{D}_t^{k+1} \tau$$
$$+ \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \Big[\varphi_u(x,y,t,u) - \varphi_u(x,y,0,u) - \frac{t}{1-\alpha} \varphi_{tu}(x,y,0,u)$$
$$- \frac{1}{2} \alpha t \mathcal{D}_t^2 \tau(x,y,t,u) \Big] u(x,y,0) + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \Big[\varphi_u(x,y,t,u) \tag{3}$$
$$- \varphi_u(x,y,0,u) - \alpha \mathcal{D}_t \tau(x,y,t,u)$$
$$+ \frac{\alpha-1}{t} \tau(x,y,t,u) + \tau_t(x,y,0,u) \Big] u_t(x,y,0) r \tag{4}$$
$$+ \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \tau_u(x,y,0,u) u_t^2(x,y,0). \tag{5}$$

where $\Theta(x,y)$ is arbitrary function. Now, we intend to compare the Lie symmetries of the Caputo FDEs with those of the R-L FDEs by providing a simple example:

Example 3.3. (the comparison of R-L and Caputo)
Let $1 < \alpha < 2$. We consider FDE ${}^C D_t^\alpha u(x,y,t) = 0$. Suppose that

(2) is an infinitesimal generator of this equation. By utilizing the α-prolongation of the vector field V on the given equation, we can determine the corresponding infinitesimals:

$$\xi(x,y,t,u) = a(x,y), \eta(x,y,t,u) = b(x,y), \tau(x,y,t,u) = c(x,y)t^2 + d(x,y)t,$$
$$\varphi(x,y,t,u) = (\alpha-1)c(x,y)tu - (\alpha-2)c(x,y)u_t(x,y,0)t^2 + A(x,y,t) + B(x,y,u),$$

where a, b, c, d, A, B are arbitrary functions and ${}^C D_t^\alpha A(x,y,t) = 0$. Now we consider the same equation with R-L fractional derivatives i.e. $D_t^\alpha u(x,y,t) = 0$. We obtain the invariance with

$$\xi(x,y,t,u) = a(x,y), \eta(x,y,t,u) = b(x,y), \tau(x,y,t,u) = c(x,y)t^2 + d(x,y)t,$$
$$\varphi(x,y,t,u) = (\alpha-1)c(x,y)tu + A(x,y,t) + e(x,y)u,$$

where a, b, c, d, e, A are arbitrary functions and $D_t^\alpha A(x,y,t) = 0$.

It is evident that the Lie group of transformations admitted by these equations is not the same. This is due to the fact that, while the basic part of the corresponding infinitesimals remain the same(ξ, η, τ), there are differences in other components. For instance, considering the infinitesimal generator

$$V = t^2 \frac{\partial}{\partial t} + \left[(\alpha-1)tu - (\alpha-2)u_t(x,y,0)t^2\right] \frac{\partial}{\partial u}$$

for the equation ${}^C D_t^\alpha u(x,y,t) = 0$, we obtain the invariant solution $u(x,y,t) = u_t(x,y,0)t$ which depends on the initial value.

4 The Classical Lie Symmetry Analysis to the 2-D Time Fractional Nonlinear Wave Equation

In this section, we will focus on a particular class of nonlinear (2+1)-dimensional wave equations that can be described by the following form:

$$\Delta : {}^C D_t^\alpha u(x,y,t) - (f(u)u_x)_x - (g(u)u_y)_y - h(u) = 0, \tag{6}$$

where ${}^C D_t^\alpha$ denotes the Caputo fractional derivative of order $1 < \alpha < 2$. Assume that (2) is an admitted infinitesimal generator for a point symmetry of the fractional wave equation, (6). To apply classical Lie symmetry analysis to Eq. (6), we require that the set of solutions of Eq. (6) is invariant under the group of point transformations (1). This leads to the determining equations, which are obtained by requiring the infinitesimal criterion of invariance: $Pr_C^{(\alpha,2)}V(\Delta)|_{\Delta=0} = 0$, where

$$
\begin{aligned}
Pr_C^{(\alpha,2)}V = V &+ \varphi^{(1,x)}\frac{\partial}{\partial u_x} + \varphi^{(1,y)}\frac{\partial}{\partial u_y} + \varphi^{(2,xx)}\frac{\partial}{\partial u_{xx}} + \varphi^{(2,xy)}\frac{\partial}{\partial u_{xy}} \\
&+ \varphi^{(2,yy)}\frac{\partial}{\partial u_{yy}} + \varphi_C^{(\alpha,t)}\frac{\partial}{\partial\, {}^C D_t^{\alpha} u},
\end{aligned}
$$

and V is the admitted infinitesimal generator (2). Applying $Pr_C^{(\alpha,2)}V$ to (6), we find infinitesimal criterion

$$
\begin{aligned}
&\varphi_C^{(\alpha,t)} - \varphi\left[f''(u)u_x^2 + f'(u)u_{xx} + g''(u)u_y^2 + g'(u)u_{yy} + h'(u)\right] \\
&- 2\varphi^{(1,x)}f'(u)u_x - 2\varphi^{(1,y)}g'(u)u_y - \varphi^{(2,xx)}f(u) - \varphi^{(2,yy)}g(u)|_{\Delta=0} = 0.
\end{aligned}
$$

The coefficients functions $\varphi^{(1,x)}, \varphi^{(2,xy)}$ are given by the following formulas

$$
\begin{aligned}
\varphi^{(1,x)} &= \mathcal{D}_x(\varphi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xx} + \eta u_{xy} + \tau u_{xt}, \\
\varphi^{(2,xy)} &= \mathcal{D}_x\mathcal{D}_y(\varphi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xxy} + \eta u_{xyy} + \tau u_{xyt},
\end{aligned}
$$

and $\varphi_C^{(\alpha,t)}$ is as in (5). As a result we arrive at the following set of determining equations for $\xi, \tau, \varphi, f, g, h$.

$$
\begin{aligned}
&h\varphi_u - \alpha\tau_t h - f\varphi_{xx} - g\varphi_{yy} - \varphi h_u + A_1 = 0,\\
&{}^C D_t^\alpha \varphi - u\, {}^C D_t^\alpha \varphi_u - {}^C\mathcal{D}_t^\alpha(\xi u_x) + \xi\, {}^C\mathcal{D}_t^\alpha u_x - {}^C\mathcal{D}_t^\alpha(\eta u_y) + \eta\, {}^C\mathcal{D}_t^\alpha u_y = 0,\\
&f\xi_{xx} - 2f\varphi_{xu} + g\xi_{yy} - 2f'\varphi_x = 0, \quad f\eta_{xx} + g\eta_{yy} - 2g\varphi_{yu} - 2g'\varphi_y = 0,\\
&-\varphi_u f' - \alpha\tau_t f' + 2f\xi_{xu} - f\varphi_{uu} + 2f'\xi_x - \varphi f'' = 0, \quad -\alpha\tau_t f + 2f\xi_x - f'\varphi = 0,\\
&-\varphi_u g' - \alpha\tau_t g' + 2g\eta_{yu} - g\varphi_{uu} + 2g'\eta_y - \varphi g'' = 0, \quad -\alpha\tau_t g + 2g\eta_y - \varphi g' = 0,\\
&2f\eta_{xu} + 2g\xi_{yu} + 2f'\eta_x + 2g'\xi_y = 0, \quad f\eta_{uu} + 2f'\eta_u = 0, \quad 2f\eta_u = 0,\\
&2f\eta_x + 2g\xi_y = 0, \quad f\xi_{uu} + 2f'\xi_u = 0, \quad 3f\xi_u = 0, \quad f\tau_{uu} + 2f'\tau_u - \alpha f'\tau_u = 0,\\
&2f\tau_u = 0, \quad 2f\tau_{xu} + 2f'\tau_x = 0, \quad f\tau_u - \alpha f\tau_u = 0, \quad f\tau_{xx} + g\tau_{yy} - \alpha h\tau_u = 0,\\
&g\eta_{uu} + 2g'\eta_u = 0, \quad 2f\tau_x = 0, \quad 3g\eta_u = 0, \quad g\xi_{uu} + 2g'\xi_u = 0, \quad 2g\xi_u = 0,\\
&g\tau_{uu} + 2g'\tau_u - \alpha g'\tau_u = 0, \quad 2g\tau_u = 0, \quad 2g\tau_{yu} + 2g'\tau_y = 0, \quad g\tau_u - \alpha g\tau_u = 0,\\
&2g\tau_y = 0, \quad \binom{\alpha}{k} D_t^k \varphi_u - \binom{\alpha}{k+1}\mathcal{D}_t^{k+1}\tau = 0, \quad \mathcal{D}_t^{k+1}\tau = 0 \qquad (7)
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 = &\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\{\varphi_u(x,y,t,u)u(x,y,0) - \varphi_u(x,y,0,u)u(x,y,0) - \tau u_t(x,y,0)\}\\
&+ \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\{\varphi_u(x,y,t,u)u_t(x,y,0) - \varphi_u(x,y,0,u)u_t(x,y,0)\\
&\qquad -\varphi_{tu}(x,y,0,u)u(x,y,0) - \alpha\mathcal{D}_t\tau u_t(x,y,0) + \mathcal{D}_t(\tau u_t)|_{t=0}\}
\end{aligned}
$$

When considering arbitrary functions f, g, and h, only a limited number of infinitesimal generators arise from solving Eq. (7). We reach at a two-parameter Lie group of point transformations with its infinitesimal generators given by

$$V_1 = \frac{\partial}{\partial x}, \qquad V_2 = \frac{\partial}{\partial y}.$$

They cause the dimension of this equation to decrease as follows:

$${}^C D_t^\alpha u(y,t) - (g(u)u_y)_y - h(u) = 0, \quad {}^C D_t^\alpha u(x,t) - (f(u)u_x)_x - h(u) = 0.$$

Therefore, we need to examine the forms of $f(u)$, $g(u)$, and $h(u)$. For example, if $f(u) = b$ and $h(u) = 0$, then

$$^{C}D_t^{\alpha} u(x,t) - b u_{xx} = 0, \tag{8}$$

have the vector fields generated by

$$V_1 = 2t\frac{\partial}{\partial t} + \alpha x \frac{\partial}{\partial x}, \qquad V_2 = \frac{\partial}{\partial x}, \qquad V_3 = u\frac{\partial}{\partial u}, \qquad V_4 = f(x,t)\frac{\partial}{\partial u}.$$

Similarity variables under V_1-generated group has the form

$$u(x,t) = G(s), \qquad s = tx^{-\dfrac{2}{\alpha}},$$

where $G(s)$ satisfies the equation

$$^{C}D_s^{\alpha} G(s) = b\left[(\frac{2}{\alpha} + \frac{4}{\alpha^2})sG'(s) + \frac{4}{\alpha^2}s^2 G''(s)\right].$$

The solutions of above equation can be written as

$$G(s) = c_1 + c_2 s + \sum_{n=1}^{\infty} \frac{(2b)^n c_2}{\alpha^{2n}\Gamma(n\alpha+2)} \left(\prod_{k=1}^{n} [(k-1)\alpha + 1]\,[(2k-1)\alpha + 2]\right) s^{n\alpha+1},$$

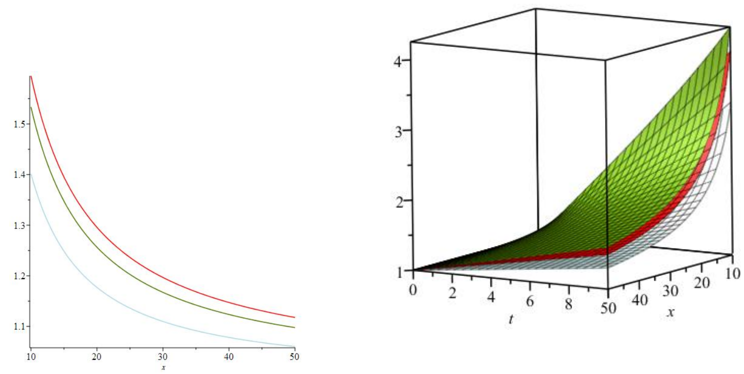
where c_1 , c_2 are arbitrary constants. Thus

$$u(x,t) = c_1 + c_2 t x^{\frac{-2}{\alpha}} + \sum_{n=1}^{\infty} \frac{(2b)^n c_2}{\alpha^{2n}\Gamma(n\alpha+2)} \left(\prod_{k=1}^{n} [(k-1)\alpha + 1]\,[(2k-1)\alpha + 2]\right) (tx^{\frac{-2}{\alpha}})^{n\alpha+1}.$$

The graphical representations of these solutions are displayed in Figure 1. In this section, we will focus on the case that $f(u) = u^n, g(u) = u^k$ and $h(u) = u^m$. We have the following results considering m, n, k.

- **Case I.** $n, k, m \in \mathbb{R}$
  In this case,

$$^{C}D_t^{\alpha} u(x,y,t) = (u^n u_x)_x + (u^k u_y)_y + u^m, \qquad 1 < \alpha < 2 \tag{9}$$



(a) **(b)**

Figure 1: The solution u in Eq. (8) at different values of α: (a) at $t = 2$; (b) for $0 < t < 10$ and $10 < x < 50$. The parameters are $b = \frac{1}{4}$, $c_1 = 1$, and $c_2 = 3$.

is invariant under the group of transformations generated by

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial y},$$
$$V_3 = \alpha(m-n-1)x\frac{\partial}{\partial x} + \alpha(m-k-1)y\frac{\partial}{\partial y} + (2m-2)t\frac{\partial}{\partial t} - 2\alpha u\frac{\partial}{\partial u}. \tag{10}$$

We want to identify and classify all group-invariant solutions by seeking the optimal system of one-dimensional subalgebras for Equation (9). This involves analyzing the Lie algebra structure associated with the problem and finding the most suitable subalgebras that provide valuable insights into the behavior and properties of the equation's solutions. According to the commutator operators $[V_i, V_j] = V_iV_j - V_jV_i$, we obtain the commutator Table 1.

Also, to compute the adjoint representation, we use the following

Table 1: Commutator table for the Lie algebra of Eq. (9)(Case I)

$[.,.]$	V_1	V_2	V_3
V_1	0	0	$\frac{\alpha(m-n-1)}{2m-2}V_1$
V_2	0	0	$-\frac{\alpha(k-m+1)}{2m-2}V_2$
V_3	$-\frac{\alpha(m-n-1)}{2m-2}V_1$	$\frac{\alpha(k-m+1)}{2m-2}V_2$	0

Lie series

$$Ad(exp(\varepsilon V_i))V_j = V_j - \varepsilon[V_i, V_j] + \frac{\varepsilon^2}{2}[V_i, [V_i, V_j]] - \cdots ,$$

where ε is a parameter. According to the above formula, we calculate the adjoint action of the generators (10) which is listed in Table.2.

Table 2: Adjoint table for the Lie algebra of Eq. (9)(Case I)

$Ad(exp(V_i), V_j)$	V_1	V_2	V_3
V_1	V_1	V_2	$-\frac{\varepsilon\alpha(m-n-1)}{2m-2}V_1+V_3$
V_2	V_1	V_2	$\frac{\varepsilon\alpha(k-m+1)}{2m-2}V_2+V_3$
V_3	$e^{\frac{\varepsilon\alpha(m-n-1)}{2m-2}}V_1$	$e^{-\frac{\varepsilon\alpha(k-m+1)}{2m-2}}V_2$	V_3

Taking into account the commutator Table 1 and adjoint Table 2, then Upon conducting an in-depth analysis, we have identified an optimal system of one-dimensional subalgebras, which are generated by the following vector fields

$$V_1, \qquad V_2, \qquad V_3.$$

We already have discussed the cases V_1 and V_2. So for the infinitesimal generator V_3 after solving characteristic equations, we introduce the following similarity variables

$$u(t,x,y) = x^{\frac{-2}{m-n-1}} F(p,q), \quad p = \frac{y^{m-n-1}}{x^{m-k-1}}, \quad q = tx^{-\frac{2m-2}{\alpha(m-n-1)}},$$

where $m \neq n+1$. Substitution these similarity variables into (9) gives the reduction

$$\begin{aligned}
{}^{C}D_q^{\alpha} F &= \frac{2(m+n+1)}{(m-n-1)^2} F^{n+1} + (m-k-1)^2 np^2 F^{n-1} F_p^2 \\
&+ \left(\frac{2m-2}{\alpha(m-n-1)}\right)^2 nq^2 F^{n-1} F_q^2 + (m-k-1)\left(\frac{4n+4}{m-n-1} + m-k\right) pF^n F_p \\
&+ \frac{2m-2}{\alpha(m-n-1)}\left[\frac{4n+4}{m-n-1} + \frac{2m-2}{\alpha(m-n-1)} + 1\right] qF^n F_q \\
&+ \frac{2n(m-k-1)(2m-2)}{\alpha(m-n-1)} pqF^{n-1} F_p F_q + k(m-n-1)^2 p^{\frac{2(m-n-2)}{m-n-1}} F^{k-1} F_p^2 \\
&+ (m-n-1)(m-n-2) p^{\frac{m-n-3}{m-n-1}} F^k F_p + (m-k-1)^2 p^2 F^n Fpp \\
&+ (m-n-1)^2 p^{\frac{2(m-n-2)}{m-n-1}} F^k F_{pp} + \frac{(4m-4)(m-k-1)}{\alpha(m-n-1)} pqF^n F_{pq} \\
&+ \left(\frac{2m-2}{\alpha(m-n-1)}\right)^2 q^2 F^n F_{qq} + F^m.
\end{aligned}$$

- **Case II.** $k = n = m-1,\ n > -\alpha,\ 1 < \alpha < 2$
  If we consider $k = n = m-1,\ n > -\alpha$, therefore

$$ {}^{C}D_t^{\alpha} u = (u^n u_x)_x + (u^n u_y)_y + u^{n+1}. \tag{11}$$

In this case, using the infinitesimal generator $V_3 = nt\frac{\partial}{\partial t} - \alpha u \frac{\partial}{\partial u}$, we define a set of similarity variables as follows:

$$u(t,x,y) = t^{-\frac{\alpha}{n}} F(x,y) \tag{12}$$

and hence Eq. (11) is reduced to the following equation

$$(\frac{\alpha}{n})(\frac{\alpha}{n}+1)\frac{\Gamma(-\frac{\alpha}{n}-1)}{\Gamma(1-\alpha-\frac{\alpha}{n})}F = nF^{n-1}F_x^2 + F^n F_{xx} + nF^{n-1}F_y^2 + F^n F_{yy} + F^{n+1}$$

- **Case III.** n, k arbitrary real numbers, $m = 0$.
  Consider the equation

$$ {}^C D_t^{\alpha} u(x,y,t) = (u^n u_x)_x + (u^k u_y)_y, \qquad 1 < \alpha < 2 \qquad (13)$$

  The symmetry algebra of this equation is generated by

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial y}, \quad V_3 = \alpha x\frac{\partial}{\partial x} + \alpha y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t},$$
$$V_4 = nx\frac{\partial}{\partial x} + ky\frac{\partial}{\partial y} + 2u\frac{\partial}{\partial u}.$$

  Using these infinitesimal generators, the optimal system of invariant solutions for (13) consists of the subalgebras

$$V_1, \quad V_2, \quad V_3, \quad aV_3 + V_4,$$

  where $a \in \{-1, 0, 1\}$.
  Indeed, according to our optimal system, we need only find the reduced equations for the one–parameter subgroups generated by: (a) V_3, (b) $aV_3 + V_4$.

  i) For the infinitesimal generator V_3, From the corresponding characteristic equations, it has invariant solution

$$u(x,y,t) = F(p,q), \qquad p = \frac{y}{x}, \qquad q = tx^{\frac{-2}{\alpha}}.$$

  Thus Eq. (13) is reduced to the following equation

$$ {}^C D_q^{\alpha} F = nF^{n-1}\left(pF_p + \frac{2}{\alpha}qF_q\right)^2 + F^n[2pF_p + p^2F_{pp} + \frac{4}{\alpha}pqF_{pq} + \frac{2}{\alpha}(\frac{2}{\alpha}+1)qF_q + \frac{4}{\alpha^2}q^2F_{qq}] + kF^{k-1}F_p^2 + F^kF_{pp}.$$

ii) For the linear combination $aV_3 + V_4$, we have the following similarity variables

$$u(x,y,t) = x^{\frac{2}{a\alpha+n}} F(p,q), \quad p = tx^{\frac{-2a}{a\alpha+n}}, \quad q = \frac{y^{a\alpha+n}}{x^{a\alpha+k}},$$

and hence Eq. (13) is reduced to the following equation

$$\begin{aligned}
{}^{C}D_q^{\alpha}F &= -\frac{2(a\alpha-n-2)}{(a\alpha+n)^2}F^{n+1} + (a\alpha+k)^2 np^2 F^{n-1}F_p^2 \\
&+ \frac{4a^2n}{(a\alpha+n)^2}q^2F^{n-1}F_q^2 + (a\alpha+k)\left(a\alpha+k+1-\frac{4n+4}{a\alpha+n}\right)pF^nF_p \\
&+ \frac{2a}{a\alpha+n}\left[\frac{2a-4n-4}{a\alpha+n}+1\right]qF^nF_q + \frac{4an(a\alpha+k)}{a\alpha+n}pqF^{n-1}F_pF_q \\
&+ (a\alpha+k)^2p^2F^nFpp + \frac{4a(a\alpha+k)}{a\alpha+n}pqF^nF_{pq} + \frac{4a^2}{(a\alpha+n)^2}q^2F^nF_{qq} \\
&+ (a\alpha+n)^2p^{\frac{2a\alpha+2n-2}{a\alpha+n}}\left[kF^{k-1}F_p^2 + F^kF_{pp}\right] \\
&+ (a\alpha+n)(a\alpha+n-1)p^{\frac{a\alpha+n-2}{a\alpha+n}}F^kF_p.
\end{aligned}$$

5 Single-wave Solutions and Dimensionality reduction in a two-Dimensional extension of the Burger's Equation

In this section, we will consider the (2+1)-dimensional time-fractional Burger's equation with a Caputo derivative as follows:

$$ {}^{C}D_t^{\alpha}u + au(u_x+u_y) + b(u_{xx}+u_{yy}) = 0, \tag{14}$$

where $0 < \alpha < 1$, and $a, b \in \mathbb{R}$. The (2+1)-dimensional fractional Burger's equation with the Caputo derivative has been studied in various fields, including physics, biology, and mathematics [12, 17]. Previous research has explored the mathematical approaches to this equation [7, 10]. In this section, we will apply Lie symmetry analysis based on the previous results to obtain some invariant solutions for this equation. Moreover, we will directly obtain the solitary solution, which could not be obtained through Lie symmetry analysis.

5.1 Lie symmetry analysis

Applying Lie symmetry analysis to Eq.14, it follows that the Lie algebra of infinitesimal symmetries of the time-fractional Burger's equation is spanned by the three vector fields.

$$V_1 = \frac{\partial}{\partial x}, \qquad V_2 = \frac{\partial}{\partial y}, \qquad V_3 = \alpha x\frac{\partial}{\partial x} + \alpha y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t} - \alpha u\frac{\partial}{\partial u}.$$

We utilize these vector fields to reduce the equation.

- **The vector field V_1**
  Invariant solution under the group with generator V_1, has the form
  $$u = F(t, y),$$
  where F satisfies the equation
  $${}^C D_t^\alpha F + aFF_y + bF_{yy} = 0,$$
  Its admitted operators are $V_{11} = \dfrac{\partial}{\partial y}$ and
  $$V_{12} = \alpha y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t} - \alpha F\frac{\partial}{\partial F}$$
  - If $V = V_{11}$, we have$F(t, y) = G(t)$ where G satisfies the equation
    $${}^C D_t^\alpha G = 0,$$
    therefore $G(t) = k_1$
  - If $V = V_{12}$, we have $F(t, y) = \dfrac{1}{y}G(s), \quad s = ty^{\frac{-2}{\alpha}}$ where G satisfies the equation
    $${}^C D_s^\alpha G - aG^2 + 2bG - \frac{2}{\alpha}asGG' + b(\frac{6}{\alpha} + \frac{4}{\alpha^2})sG' + \frac{4}{\alpha^2}bs^2G'' = 0$$

- **The vector field V_2**
An invariant solution under the group with generator V_2 can be expressed in the following form

$$u = F(t,x),$$

where F satisfies the equation

$$^{C}D_t^{\alpha} F + aFF_x + bF_{xx} = 0,$$

Its admitted operators are $V_{21} = \dfrac{\partial}{\partial x}$ and

$$V_{22} = \alpha x \frac{\partial}{\partial y} + 2t \frac{\partial}{\partial t} - \alpha F \frac{\partial}{\partial F}$$

  - If $V = V_{21}$, we have$F(t,x) = G(t)$ where G satisfies the equation

$$^{C}D_t^{\alpha} G = 0,$$

therefore $G(t) = k_1$

  - If $V = V_{22}$, we have $F(t,x) = \dfrac{1}{x} G(s), s = tx^{\frac{-2}{\alpha}}$ where G satisfies the equation

$$^{C}D_s^{\alpha} G - aG^2 + 2bG - \frac{2}{\alpha} asGG' + b(\frac{6}{\alpha} + \frac{4}{\alpha^2}) sG' + \frac{4}{\alpha^2} bs^2 G'' = 0$$

- **The vector field V_3**
An invariant solution under the group with generator V_3 can be expressed in the following form

$$u(t,x,y) = \frac{1}{x} F(p,q), \qquad p = \frac{y}{x}, \qquad q = tx^{-\frac{2}{\alpha}}$$

where F satisfies the equation

$$^{C}D_q^{\alpha} F + 2bF - aF^2 + [4bp - a(p-1)F]\, F_p + \left[b\left(\frac{6}{\alpha} + \frac{4}{\alpha^2}\right) q - \frac{2}{\alpha} aqF \right] F_q + b(p^2+1) F_{pp} + \frac{4}{\alpha} bpqF_{pq} + \frac{4}{\alpha^2} bq^2 F_{qq} = 0$$

Its admitted non-classical infinitesimal symmetries are

$$V = (\alpha p + \nu)\frac{\partial}{\partial p} + 2q\frac{\partial}{\partial q} - \alpha F\frac{\partial}{\partial F},$$

where ν is an arbitrary constant. Thus we have the following similarity variables

$$F(p,q) = \frac{1}{\alpha p + \nu}G(s), \qquad s = q(\alpha p + \nu)^{-\frac{2}{\alpha}}$$

where G satisfies the equation

$$\begin{aligned} &{}^{C}D_s^{\alpha}G - a(\alpha + \nu)G^2 - a(2 + \frac{2}{\alpha}\nu)sGG' + 2b(\nu^2 + \alpha^2)G \\ &+ 2b(3\alpha + 2)(\frac{\nu^2}{\alpha^2} + 1)sG' + 4b(\frac{\nu^2}{\alpha^2} + 1)s^2G'' = 0. \end{aligned} \tag{15}$$

If $\nu = -\alpha$ or $\nu = i\alpha$ where i is the complex unit, By the equation (15), respectively, we obtain the simpler equations

$$\begin{aligned} &{}^{C}D_s^{\alpha}G + 4b\alpha^2G + 4b(3\alpha + 2)sG' + 8bs^2G'' = 0, \\ &{}^{C}D_s^{\alpha}G - a\alpha(1 + i)G^2 - 2a(1 + i)sGG' = 0. \end{aligned}$$

5.2 Single-wave solutions

Consider the fractional differential equation of the form:

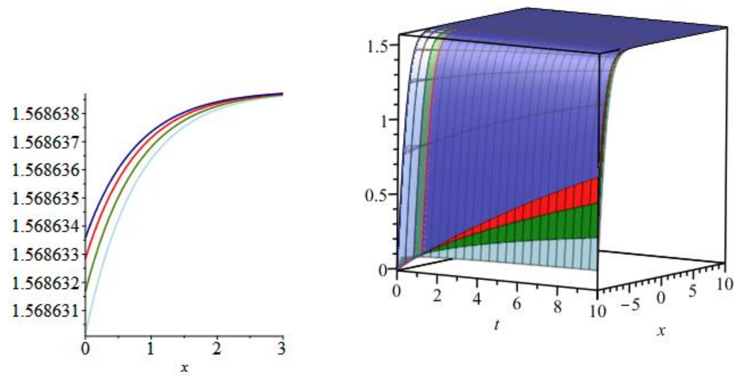
$${}^{C}D_t^{\alpha}u = au(u_x + u_y) + b(u_{xx} + u_{yy}), \qquad 0 < \alpha < 1 \tag{16}$$

By employing the wave transformation technique, we can express the solution in the form of traveling waves or other wave-like structures,

$$u(x,y,t) = u(\zeta), \qquad \zeta = kx + ly + m\frac{t^{\alpha}}{\Gamma(\alpha + 1)} \tag{17}$$

where k, l and m are real constants. Applying the transformation (17) to Eq. (16), the following equations can be obtained, as

$$a(k + l)uu' - mu' + b(k^2 + l^2)u'' = 0 \tag{18}$$



(a) **(b)**

Figure 2: The solitary solution $u(\zeta)$, where $\zeta = kx + ly + m\frac{t^{\alpha}}{\Gamma(\alpha+1)}$, in Eq. (16) at different values of α: (a) at $t = 6$ and $y = 3$; (b) for $0 < t < 10$ and $y = 3$. The parameters are $a = 2$, $b = 1$, $k = 1$, $l = 3$, and $m = 0.1$.

where the prime notation is used to denote the derivative with respect to ζ. we obtain the solutions of Eq.(18)

$$u(\zeta) = \frac{\sqrt{2(bc_1a(k^2+l^2)(k+l))}}{a(k+l)} \tanh \frac{(c_2+\zeta)\sqrt{2(bc_1a(k^2+l^2)(k+l))}}{2b(k^2+l^2)} - \frac{m}{a(k+l)}$$

where c_1 and c_2 are arbitrary smooth functions. Inserting $\zeta = kx + ly + m\frac{t^{\alpha}}{\Gamma(\alpha+1)}$, we obtain the single-wave solutions of Eq.(16) as follows

$$u(x, y, t) = \frac{\sqrt{2(bc_1a(k^2+l^2)(k+l))}}{a(k+l)} \times \tanh \frac{(c_2 + kx + ly + m\frac{t^{\alpha}}{\Gamma(\alpha+1)})\sqrt{2(bc_1a(k^2+l^2)(k+l))}}{2b(k^2+l^2)} - \frac{m}{a(k+l)}$$

The graph of this solution can be found in Figure 2

6 Conclusion

In this paper, we have successfully demonstrated the application of Lie symmetry analysis to a class of (2+1)-dimensional fractional wave-diffusion equations involving Caputo fractional derivatives. The non-local nature of fractional operators presents challenges in obtaining exact solutions for fractional differential equations; however, our study indicates that these challenges can be overcome using the systematic approach provided by Lie symmetry analysis. By deriving Lie point symmetries and presenting similarity transformations, we have reduced the original three-dimensional wave-diffusion equation to a more manageable two-dimensional form. These results have significant implications for comprehending and modeling the corresponding physical phenomena encountered in scientific research and applied mathematics. Our findings contribute to the broader objective of addressing complex fractional differential equations and offer a promising path for future investigations into nonlinear wave-diffusion systems with fractional derivative components. Furthermore, the obtained similarity transformations and reduced equations may serve as a foundation for further theoretical analysis and practical applications across various disciplines. Future research will concentrate on deriving prolongation formulas for generalized fractional derivatives, particularly those with Sonin kernels[24, 25], and examining equations involving such derivatives, thereby expanding the scope of our understanding and capability in this complex and rapidly evolving field.

7 Appendix

In this section, we prove in details the results of Theorem 3.1. The following proposition plays crucial role in our computations.

Proposition 7.1. *Let $m-1<\alpha<m$ and $m\in\mathbb{N}$. Suppose $u=u(x,y,t)$, and $\varphi=\varphi(x,y,t,u)$ be analytic functions, and ${}^{C}\mathcal{D}_t^{\alpha}\varphi$ denote*

the total derivative of φ, then we have

$$
\begin{aligned}
{}^{C}\mathcal{D}_t^{\alpha}\varphi = {}^{C}D_t^{\alpha}\varphi + \varphi_u\,{}^{C}D_t^{\alpha}u - u\,{}^{C}D_t^{\alpha}\varphi_u + \sum_{k=1}^{\infty}\binom{\alpha}{k}D_t^{k}\varphi_u D_t^{\alpha-k}u \\
+ \sum_{k=0}^{m-1}\frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)}\left[\varphi_u D_t^{k}u\big|_{t=0} - D_t^{k}(u\varphi_u)\big|_{t=0}\right] + \nu,
\end{aligned}
$$

with $\varphi_u = \frac{\partial\varphi}{\partial u}$ *and* $D_t^m = \frac{\partial^m}{\partial t^m}$*, where*

$$
\begin{aligned}
\nu = \sum_{n=m}^{\infty}\sum_{j=2}^{n}\sum_{k=2}^{j}\sum_{r=0}^{k-1}\binom{\alpha-m}{n-m}\binom{n}{j}\binom{k}{r}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} \\
\frac{1}{k!}(-u)^{r}D_t^{j}u^{k-r}D_t^{n-j}\left(D_u^{k}\varphi\right).
\end{aligned}
$$

Proof. Using Proposition 2.3, and the definition of the total derivative $\mathcal{D}_t^n\varphi$, we have

$$
\begin{aligned}
{}^{C}\mathcal{D}_t^{\alpha}\varphi(x,y,t,u(x,y,t)) &= \sum_{n=m}^{\infty}\binom{\alpha-m}{n-m}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}\mathcal{D}_t^{n}\varphi(x,y,t,u(x,y,t)) \\
&= \sum_{n=m}^{\infty}\binom{\alpha-m}{n-m}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} \\
&\quad \sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{r=0}^{k}\binom{n}{j}\binom{k}{r}\frac{1}{k!}(-u)^{r}D_t^{j}u^{k-r}D_t^{n-j}\left(D_u^{k}\varphi\right).
\end{aligned}
$$

By rearranging the indexes, recalling Proposition 2.3, and some technical calculations we deduce

$$
\begin{aligned}
{}^{C}\mathcal{D}_t^{\alpha}\varphi = {}^{C}D_t^{\alpha}\varphi - u\sum_{n=m}^{\infty}\binom{\alpha-m}{n-m}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}D_t^{n}\varphi_u \\
+ \sum_{n=m}^{\infty}\binom{\alpha-m}{n-m}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}\sum_{j=0}^{n}\binom{n}{j}D_t^{j}uD_t^{n-j}\left(\varphi_u\right) + \nu,
\end{aligned}
$$

where ν is as in the Proposition. Now using the classical Leibniz's formula, we conclude

$$
\begin{aligned}
{}^{C}\mathcal{D}_t^{\alpha}\varphi = & {}^{C}D_t^{\alpha}\varphi - u\sum_{n=m}^{\infty}\binom{\alpha-m}{n-m}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}D_t^n\varphi_u \\
& +\sum_{n=m}^{\infty}\binom{\alpha-m}{n-m}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}D_t^n(u\varphi_u) \\
& +\nu \\
& = {}^{C}D_t^{\alpha}\varphi - u\,{}^{C}D_t^{\alpha}\varphi_u + {}^{C}D_t^{\alpha}(u\varphi_u) + \nu,
\end{aligned}
$$

where for the practical use, in view of Proposition 2.4, we obtain

$$
{}^{C}\mathcal{D}_t^{\alpha}\varphi = {}^{C}D_t^{\alpha}\varphi - u\,{}^{C}D_t^{\alpha}\varphi_u + D_t^{\alpha}(u\varphi_u) - \sum_{k=0}^{m-1}\frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)}D_t^k(u\varphi_u)|_{t=0} + \nu.
$$

By employing Leibniz's formula for R-L fractional derivative, we have

$$
\begin{aligned}
{}^{C}\mathcal{D}_t^{\alpha}\varphi = & {}^{C}D_t^{\alpha}\varphi - u\,{}^{C}D_t^{\alpha}\varphi_u + \varphi_u D_t^{\alpha}u + \sum_{k=1}^{\infty}\binom{\alpha}{k}D_t^k\varphi_u D_t^{\alpha-k}u \\
& - \sum_{k=0}^{m-1}\frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)}D_t^k(u\varphi_u)|_{t=0} + \nu.
\end{aligned}
$$

Recalling Proposition 2.4 yields

$$
\begin{aligned}
{}^{C}\mathcal{D}_t^{\alpha}\varphi = & {}^{C}D_t^{\alpha}\varphi - u\,{}^{C}D_t^{\alpha}\varphi_u + \varphi_u\,{}^{C}D_t^{\alpha}u + \varphi_u\sum_{k=0}^{m-1}\frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)}D_t^k u|_{t=0} \\
& + \sum_{k=1}^{\infty}\binom{\alpha}{k}D_t^k\varphi_u D_t^{\alpha-k}u - \sum_{k=0}^{m-1}\frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)}D_t^k(u\varphi_u)|_{t=0} + \nu,
\end{aligned}
$$

which is our desired result. □

Proof of Theorem 3.1. In view of the analytic assumption on u and utilizing Proposition 2.3, we have

$$
{}^{C}D_{\bar{t}}^{\alpha}\bar{u}(\bar{x},\bar{y},\bar{t}) = \sum_{k=m}^{\infty}\binom{\alpha-m}{k-m}\frac{\bar{t}^{k-\alpha}}{\Gamma(k+1-\alpha)}D_{\bar{t}}^k\bar{u}(\bar{x},\bar{y},\bar{t}).
$$

Using (1), (2), and applying the Taylor's series expansion for the $\bar{t}^{k-\alpha}$, we have

$$
\begin{aligned}
{}^{C}D_{\bar{t}}^{\alpha}\bar{u}(\bar{x},\bar{y},\bar{t}) = & \sum_{k=m}^{\infty}\binom{\alpha-m}{k-m}\frac{1}{\Gamma(k+1-\alpha)}\left[t^{k-\alpha}+\varepsilon(k-\alpha)t^{k-\alpha-1}\tau+O(\varepsilon^2)\right] \\
& \times\left[D_t^k u(x,y,t)+\varepsilon\left(\mathcal{D}_t^k\varphi-\sum_{n=1}^{k}\binom{k}{n}\mathcal{D}_t^n\xi D_t^{k-n}u_x\right.\right. \\
& \left.\left.-\sum_{n=1}^{k}\binom{k}{n}\mathcal{D}_t^n\eta D_t^{k-n}u_y-\sum_{n=1}^{k}\binom{k}{n}\mathcal{D}_t^n\tau D_t^{k+1-n}u\right)+O(\varepsilon^2)\right] \\
& =\sum_{k=m}^{\infty}\binom{\alpha-m}{k-m}\frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)}D_t^k u+\varepsilon\sum_{k=m}^{\infty}\binom{\alpha-m}{k-m}\frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)}\mathcal{D}_t^k\varphi \\
& -\varepsilon\sum_{k=m}^{\infty}\binom{\alpha-m}{k-m}\frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)}\sum_{n=1}^{k}\binom{k}{n}\mathcal{D}_t^n\xi D_t^{k-n}u_x \\
& -\varepsilon\sum_{k=m}^{\infty}\binom{\alpha-m}{k-m}\frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)}\sum_{n=1}^{k}\binom{k}{n}\mathcal{D}_t^n\eta D_t^{k-n}u_y \\
& -\varepsilon\sum_{k=m}^{\infty}\binom{\alpha-m}{k-m}\frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)}\sum_{n=1}^{k}\binom{k}{n}\mathcal{D}_t^n\tau D_t^{k+1-n}u \\
& +\varepsilon\sum_{k=m}^{\infty}\binom{\alpha-m}{k-m}\frac{t^{k-\alpha-1}}{\Gamma(k-\alpha)}\tau D_t^k u+O(\varepsilon^2),
\end{aligned}
$$

we added some statements to get the summations with the lower index $n=0$, and also using the Leibniz's formula for integer-order derivatives and Proposition 2.3, we conclude

$$
\begin{aligned}
& \sum_{k=m}^{\infty}\binom{\alpha-m}{k-m}\frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)}\sum_{n=0}^{k}\binom{k}{n}\mathcal{D}_t^n\xi D_t^{k-n}u_x \\
& =\sum_{k=m}^{\infty}\binom{\alpha-m}{k-m}\frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)}\mathcal{D}_t^k(\xi u_x)={}^{C}\mathcal{D}_t^{\alpha}\left(\xi u_x\right).
\end{aligned}
$$

The same result is valid for ${}^{C}\mathcal{D}_t^{\alpha}\left(\eta u_y\right), {}^{C}\mathcal{D}_t^{\alpha}\left(\tau u_t\right)$. Thus we get

$$
\begin{aligned}
{}^{C}D_{\bar{t}}^{\alpha}\bar{u}(\bar{x},\bar{y},\bar{t}) &= {}^{C}D_t^{\alpha}u(x,y,t)+\varepsilon\, {}^{C}\mathcal{D}_t^{\alpha}\varphi-\varepsilon\, {}^{C}\mathcal{D}_t^{\alpha}\left(\xi u_x\right)+\varepsilon\xi\, {}^{C}D_t^{\alpha}u_x-\varepsilon\, {}^{C}\mathcal{D}_t^{\alpha}\left(\eta u_y\right)\\
&+\varepsilon\eta\, {}^{C}D_t^{\alpha}u_y-\varepsilon\, {}^{C}\mathcal{D}_t^{\alpha}\left(\tau u_t\right)+\varepsilon\tau\, {}^{C}D_t^{\alpha}u_t+\varepsilon\sum_{k=m}^{\infty}\binom{\alpha-m}{k-m}\frac{t^{k-\alpha-1}}{\Gamma(k-\alpha)}\tau D_t^{k}u+O(\varepsilon^2).
\end{aligned}
$$

Now a change of the index, and using the relation

$$
\binom{\alpha-m}{k}\frac{1}{\Gamma(k+m-\alpha)}=\frac{(-1)^k}{k!\Gamma(m-\alpha)},
$$

we have

$$
\sum_{k=m}^{\infty}\binom{\alpha-m}{k-m}\frac{t^{k-\alpha-1}}{\Gamma(k-\alpha)}\tau D_t^{k}u=\frac{\tau t^{m-\alpha-1}}{\Gamma(m-\alpha)}\sum_{k=0}^{\infty}\frac{(-t)^k}{k!}D_t^{k+m}u.
$$

At this stage, we consider the function $D_{\theta}^{m}u(x,y,\theta)$ and arrive at the following formula by employing the Taylor's expansion of this function around $\theta=t$,

$$
D_{\theta}^{m}u(x,y,\theta)=\sum_{k=0}^{\infty}\frac{(\theta-t)^k}{k!}D_t^{k+m}u(x,y,t).
$$

Now let $\theta=0$, then we deduce

$$
D_t^{m}u(x,y,t)\big|_{t=0}=\sum_{k=0}^{\infty}\frac{(-t)^k}{k!}D_t^{k+m}u(x,y,t),
$$

which results in

$$
\begin{aligned}
\varphi_C^{(\alpha,t)} &= {}^{C}\mathcal{D}_t^{\alpha}\varphi-{}^{C}\mathcal{D}_t^{\alpha}\left(\xi u_x\right)+\xi\, {}^{C}D_t^{\alpha}u_x-{}^{C}\mathcal{D}_t^{\alpha}\left(\eta u_y\right)+\eta\, {}^{C}D_t^{\alpha}u_y\\
&-{}^{C}\mathcal{D}_t^{\alpha}\left(\tau u_t\right)+\tau\, {}^{C}D_t^{\alpha}u_t+\tau\frac{t^{m-\alpha-1}}{\Gamma(m-\alpha)}D_t^{m}u(x,y,t)\big|_{t=0}.
\end{aligned}
$$

Proposition 2.4 and the fact $\tau(x,y,0,u(x,y,0))=0$ completes the proof. Recalling Propositions 2.2, 2.4, 7.1, and also $\tau(x,y,0,u(x,y,0))=0$, we conclude the following description which is an alternative to $\varphi_C^{(\alpha,t)}$

$$\begin{aligned}\varphi_C^{(\alpha,t)} &= {}^C D_t^\alpha \varphi + \varphi_u\, {}^C D_t^\alpha u - u\, {}^C D_t^\alpha \varphi_u + \nu - {}^C \mathcal{D}_t^\alpha\, (\xi u_x) \\ &+ \xi\, {}^C D_t^\alpha u_x - {}^C \mathcal{D}_t^\alpha\, (\eta u_y) + \eta\, {}^C D_t^\alpha u_y - \alpha \mathcal{D}_t \tau\, {}^C D_t^\alpha u \\ &+ \sum_{k=1}^{\infty} \binom{\alpha}{k} D_t^k \varphi_u D_t^{\alpha-k} u - \sum_{k=1}^{\infty} \binom{\alpha}{k+1} \mathcal{D}_t^{k+1} \tau D_t^{\alpha-k-1} u_t \\ &+ \sum_{k=0}^{m-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} \left[\varphi_u D_t^k u\big|_{t=0} - D_t^k (u\varphi_u)\big|_{t=0} \right] \\ &- \sum_{k=0}^{m-2} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} \left[\left(\frac{\alpha t \mathcal{D}_t \tau}{k+1-\alpha} + \tau \right) D_t^{k+1} u\big|_{t=0} - \frac{t}{k+1-\alpha} \mathcal{D}_t^{k+1} (\tau u_t)\big|_{t=0} \right].\end{aligned}$$

References

[1] M. Ait Ichou, A. Ezziani, A mixed finite element approach for a factional viscoelastic wave propagation in–time–domain, Indian *J. Pure Appl. Math.* (2023) doi: 10.1007/s13226-023-00461-8

[2] S.S. Akhiev, T. Özer, Symmetry groups of the equations with non-local structure and an application for the collisionless Boltzmann equation, *Int. J. Eng. Sci.* 43 (2005) 121–137.

[3] Y. AryaNejad, Lie Symmetries of Schrödinger Equation on a Sphere, *Journal of Mathematical Extension* 17 (2023)(4) 1–20.

[4] F. Bahrami, R. Najafi, M.S. Hashemi, On the invariant solutions of space/time–fractional diffusion equations, *Indian J. Phys.* 91 (2017) 1571–1579.

[5] S. Beckers, M. Yamamoto, Regularity and unique existence of solution to linear diffusion equation with multiple time–fractional derivatives, *Control and Optimization with PDE Constraints*, 164 (2013) 45–55.

[6] A. Bueno-Orovio, D. Kay, V. Grau, B. Rodriguez, K. Burrage, Fractional diffusion models of cardiac electrical propagation: role of

structural heterogeneity in dispersion of repolarization, *J. R. Soc. Interface* 11 (97) (2014).

[7] W. Cao, Q. Xu, Z. Zheng, Solution of two-dimensional time-fractional Burgers equation with high and low Reynolds numbers. *Adv Differ Equ* 2017, 338 (2017).

[8] A. Chatterjee, M.M. Panja, U. Basu, D. Datta, B. N. Mandal, Solving one-dimensional advection diffusion transport equation by using CDV wavelet basis, Indian *J. Pure Appl. Math.* 52 (2021) 872–896.

[9] K. Diethelm, *The Analysis of Fractional Differential Equations: An Application–Oriented Exposition Using Differential Operators of Caputo Type*, Springer–Verlag, 2010.

[10] V.D. Djordjevica, T.M. Atanackovic, Similarity solutions to the nonlinear heat conduction and Burgers/Korteweg de Vries fractional equations, *J. Comput. Appl.* Math., 222 (2) (2008) 701-714.

[11] J. Duan, L. Jing, The solution of the time–space fractional diffusion equation based on the Chebyshev collocation method, Indian *J. Pure Appl. Math.* (2023) doi: 10.1007/s13226-023-00495-y

[12] R. Garra, Fractional-calculus model for temperature and pressure waves in fluid saturated porous rocks, *Phys. Rev. E*, 84 (3) (2011) 1-6.

[13] R.K. Gazizov, A.A. Kasatkin, S.Y. Lukashchuk, Continuous transformation groups of fractional differential equations, *Vestnik USATU 9* (21) (2007) 125–135.

[14] H. Hajinezhad, A. R. Soheili, A Finite Difference Approximation for the Solution of the Space Fractional Diffusion, *Journal of Mathematical Extension* 18 (2024)(1) 1–14.

[15] M.S. Hashemi, F. Bahrami, R. Najafi, Lie symmetry analysis of steady–state fractional reaction–convection–diffusion equation, *Optik* 138 (2017) 240–249.

[16] M. Inc, A. Yusuf, A.I. Aliyu, D. Baleanu, Lie symmetry analysis, explicit solutions and conservation laws for the space–time fractional nonlinear evolution equations, *Physica A* 496 (2018) 371–383.

[17] N. Iqbal, M.T. Chughtai, R. Ullah, Fractional study of the non-Linear Burgers' equations via a semi-analytical technique. Fractal Fract. 7(2), 103 (2023).

[18] M. Jafari, A. Zaeim, M. Gandom, On Similarity Reductions and Conservation Laws of the Two Non-linearity Terms Benjamin-Bona-Mahoney Equation, *Journal of Mathematical Extension* 17 (2023)(1) 1–22.

[19] A. Khan, T.S. Khan, M.I. Syam, H. Khan, Analytical solutions of time–fractional wave equation by double Laplace transform method, *Eur. Phys. J. Plus* 134 (4) (2019) 163.

[20] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Application of Fractional Differential Equations*, Elsevier, 2006.

[21] K. Krishnakumar, A. Durga Devi, V. Srinivasan, P.G.L. Leach , Optimal system, similarity solution and Painlevé test on generalized modified Camassa–Holm equation, Indian *J. Pure Appl. Math.* 54 (2023) 547–557.

[22] F. Liu, P. Zhuang, I. Turner, V. Anh, K. Burrage, A semi-alternating direction method for a 2-D fractional FitzHugh-Nagumo monodomain model on an approximate irregular domain, *J. Comput. Phys.* 293 (2015) 252–263.

[23] Y. Luchko, Some uniqueness and existence results for the initial–boundary–value problems for the generalized time–fractional diffusion equation, *Comput. Math. Appl.* 59 (2010) 1766–1772.

[24] Y. Luchko, The 1st level general fractional derivatives and some of their properties, *J. Math. Sci.* 266 (2022) 709–722.

[25] Y. Luchko, General fractional integrals and derivatives and their applications, *Physica D* 455 (2023) 133906.

[26] R.L. Magin, O. Abdullah, D. Baleanu, X.J. Zhou, Anomalous diffusion expressed through fractional order differential operators in the Bloch-Torrey equation, *J. Magn. Reson.* 190 (2) (2008) 255–270.

[27] K.S. Miller and B. Ross, An introduction to the fractional calculus and fractional differential equations. *Wiley*, New York, 1993.

[28] M. Molati, H. Murakawa, Exact solutions of nonlinear diffusion–convection–reaction equation: A Lie symmetry analysis approach, *Commun. Nonlinear Sci. Numer. Simulat.* 67 (2019) 253–263.

[29] R. Najafi, F. Bahrami, M.S. Hashemi, Classical and nonclassical Lie symmetry analysis to a class of nonlinear time–fractional differential equations, *Nonlinear Dynam.* 87 (2017) 1785–1796.

[30] R. Najafi, Approximate nonclassical symmetries for the time-fractional KdV equations with the small parameter, *Comput. Meth. Differ. Equ.* 8 (2020) 111–118.

[31] R. Najafi, Group–invariant solutions for time–fractional Fornberg-Whitham equation by Lie symmetry analysis, *Comput. Meth. Differ. Equ.* 8 (2020) 251–258.

[32] R. Najafi, E. Çelik, N. Uyanik, Invariant Solutions and Conservation Laws of the Time–Fractional Telegraph Equation, *Adv. Math. Phys.* 2023 (2023) Article ID 1294070.

[33] C.Y. Qin, S.F. Tian, X.B. Wang T.T. Zhang, Lie symmetry analysis, conservation laws and analytical solutions for a generalized time–fractional modified KdV equation, *Wave Random Complex.* 29 (2019) 456–476.

[34] M. Rosa, J.C. Camacho, M. Bruzón, M.L.Gandarias, Lie symmetries and conservation laws for a generalized Kuramoto-Sivashinsky equation, *Math. Meth. Appl. Sci.* 41 (2018) 7295–7303.

[35] S. Sahoo, S.S. Ray, The conservation laws with Lie symmetry analysis for time fractional integrable coupled KdV–mKdV system, *Int. J. Nonlinear Mech.* 98 (2018) 114–121.

[36] K. Sakamoto, M. Yamamoto, Initial value/boundary value problems for fractional diffusion–wave equations and applications to some inverse problems, *J. Math. Anal. Appl.* 382 (2011) 426–447.

[37] M. Singh, S.-Fu Tian, Lie symmetries, group classification and conserved quantities of dispersionless Manakov–Santini system in (2+1)–dimension, Indian *J. Pure Appl. Math.* 54 (2023) 312–329.

[38] A.-M. Yang, X.-J. Yang, Z.-B. Li, Local fractional series expansion method for solving wave and diffusion equations on Cantor sets, *Abstract Appl. Anal.* 2013 (2013) Article ID 351057.

Fariba Bahrami
Professor of Mathematics
Department of Applied Mathematics
University of Tabriz
Tabriz, Iran
E-mail: fbahram@tabrizu.ac.ir

Ramin Najafi
Assistant Professor of Mathematics
Department of Mathematics
Maku Branch, Islamic Azad University
Maku, Iran
E-mail: Ra.Najafi@iau.ac.ir, raminnajafi984@gmail.com

Parisa Vafadar
PhD Student of Mathematics
Department of Applied Mathematics
University of Tabriz
Tabriz, Iran
E-mail: p.vafadar@gmail.com