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The Upper Bound for GMRES on Normal Tridiagonal Toeplitz Linear System

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Abstract. The Generalized Minimal Residual method (GMRES) is often used to solve a large and sparse system $Ax = b$. This paper establishes error bound for residuals of GMRES on solving an $N \times N$ normal tridiagonal Toeplitz linear system. This problem has been studied previously by Li [R.-C. Li, Convergence of CG and GMRES on a tridiagonal Toeplitz linear system, BIT 47 (3) (2007) 577-599.], for two special right-hand sides $b = e_1, e_N$. Also, Li and Zhang [R.-C. Li, W. Zhang, The rate of convergence of GMRES on a tridiagonal Toeplitz linear system, Numer. Math. 112 (2009) 267-293.] for non-symmetric matrix A, presented upper bound for GMRES residuals. But in this paper we establish the upper bound on normal tridiagonal Toeplitz linear systems for special right-hand sides $b = b_{(l)}e_l$, for $1 \leqslant l \leqslant N$.

AMS Subject Classification: 65F10
Keywords and Phrases: GMRES, tridiagonal Toeplitz matrix, linear system

1. Introduction

Iterative methods are used to solve large sparse systems of linear equations $Ax = b$. The Generalized Minimal Residual (GMRES) method is such an algo-

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rithm and is often used for solving a linear system

$$Ax = b,$$

where A is an $N \times N$ matrix, and b is a vector with dimension N.
The basic idea is to seek approximate solutions, with minimize the residual norm, within Krylov subspaces. Specifically, the kth approximation x_k is sought so that the kth residual $r_k = b - Ax_k$ satisfies [14]

$$\|r_k\|_2 = \min_{x \in x_0 + \mathcal{K}_k(A, r_0)} \|b - Ax\|_2,$$

where the Krylov subspace $\mathcal{K}_k(A, r_0)$ of A on r_0 is defined as

$$\mathcal{K}_k(A, r_0) \stackrel{\text{def}}{=} span\{r_0, Ar_0, ..., A^{k-1}r_0\}, \tag{1}$$

and generic norm $\|.\|_2$ is the usual l_2 norm of a vector or the spectral norm of a matrix.
This paper is concerned with the convergence analysis of GMRES on a linear system $Ax = b$, whose cofficient matrix A is a tridiagonal Toeplitz matrix

$$A = \begin{pmatrix} \lambda & \mu & & \\ \nu & \ddots & \ddots & \\ & \ddots & \ddots & \mu \\ & & \nu & \lambda \end{pmatrix}, \tag{2}$$

where λ, μ, and ν are assumed nonzero and possibly complex.
Tridiagonalz Toeplitz linear systems naturally arise from a convection-diffusion problem [13]. This type of matrices are appeared in a variety of applications, such as image processing, numerical differential equations and integral equations, time series analysis, and control theory. When $|\mu| = |\nu|$, A is normal, including symmetric and symmetric positive definite as subcases, R.-C. Li [8] obtained the exact expressions for two special right-hand sides $b = e_1, e_N$. Also, R.-C. Li and W. Zhang [10] for non-symmetric matrix A, presented upper bound for GMRES residual. But in this paper we establish the upper bound on normal tridiagonal Toeplitz linear systems for special right-hand sides $b = b_{(l)}e_l$, $1 \leqslant l \leqslant N$.

Notation. Throughout this paper, I_N is the $N \times N$ identity matrix, and e_j is its jth column.
We shall also adopt MATLAB-like convention to access the entries and matrices. For a vector u and a matrix X, $u_{(j)}$ is jth entry of u, $X_{(:,j)}$ is jth column of X, $X_{(:,i:j)}$ consists of intersections of all rows and column i to column j, and $diag(u)$ is the diagonal matrix with $(diag(u))_{(j,j)} = u_{(j)}$. Finally Π_k denotes the set of polynomials of degree at most k.

2. Basic Concepts

Let $N \times N$ tridiagonal Toeplitz matrix A be given as in (2), where λ, μ, and ν are assumed nonzero and possibly complex, and $|\mu| = |\nu|$. Then A is normal. In fact $|\mu| = |\nu|$ is a sufficient and necessary condition for A to be normal.
In fact [15]

$$A = X\Lambda X^{-1},\ \Lambda = diag(\lambda_1, \ldots, \lambda_N),\ X = SZ, \tag{3}$$

$$\lambda_j = \lambda - 2\sqrt{\mu\nu}\, t_j,\ t_j = cos\,\theta_j, \theta_j = \frac{j\pi}{N+1},\ S = diag(\xi^0, \ldots, \xi^{-N+1}), \tag{4}$$

$$\xi = -\frac{\sqrt{\mu\nu}}{\nu},\ Z_{(:,j)} = \sqrt{\frac{2}{N+1}}(\sin j\theta_1, \ldots, \sin j\theta_N)^T.$$

It can be verified that $Z^T Z = I_N$, and $Z^T = Z$.
Let

$$\omega \overset{\text{def}}{=} -2\sqrt{\mu\nu}, \quad \tau \overset{\text{def}}{=} -\frac{\lambda}{\sqrt{\mu\nu}},$$

and by (4), we have

$$\lambda_j = \omega(t_j - \tau),\ (1 \leqslant j \leqslant N). \tag{5}$$

Without loss of generality, for GMRES, we take initially $x_0 = 0$, and thus the kth approximation x_k is sought so that the kth residual $r_k = b - Ax_k$ satisfies

$$\|r_k\|_2 = \min_{x \in \mathcal{K}_k} \|b - Ax\|_2,$$

where the Krylov subspace $\mathcal{K}_k \equiv \mathcal{K}_k(A, b)$ of A on b is defined as (1). We can write

$$\|r_k\|_2 = \min_{\phi_k \in \Pi_k,\ \phi_k(0)=1} \|\phi_k(A)\ b\|_2.$$

From (3), we have

$$\|\ r_k\ \|_2 = \min_{u(1)=1} \|SZdiag(ZS^{-1}b)V_{k+1,N}^T u\ \|_2, \tag{6}$$

where $V_{k+1,N}$ is the $(k+1) \times N$ rectangular Vandermonde matrix

$$V_{k+1,N} \overset{\text{def}}{=} \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_N \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^k & \lambda_2^k & \ldots & \lambda_N^k \end{pmatrix}.$$

Recall Chebyshev polynomials of the second kind:

$$U_m(t) = \begin{cases} \frac{\sin((m+1)\arccos t)}{\sin(\arccos t)} & \text{for real t and } |t| \leqslant 1, \\ \frac{(t+\sqrt{t^2-1})^{m+1}-(t-\sqrt{t^2-1})^{m+1}}{2\sqrt{t^2-1}} & \text{else.} \end{cases}$$

and define the mth Translated Chebyshev polynomial of the second kind in z of degree m by

$$U_m(z;\omega,\tau) \stackrel{\text{def}}{=} U_m(z/\omega+\tau) = a_{m\,m}z^m + a_{m-1\,m}z^{m-1} + \ldots + a_{1\,m}z + a_{0\,m}. \quad (7)$$

Define also R_m, and $\mathbf{U}_N$ by

$$R_m \stackrel{\text{def}}{=} \begin{pmatrix} a_{00} & a_{01} & \dots & a_{0\,m-1} \\ & a_{11} & \dots & a_{1\,m-1} \\ & & \ddots & \vdots \\ & & & a_{m-1\,m-1} \end{pmatrix}, \quad (8)$$

$$\mathbf{U}_N \stackrel{\text{def}}{=} \begin{pmatrix} U_0(t_1) & U_0(t_2) & \dots & U_0(t_N) \\ U_1(t_1) & U_1(t_2) & \dots & U_1(t_N) \\ \vdots & \vdots & \ddots & \vdots \\ U_{N-1}(t_1) & U_{N-1}(t_2) & \dots & U_{N-1}(t_N) \end{pmatrix}. \quad (9)$$

Lemma 2.1. *Let R_m and $\boldsymbol{U}_N$ define as (8), and (9), then*

$$V_{k+1,N}^T = \boldsymbol{U}_{k+1,N}^T R_{k+1}^{-1}. \quad (10)$$

Proof. The proof is cosequence from (5) and (7). Also, see [10]. □

The next lemma was proven in [6].

Lemma 2.2. *If W has full column rank, then*

$$\min_{u(1)=1} \|Wu\|_2 = [e_1^T(W^*W)^{-1}e_1]^{-1/2}.$$

In particular if W is nonsingular, $\min_{u(1)=1}\|Wu\|_2 = \|W^{-*}e_1\|_2^{-1}$.

The next theorem gives the upper bounds for GMRES residuals and was proven in [10].

Theorem 2.3. *For $Ax = b$, where A is a non-normal tridiagonal Toeplitz as in (2) with nonzero (real or complex) parameters λ, μ, and ν. Then the kth GMRES residual r_k satisfies for $1 \leqslant k < N$*

$$\frac{\|r_k\|_2}{\|r_0\|_2} \leqslant \sqrt{k+1}\left[\sum_{j=0}^{k} |\zeta|^{2j} |T_j(\tau)|^2\right]^{-1/2},$$

where T_j is the jth Chebyshev polynomial of the first kind, and

$$\zeta = min\left\{|\xi|, |\xi|^{-1}\right\}.$$

Theorem 2.3 gives upper bound for GMRES residual on non-symmetric tridiagonal Toeplitz linear systems $Ax = b$. Our mail goal in this paper is the next Theorem, where gives upper bound for GMRES residual on normal tridiagonal Toeplitz linear systems $Ax = b$, when $b = b_{(l)}e_l$. If $|\mu| = |\nu|$, then A is normal, and for case $\mu = -\nu$, A becomes both normal and non-symmetric. Therefore for this case, we compare Theorem 2.3 and next Theorem in our examples.

Theorem 2.4. *For $Ax = b$, where A is a normal tridiagonal Toeplitz as in (2) with nonzero (real or complex) parameters λ, μ, and ν, and $b = b_{(l)}e_l$ $(1 \leqslant l \leqslant N)$, then the k-th GMRES residual r_k satisfies for $1 \leqslant k \leqslant N$*

$$\|r_k\|_2 \leqslant | b_{(l)}| \max_{\theta_i}(\frac{\sin l\theta_i}{\sin\theta_i})[\sum_{j=0}^{k}|U_j(\tau)|^2]^{-1/2}.$$

Proof. From (6) and (10), we have

$$\|r_k\|_2 \leqslant \| S\|_2 \parallel M_{(:,1:k+1)}\|_2 \min_{u(1)=1} \|R_{k+1}^{-1}\|_2 \tag{11}$$

where $M = Zdiag(ZS^{-1}b)\mathbf{U}_N^T$. Because $|\mu| = |\nu|$, we have $\| S\|_2 = 1$. By Lemma 2.2, we have

$$\min_{u(1)=1} \|R_{k+1}^{-1}\|_2 = [\sum_{j=0}^{k}|U_j(\tau)|^2]^{-1/2}. \tag{12}$$

To compute $\| M_{(:,1:k+1)}\|_2$, we shall investigate M first. Thus

$$M = Zdiag(ZS^{-1}b) = \sum_{l=1}^{N} b_{(l)}\xi^{l-1}M_l,$$

where $M_l = Zdiag(Z_{(:,l)}\mathbf{U}_N^T)$. But, we can also write

$$M_l = Z^T D_l Z,$$

where

$$D_l = diag(\frac{\sin l\theta_1}{\sin\theta_1}, \frac{\sin l\theta_2}{\sin\theta_2}, \ldots, \frac{\sin l\theta_N}{\sin\theta_N}).$$

Hence

$$\| M_{(:,1:k+1)}\|_2 \leqslant | b_{(l)}| \max_{\theta_i}(\frac{\sin l\theta_i}{\sin\theta_i}). \tag{13}$$

Therefore the proof is now completed by combining (11), (12), and (13). □

In the following, we give examples, which show the upper bound by Theorem 2.4.

Example 2.5. Let

$$A = \begin{pmatrix} 3 & 2.5 & & \\ -2.5 & \ddots & \ddots & \\ & \ddots & \ddots & 2.5 \\ & & -2.5 & 3 \end{pmatrix}_{50\times 50},$$

and $b = e_{49}$. Because A is both normal and non-symmetric, we compare Theorem 2.3 and our upper bounds by Theorem 2.4. Figure 1 plot GMRES residuals and their upper bounds by Theorem 2.4 and Theorem 2.3. We see that our upper bounds are smaller than Theorem 2.3. It seems that our upper bounds are very good. Thus we can approximate GMRES residuals by Theorem 2.4.

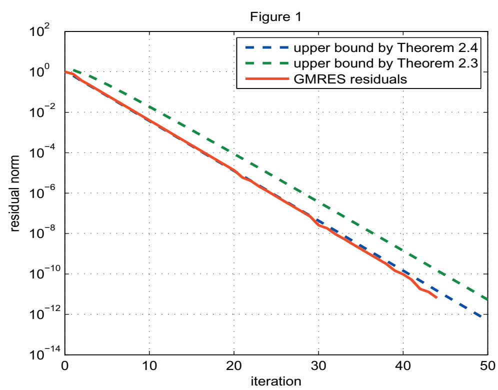

Figure 1: GMRES residuals, and their upper bounds by Theorem 2.4.

Example 2.6. Let

$$A = \begin{pmatrix} 2 & 0.74 & & \\ -0.74 & \ddots & \ddots & \\ & \ddots & \ddots & 0.74 \\ & & -0.74 & 2 \end{pmatrix}_{50\times 50},$$

and $b = 2e_{48}$. Because A is both normal and non-symmetric, we compare Theorem 2.3 and our upper bounds by Theorem 2.4. Figure 2 plot GMRES residuals and their upper bounds by Theorem 2.4 and Theorem 2.3. We see that our upper bounds are smaller than Theorem 2.4.

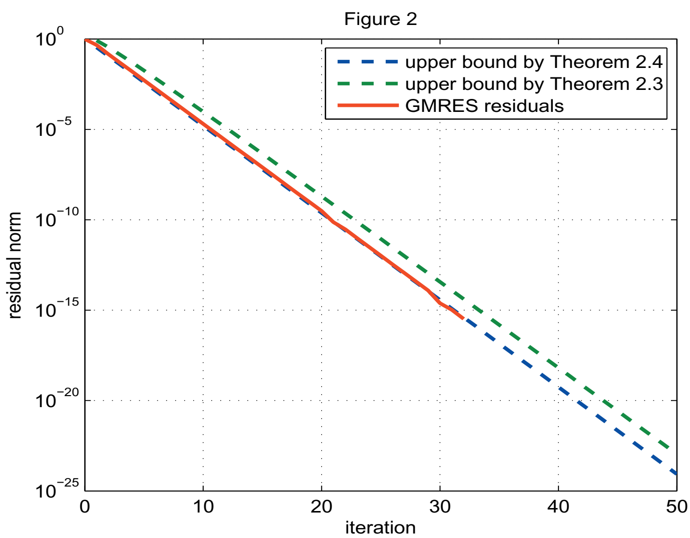

Figure 2: GMRES residuals, our upper bounds by Theorems 2.4 and upper bounds by Theorem 2.3.

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