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# On Extended H-Proper Hypersurfaces with Three Curvatures in the Lorentz 5-Pseudosphere

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**Abstract.** Based on previous researches, the **H**-proper hypersurfaces of Riemannian space forms include a wide range of minimal ones. In this paper, we study an extended version of **H**-properness condition on Lorentz hypersurfaces with three distinct principal curvatures in the 5-dimensional Lorentz pseudosphere. The main goal is to prove the 1-minimality of such a hypersurface.

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# 1 Introduction

The subject of $\mathbf{H}$-proper submanifolds is an interesting research topic in mathematical physics, which deals with the bienergy functional and its critical points arisen form the tension field. Also, among the differential geometric research subjects, the study of constant mean curvature submanifolds is of great importance. In this context, the subject of biharmonic submanifolds has received much attentions. By definition, a hypersurface is $\mathbf{H}$-proper if it satisfies the condition $\Delta\mathbf{H} = a\mathbf{H}$ for a constant real number $a$, where $\Delta$ is the Laplace operator and $\mathbf{H}$ is the ordinary mean curvature vector field of hypersurface.

It is proven that every $\mathbf{H}$-proper hypersurface of a Riemannian space form has constant mean curvature. This is a question that has remained unanswered in some cases and is closely related to a well-known conjecture of Bang-Yen Chen which says that every submanifold of an Euclidean space with harmonic mean curvature vector field has zero mean curvature [7]. It has several improvements (for instance) in [1, 6]. In this field, Defever has proved that the mean curvature of a hypersurface in $\mathbb{E}^4$ is constant if its mean curvature vector is proper ([8]). In the context of hypersurfaces in semi-Riemannian manifolds, it has been studied in the last two decades (see [5, 2, 14]).

We take an extend version of this condition by putting the second mean curvature $\mathbf{H}_2$ instead of $\mathbf{H}$ and the Cheng-Yau operator C instead of the Laplace operator. The operator C denotes the linear operator arisen from the first variation of the second mean curvature (see [3, 11, 13]). We study the $\mathbf{H}_2$-proper timelike (i.e. Lorentzian) hypersurfaces of Lorentz 5-pseudosphere.

# 2 Prerequisite Concepts

Here are some concepts and notations, required in the rest of the article, taken from [10, 11, 15]. We use the semi-Euclidean $q$-space $\mathbb{E}^q_\xi$ of index $\xi = 1, 2$, equipped with the product defined by $\langle \mathbf{v}, \mathbf{u}\rangle = -\sum_{i=1}^{\xi} v_i u_i + \sum_{i=\xi+1}^{q} v_i u_i$, for each vectors $\mathbf{v} = (v_1, \ldots, v_q)$ and $\mathbf{u} = (u_1, \ldots, u_q)$ in $\mathbb{R}^q$. In fact, we deal with the 5-dimensional Lorentz space forms with the following common notation

$$\mathbb{M}^5_1(c) = \begin{cases} \mathbb{S}^5_1(r) & (\text{if } c = 1/r^2) \\ \mathbb{L}^5 = \mathbb{E}^4_1 & (\text{if } c = 0) \\ \mathbb{H}^5_1(-r) & (\text{if } c = -1/r^2), \end{cases}$$

where, for $r > 0$, $\mathbb{S}^5_1(r) = \{\mathbf{v} \in \mathbb{E}^6_1 | \langle \mathbf{v}, \mathbf{v} \rangle = r^2\}$ denotes the 5-dimensional $r$-radius pseudosphere, and $\mathbb{H}^5_1(-r) = \{\mathbf{v} \in \mathbb{E}^6_2 | \langle \mathbf{v}, \mathbf{v} \rangle = -r^2, v_1 > 0\}$ denotes the $(-r)$-radius pseudo-hyperbolic 5-space. In the canonical cases $c = \pm 1$, we get the de canonical 5-pseudosphere 5-space $\mathbb{S}^5_1 := \mathbb{S}^5_1(1)$ and pseudo-hyperbolic 5-space $\mathbb{H}^5_1 = \mathbb{H}^5_1(-1)$. Also, for $c = 0$ we get the Lorentz-Minkowski 5-space $\mathbb{L}^5 := \mathbb{E}^5_1$.

We consider a Lorentzian (timelike) hypersurface $M^4_1$ of $\mathbb{S}^5_1$ defined by an isometric immersion $\mathbf{x} : M^4_1 \to \mathbb{S}^5_1$. The set of all smooth tangent vector fields on $M^4_1$ is denoted by $\chi(M^4_1)$. According to the Lorentz metric on $M^4_1$ induced from $\mathbb{S}^5_1$, we can determine the possible states for a base of the tangent space of $M^4_1$. For a detailed study, one can refer to the references [9, 10, 12]. In general, a basis $\Omega := \{w_1, w_2, w_3, w_4\}$ of a Lorentz linear 4-space is said to be *orthonormal* if it satisfies equalities $\langle w_1, w_1 \rangle = -1$, $\langle w_2, w_2 \rangle = \langle w_3, w_3 \rangle = \langle w_4, w_4 \rangle = 1$ and $\langle w_i, w_j \rangle = 0$ for each $i \neq j$. Also, $\Omega$ is called *pseudo-orthonormal* if it satisfies $\langle w_1, w_1 \rangle = \langle w_2, w_2 \rangle = 0$, $\langle w_1, w_2 \rangle = -1$ and $\langle w_i, w_j \rangle = \delta^j_i$ for $j = 3, 4$ and $i = 1, \ldots, 4$. As usual, $\delta$ is the Kronecker delta.

Associated to a basis chosen on $M^4_1$, the second fundamental form (shape operator) S has four different matrix forms. When the metric on $M^4_1$ has diagonal form $\mathcal{G}_1 := \text{diag}[-1, 1, 1, 1]$, then S is of form $\mathcal{D}_1 = \text{diag}[\lambda_1, \lambda_2, \lambda_3, \lambda_4]$ or

$$\mathcal{D}_2 = \text{diag}[\begin{bmatrix} \lambda_1 & \lambda_2 \\ -\lambda_2 & \lambda_1 \end{bmatrix}, \lambda_3, \lambda_4], \quad (\lambda_2 \neq 0).$$

In the non-diagonal metric case $\mathcal{G}_2 = \text{diag}[\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, 1, 1]$ the shape operator is of form

$\mathcal{D}_3 = \text{diag}[\begin{bmatrix} \lambda_1 + \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \lambda_1 - \frac{1}{2} \end{bmatrix}, \lambda_2, \lambda_3]$ or

$\mathcal{D}_4 = \text{diag}[\begin{bmatrix} \lambda_1 & 0 & \frac{\sqrt{2}}{2} \\ 0 & \lambda_1 & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \lambda_1 \end{bmatrix}, \lambda_2]$.

When $S = \mathcal{D}_k$, we say that $M^4_1$ is a $\mathcal{D}_k$-*hypersurface*.

**Definition 2.1.** *We define the ordered quadruple $\{\kappa_1;\kappa_2;\kappa_3;\kappa_4\}$ of principal curvatures as follows:*

$$\{\kappa_1;\kappa_2;\kappa_3;\kappa_4\} = \begin{cases} \{\lambda_1;\lambda_2;\lambda_3;\lambda_4\} & (\text{if } S=\mathcal{D}_1) \\ \{\lambda_1+i\lambda_2;\lambda_1-i\lambda_2;\lambda_3;\lambda_4\} & (\text{if } S=\mathcal{D}_2) \\ \{\lambda_1;\lambda_1;\lambda_2;\lambda_3\} & (\text{if } S=\mathcal{D}_3) \\ \{\lambda_1;\lambda_1;\lambda_1;\lambda_2\} & (\text{if } S=\mathcal{D}_4). \end{cases}$$

Clearly, the characteristic polynomial of S is $\mathcal{Q}(t)=\sum_{j=0}^{4}(-1)^j s_j t^{4-j}$, where $s_j := \sum_{1\le j_1<\ldots<j_i\le 4}\kappa_{j_1}\ldots\kappa_{j_i}$ for $j=1,\ldots,4$ and $s_0:=1$. As usual, using $s_j$ (for $j=1,\ldots,4$) we define $H_j$ (i.e. the *jth mean curvature*) of $M_1^4$ by equality $\binom{4}{j}H_j = s_j$. In special case, $H_1$ is the ordinary mean curvature $H$. The second mean curvature $H_2$ and the normalized scalar curvature $R$ satisfy the equality $H_2 := n(n-1)(1-R)$.

The hypersurfaces $M_1^4$ is called *j-minimal*, if its $(j+1)$th mean curvature is equal to zero. $M_1^4$ is called *isoparametric* if its shape operator is either of diagonal type $\mathcal{D}_1$ with constant eigenvalues or of non-diagonal types $\mathcal{D}_k$ $(k=2,3,4)$ and the coefficients of its minimal polynomial are constant.

As well-known, the sequence $\{\mathrm{N}_j\}_{j=0}^4$ of Newton transformations is considered as $\mathrm{N}_0 = I$ (i.e. the identity map) and $\mathrm{N}_j = s_j\mathrm{I} - \mathrm{S}\circ\mathrm{N}_{j-1}$ for $j=1,\ldots,4$. Using its explicit formula, $\mathrm{N}_j = \sum_{i=0}^{j}(-1)^i s_{j-i}\mathrm{S}^i$ (where $\mathrm{S}^0=\mathrm{I}$) and the Cayley-Hamilton theorem (which stats that any operator is annihilated by its characteristic polynomial) we have $\mathrm{N}_4 = 0$ (see [4, 11]).

In the rest, we use the following notation for $k=1,2,3$,

$$\mu_{i_1,\ldots,i_m;k} = \sum_{1\le j_1<\ldots<j_k\le 4; j_l\ne i_1,\ldots,i_m} \kappa_{j_1}\cdots\kappa_{j_k}, \quad (i_1,\ldots,i_m\in\{1,2,3,4\}).$$

Clearly, $\mathrm{N}_j$ has four possible matrix forms since it is defined in terms of S. If $\mathrm{S}=\mathcal{D}_1$, then for $j=1,2,3$, we have $\mathrm{N}_j = diag[\mu_{1;j},\cdots,\mu_{4;j}]$.

When $\mathrm{S}=\mathcal{D}_2$, we have

$\mathrm{N}_1 = \mathrm{diag}[\left[\begin{smallmatrix}\lambda_1+\lambda_3+\lambda_4 & -\lambda_2 \\ \lambda_2 & \lambda_1+\lambda_3+\lambda_4\end{smallmatrix}\right], 2\lambda_1+\lambda_4, 2\lambda_1+\lambda_3]$,

$\mathrm{N}_2 = \mathrm{diag}[\left[\begin{smallmatrix}\lambda_1(\lambda_3+\lambda_4)+\lambda_3\lambda_4 & -\lambda_2(\lambda_3+\lambda_4) \\ \lambda_2(\lambda_3+\lambda_4) & \lambda_1(\lambda_3+\lambda_4)+\lambda_3\lambda_4\end{smallmatrix}\right], \lambda_1^2+\lambda_2^2+2\lambda_1\lambda_4, \lambda_1^2+\lambda_2^2+2\lambda_1\lambda_3]$,

When $\mathrm{S}=\mathcal{D}_3$, we have

$\mathrm{N}_1 = \mathrm{diag}[\begin{bmatrix} \lambda_2+\lambda_3+\lambda_1-\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \lambda_2+\lambda_3+\lambda_1+\frac{1}{2} \end{bmatrix}, 2\lambda_1+\lambda_3, 2\lambda_1+\lambda_2]$,

$\mathrm{N}_2 = \mathrm{diag}[\begin{bmatrix} \lambda_2\lambda_3+(\lambda_1-\frac{1}{2})(\lambda_2+\lambda_3) & -\frac{1}{2}(\lambda_2+\lambda_3) \\ \frac{1}{2}(\lambda_2+\lambda_3) & \lambda_2\lambda_3+(\lambda_1+\frac{1}{2})(\lambda_2+\lambda_3) \end{bmatrix}, \lambda_1(\lambda_1+2\lambda_3), \lambda_1(\lambda_1+2\lambda_2)]$,

In the case $\mathrm{S} = \mathcal{D}_4$, we have

$\mathrm{N}_1 = \mathrm{diag}[\begin{bmatrix} 2\lambda_1+\lambda_2 & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 2\lambda_1+\lambda_2 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 2\lambda_1+\lambda_2 \end{bmatrix}, 3\lambda_1]$,

$\mathrm{N}_2 = \mathrm{diag}[\begin{bmatrix} 2\lambda_1\lambda_2+\lambda_1^2-\frac{1}{2} & -\frac{1}{2} & -\frac{\sqrt{2}}{2}(\lambda_1+\lambda_2) \\ \frac{1}{2} & 2\lambda_1\lambda_2+\lambda_1^2+\frac{1}{2} & \frac{\sqrt{2}}{2}(\lambda_1+\lambda_2) \\ \frac{\sqrt{2}}{2}(\lambda_1+\lambda_2) & \frac{\sqrt{2}}{2}(\lambda_1+\lambda_2) & 2\lambda_1\lambda_2+\lambda_1^2 \end{bmatrix}, 3\lambda_1^2]$.

Some important identities occur in four cases as follow ([11]).

$\mu_{j,1} = 4H_1-\kappa_j, \quad \mu_{j,2} = 6H_2-\kappa_j\mu_{j,1} = 6H_2-4\kappa_jH_1+\kappa_j^2, \ (j=1,\ldots,4),$

$tr(\mathrm{N}_1) = 12H_1, \ tr(\mathrm{N}_2) = 12H_2, \ tr(\mathrm{N}_1\circ\mathrm{S}) = 12H_2, \ tr(\mathrm{N}_2\circ\mathrm{S}) = 12H_3,$

$tr\mathrm{S}^2 = 4(4H_1^2-3H_2), \ \ tr(\mathrm{N}_1\circ\mathrm{S}^2) = 12(2H_1H_2-H_3),$

$tr(\mathrm{N}_2\circ\mathrm{S}^2) = 4(4H_1H_3-H_4).$

**Definition 2.2.** *The Cheng-Yau operator on $M_1^4$ is defined as $C(h) = tr(N_j\circ\nabla^2 h)$ for each $h\in\mathcal{C}^\infty(M_1^4)$, where $\langle\nabla^2h(V), W\rangle = Hess^h(V, W)$ for each $V, W\in\chi(M_1^4)$.*

The Cheng-Yau operator has an explicit version according to an orthonormal basis $\{w_1,\cdots,w_4\}$, given by

$$\mathrm{C}(f) = \sum_{i=1}^{4}\nu_i\mu_{i,1}(w_iw_if-\nabla_{w_i}w_if),$$

where, $\nu_1=-1$ and $\nu_i=1$ for $i=2,3,4$.

On every orientable Lorentzian hypersurface $\mathbf{x}: M_1^4\to\mathbb{S}_1^5$, one can choose a unit spacelike normal vector field $\mathbf{n}$ and its related shape operator. So, every vector $\mathbf{z}$ on $M_1^4$ decomposes as $\mathbf{z}=\mathbf{z}^T+\mathbf{z}^{\mathbf{n}}$ where $\mathbf{z}^T$ and $\mathbf{z}^{\mathbf{n}}$ are its tangent and normal components ([4, 11]). So we have

$\nabla<\mathbf{x},\mathbf{z}>=\mathbf{z}^T, \quad \nabla<\mathbf{n},\mathbf{z}>=-S\mathbf{z}^T.$

Here we outline the key formulas we need to present the main concepts.

$$\mathrm{C}\mathbf{x} = 12H_2\mathbf{n}-12cH_1\mathbf{x}, \tag{1}$$

$$\mathrm{C}\mathbf{H}_2 = 2[\mathrm{N}_2\nabla H_2-9H_2\nabla H_2]+[\mathrm{C}H_2-12H_2(2H_1H_2-H_3)]\mathbf{n}. \tag{2}$$

**Definition 2.3.** *A hypersurface $M_1^4$ in $\mathbb{M}_1^5(c)$ is said to be $\boldsymbol{H}_2$-proper if its second mean curvature vector field satisfies $C\boldsymbol{H}_2 = a\boldsymbol{H}_2$, for a constant number $a$. Clearly, this condition has a simpler expression by two equations as:*

$$\begin{aligned}&\text{(i) } CH_2 = H_2(a + 24H_1H_2 - 12H_3),\\ &\text{(ii) } N_2\nabla H_2 = 9H_2\nabla H_2.\end{aligned} \tag{3}$$

**Example 2.4.** Let $0 < t, u < 1$, $t^2 + u^2 < 1$ and $\Gamma = \mathbb{S}_1^1(t) \times \mathbb{S}^1(u) \times \mathbb{S}^1(\sqrt{1 - t^2 - u^2}) \subset \mathbb{S}_1^5$ defined as

$$\Gamma = \{(z_1, \ldots, z_6) \in \mathbb{L}^6 | - z_1^2 + z_2^2 = t^2, z_3^2 + z_4^2 = u^2, z_5^2 + z_6^2 = 1 - t^2 - u^2\},$$

having three distinct principal curvatures $\kappa_1 = \frac{-\sqrt{1-t^2-u^2}}{t}$, $\kappa_2 = \frac{\sqrt{1-t^2-u^2}}{u}$ and $\kappa_3 = \kappa_4 = \frac{\sqrt{t^2+u^2}}{\sqrt{1-t^2-u^2}}$. Clearly, $\Gamma$ is $\mathbf{H}_2$-proper and all of its mean curvatures are constant.

The structure equations of $\mathbb{S}_1^5$ are given by $d\omega_i = \sum_{j=1}^{5} \omega_{ij} \wedge \omega_j$, $\omega_{ij} + \omega_{ji} = 0$ and $d\omega_{ij} = \sum_{l=1}^{4} \omega_{il} \wedge \omega_{lj}$. Restricted to $M_1^4$, we have $\omega_5 = 0$. So, and then, $d\omega_5 = \sum_{i=1}^{4} \omega_{5,i} \wedge \omega_i = 0$.

A lemma due to Cartan gives the decomposition $\omega_{5,i} = \sum_{j=1}^{4} h_{ij}\omega_j$ for smooth functions $h_{ij}$ satisfying the equality $B = \sum_{i,j} h_{ij}\omega_i\omega_j w_5$ where $B$ is the second fundamental form of $M_1^4$. The mean curvature $H$ is given by $H = \frac{1}{4} \sum_{i=1}^{4} h_{ii}$. So, the structure equations of $M_1^4$ are

$$d\omega_i = \sum_{j=1}^{4} \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = \sum_{k=1}^{4} \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l=1}^{4} R_{ijkl}\omega_k \wedge \omega_l,$$

for $j, i = 1, \ldots, 4$. Let $h_{ijk}$ be covariant derivation of $h_{ij}$. So, we have

$$dh_{ij} = \sum_{k=1}^{4} h_{ijk}\omega_k + \sum_{k=1}^{4} h_{kj}\omega_{ik} + \sum_{k=1}^{4} h_{ik}\omega_{jk},$$

and from Codazzi equation we have $h_{ijk} = h_{ikj}$.

One can choose $w_1, \ldots, w_4$ such that $h_{ij} = \kappa_i \delta_{ij}$. On the other hand, the Levi-Civita connection of $M_1^4$ satisfies $\nabla_{w_i} w_j = \sum_k \omega_{jk}(w_i) w_k$, and we have $w_i(k_j) = \omega_{ij}(w_j)(\kappa_i - \kappa_j)$ and

$$\omega_{ij}(w_l)(\kappa_i - \kappa_j) = \omega_{il}(w_j)(\kappa_i - \kappa_l)$$

whenever $i, j, l$ are distinct.

## 3 $\mathcal{D}_1$-Hypersurfaces

The $\mathbf{H}_2$-proper timelike $\mathcal{D}_1$-hypersurfaces in the Lorentz 5-pseudosphere with constant mean curvature are examined from different points of view.

**Theorem 3.1.** *Suppose that $\boldsymbol{x} : M_1^4 \to \mathbb{S}_1^5$ is a $\boldsymbol{H}_2$-proper $\mathcal{D}_1$-hypersurface which has constant mean curvature and non-constant second mean curvature. Then, one of its non-constant principal curvatures has multiplicity one.*

**Proof.** By assumption, $\mathrm{U} = \{q \in M_1^4 | \nabla H_2(q) \neq 0\}$ is nonempty. We consider a connected component of U. The condition (3)(ii) gives that $w_1 := \frac{\nabla H_2}{||\nabla H_2||}$ is an eigenvector of $\mathrm{N}_2$ with eigenvalue $9H_2$, on U. We choose an orthonormal basis $\{w_1, \ldots, w_4\}$ satisfying $\mathrm{S}w_i = \lambda_i w_i$, $\mathrm{N}_2 w_i = \mu_{i,2} w_i$, (for $i = 1, \ldots, 4$), we have $\mu_{1,2} = 9H_2$, which by (2) gives

$$H_2 = \frac{1}{3}\lambda_1(\lambda_1 - 4H_1). \tag{4}$$

From equality $\nabla H_2 = \sum\limits_{i=1}^{4} w_i(H_2) w_i$, we get

$$w_1(H_2) \neq 0, \quad w_i(H_2) = 0, \quad i = 2, 3, 4. \tag{5}$$

which by (4) give

$$w_1(\lambda_1) \neq 0, \quad w_i(\lambda_1) = 0 \quad i = 2, 3, 4. \tag{6}$$

So, $\lambda_1$ is non-constant. Now, putting $\nabla_{w_i} w_j = \sum_{k=1}^{4} \omega_{ij}^k w_k$ (for $i, j = 1, \ldots, 4$), by $w_k \langle w_i, w_j \rangle = 0$, we get $\nu_j \omega_{ki}^j = -\nu_i \omega_{kj}^i$ (for $i, j, k \in \{1, \ldots, 4\}$). On the other hand, by Codazzi equation, for distinct $i, j, k$, we get

$$w_j(\lambda_i) = (\lambda_j - \lambda_i)\omega_{ij}^i, \quad (\lambda_j - \lambda_i)\omega_{kj}^i = (\lambda_k - \lambda_i)\omega_{jk}^i. \tag{7}$$

Since $w_1(\lambda_1) \neq 0$, we can show that $\lambda_j \neq \lambda_1$ for $j = 2, 3, 4$. If $\lambda_j = \lambda_1$ for some integer $j \neq 1$, then $w_1(\lambda_j) = w_1(\lambda_1) \neq 0$. Also, by (7) we get $0 = (\lambda_1 - \lambda_j)\omega_{j1}^j = w_1(\lambda_j) = w_1(\lambda_1)$. This is a contradiction. Therefore, $\lambda_1$ is non-constant and its multiplicity is one. □

**Theorem 3.2.** *Suppose that $\boldsymbol{x} : M_1^4 \to \mathbb{S}_1^5$ is a $\boldsymbol{H}_2$-proper $\mathcal{D}_1$-hypersurface such that whose mean curvature is constant and second mean curvature is non-constant. If it has exactly three distinct principal curvatures, then according to the orthonormal basis $\{w_1, \ldots, w_4\}$ of principal vectors of $M_1^4$ associated to principal curvatures $\lambda_1, \lambda_2 = \lambda_3, \lambda_4$, the following equalities occur:*

$$\begin{aligned}
&(i)\nabla_{w_1} w_1 = 0, \ \nabla_{w_2} w_1 = \alpha w_2, \ \nabla_{w_3} w_1 = \alpha w_3, \ \nabla_{w_4} w_1 = -\beta w_4, \\
&(ii)\nabla_{w_2} w_2 = -\alpha w_1 + \omega_{22}^3 w_3 + \gamma w_4, \ \nabla_{w_i} w_2 = \omega_{i2}^3 w_3 \quad for \ i = 1, 3, 4 \ ; \\
&(iii)\nabla_{w_3} w_3 = -\alpha w_1 - \omega_{32}^3 w_3 + \gamma w_4, \quad \nabla_{w_i} w_3 = -\omega_{i2}^3 w_2 \quad for \ i = 1, 2, 4 \ , \\
&(iv)\nabla_{w_1} w_4 = 0, \ \nabla_{w_2} w_4 = -\gamma w_2, \ \nabla_{w_3} w_4 = -\gamma w_3, \ \nabla_{w_4} w_4 = \beta w_1,
\end{aligned} \tag{8}$$

*where* $\alpha := \frac{w_1(\lambda_2)}{\lambda_1 - \lambda_2}$, $\beta := \frac{w_1(\lambda_1 + 2\lambda_2)}{\lambda_1 - \lambda_4}$, $\gamma := \frac{w_4(\lambda_2)}{\lambda_2 - \lambda_4}$ .

**Proof.** Using Theorem 3.1, we get the equalities (6) and (7) which give that the multiplicity of $\lambda_1$ is one. Also, in direct ways we get $[w_2, w_3](\lambda_1) = [w_3, w_4](\lambda_1) = [w_2, w_4](\lambda_1) = 0$, which yields

$$\omega_{23}^1 = \omega_{32}^1, \quad \omega_{34}^1 = \omega_{43}^1, \quad \omega_{24}^1 = \omega_{42}^1. \tag{9}$$

Due to the triplet of principal curvatures of $M_1^4$, we assume $\lambda_3 = \lambda_2$, and so $\lambda_4 = 4H_1 - \lambda_1 - 2\lambda_2$. By considering distinct $i$, $j$ and $k$ in equalities

(7), we obtain $w_2(\lambda_2) = w_3(\lambda_2) = 0$ and then,

$$\begin{aligned}
&(i)\ \omega_{11}^1 = \omega_{12}^1 = \omega_{13}^1 = \omega_{14}^1 = \omega_{31}^2 = \omega_{21}^3 = \omega_{34}^2 = \omega_{24}^3 = \omega_{42}^4 = \omega_{43}^4 = 0,\\
&(ii)\ \omega_{21}^2 = \omega_{31}^3 = \frac{w_1(\lambda_2)}{\lambda_1 - \lambda_2},\ \omega_{41}^4 = \frac{-w_1(\lambda_1 + 2\lambda_2)}{\lambda_1 - \lambda_4},\ \omega_{24}^2 = \omega_{34}^3 = \frac{-w_4(\lambda_2)}{\lambda_2 - \lambda_4},\\
&(iii)\ (\lambda_1 - \lambda_4)\omega_{24}^1 = (\lambda_1 - \lambda_2)\omega_{42}^1,\ (\lambda_1 - \lambda_4)\omega_{34}^1 = (\lambda_1 - \lambda_2)\omega_{43}^1.
\end{aligned} \tag{10}$$

From (9) and (10) we get $\omega_{24}^1 = \omega_{42}^1, = \omega_{34}^1 = \omega_{43}^1 = \omega_{12}^4 = \omega_{13}^4 = 0$. Therefore, all claimed equalities obtain from the above results. □

**Theorem 3.3.** *Suppose that $\boldsymbol{x} : M_1^4 \to \mathbb{S}_1^5$ is a $\boldsymbol{H}_2$-proper $\mathcal{D}_1$-hypersurface such that whose mean curvature is constant and second mean curvature is non-constant. If it has exactly three distinct principal curvatures $\lambda_1, \lambda_2 = \lambda_3, \lambda_4$ according to an orthonormal basis $\{w_1, \ldots, w_4\}$ of principal directions satisfying $w_4(\lambda_2) = 0$, then the following equality occurs:*

$$w_1(\lambda_2)w_1(\lambda_1 + 2\lambda_2) = \frac{1}{2}\lambda_2(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_1)(2\lambda_1 + 4\lambda_2 + \lambda_4). \tag{11}$$

**Proof.** By considering different choices of vectors $w_1$, $w_2$, $we_3$ in the Gauss curvature tensor $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, and using Theorem 3.2, we obtain:

$$\begin{aligned}
&(i)\ w_1(\alpha) + \alpha^2 = -\lambda_1\lambda_2, \quad \beta^2 - w_1(\beta) = -\lambda_1\lambda_4;\\
&(ii)\ w_1\left(\frac{w_4(\lambda_2)}{\lambda_2 - \lambda_4}\right) + \alpha\frac{w_4(\lambda_2)}{\lambda_2 - \lambda_4} = 0;\\
&(iii)\ w_4(\alpha) - (\alpha + \beta)\frac{w_4(\lambda_2)}{\lambda_2 - \lambda_4} = 0;\\
&(iv)\ w_4\left(\frac{w_4(\lambda_2)}{\lambda_2 - \lambda_4}\right) + \alpha\beta - \left(\frac{w_4(\lambda_2)}{\lambda_2 - \lambda_4}\right)^2 = \lambda_2\lambda_4.
\end{aligned} \tag{12}$$

Now, from (3)(ii), applying Proposition (3.2), we obtain

$$\begin{aligned}
&(\lambda_1 - 4H_1)w_1w_1(H_2) - (2(\lambda_2 - 4H_1)\alpha + (\lambda_1 + 2\lambda_2)\beta)w_1(H_2)\\
&= 12H_2(2H_1H_2 - H_3),
\end{aligned} \tag{13}$$

where $\alpha := \frac{w_1(\lambda_2)}{\lambda_1 - \lambda_2}$ and $\beta := \frac{w_1(\lambda_1 + 2\lambda_2)}{\lambda_1 - \lambda_4}$.

Also, by equalities (5) and (8), we get

$$w_i w_1(H_2) = 0, (i = 2, 3, 4). \tag{14}$$

Also, by derivation of $\alpha$ and $\beta$ along $w_4$, we get

$$(\lambda_1 - \lambda_2)w_4(\alpha) - \alpha w_4(\lambda_2) = w_4 w_1(\lambda_2) = \frac{1}{2}(\lambda_1 - \lambda_4)w_4(\beta) + \beta w_4(\lambda_2),$$

then

$$\frac{1}{2}(\lambda_1 - \lambda_4)w_4(\beta) = (\lambda_1 - \lambda_2)w_4(\alpha) - (\alpha + \beta)w_4(\lambda_2),$$

which, by using (12), implies

$$w_4(\beta) = \frac{-8w_4(\lambda_2)(\alpha + \beta)(\lambda_2 - H_1)}{(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_4)}.$$

Differentiating (13) along $w_4$ and applying the last result and equalities (14) and (12), we obtain $w_4(\lambda_2) = 0$ or

$$\frac{4(\alpha + \beta)\gamma w_1(H_2)}{\lambda_1 - \lambda_4} = 6H_2(\lambda_2 - \lambda_4)^2, \tag{15}$$

where $\gamma = -8H_1\lambda_1 + {\lambda_1}^2 + 3\lambda_1\lambda_2 - 12H_1\lambda_2 + 16H_1^2$.

We claim that $w_4(\lambda_2) = 0$ because the equality (15) dos'nt occur. Its reason is as follows. By differentiating (15) along $w_4$, we get

$$\frac{[6\gamma(\lambda_2 - H_1) + (3\lambda_1 - 12H_1)(\lambda_1 + \lambda_2 - 2H_1)(\lambda_1 + 3\lambda_2 - 4H_1)]\, w_1(H_2)}{(\lambda_1 + \lambda_2 - 2H_1)^2}$$
$$= 18\frac{H_2(4H_1 + \lambda_1 + 3\lambda_2)^2}{\alpha + \beta}. \tag{16}$$

Eliminating $w_1(H_2)$ from (15) and (16), we obtain

$$\gamma(2\lambda_1 - 2H_1) = (\lambda_1 - 4H_1)(\lambda_1 + \lambda_2 - 2H_1)(-4H_1 + \lambda_1 + 3\lambda_2). \tag{17}$$

Finally, along $w_4$ we differentiate (17), which gives $4H_1 = \lambda_1$. That is impossible since $\lambda_1$ dos'nt have constant value. Consequently, $w_4(\lambda_2) = 0$. Therefore, the main result is implied from the latest equality. □

**Theorem 3.4.** *Suppose that $\boldsymbol{x}: M_1^4 \to \mathbb{S}_1^5$ is a $\boldsymbol{H}_2$-proper $\mathcal{D}_1$-hypersurface with constant mean curvature. If $M_1^4$ has exactly three distinct principal curvatures, then it is 1-minimal.*

**Proof.** As previous theorems, we use the orthonormal basis $\{w_1, \dots, w_4\}$ of principal directions according to principal curvatures $\lambda_1, \lambda_2 = \lambda_3, \lambda_4$. By differentiating (4) along $w_1$ we have

$$w_1(H_2) = \frac{4}{3}(2H_1 - \lambda_1)w_1(\lambda_2) + \frac{4}{3}(\lambda_1 + \lambda_2 - 2H_1)(\lambda_1 - 2H_1)\beta, \quad (18)$$

where $\beta := \frac{w_1(\lambda_1+2\lambda_2)}{\lambda_1-\lambda_4}$. By Theorem 3.3 and equalities (12) and (18), we get

$$\begin{aligned} w_1w_1(H_2) &= \frac{4}{3}\lambda_1\lambda_2(\lambda_1 - \lambda_2)(\lambda_1 + 2H_1) \\ &+ \frac{4}{3}(4H_1 - \lambda_1 - 2\lambda_2)(\lambda_1 - 2H_1)(4\lambda_1\lambda_2 + {\lambda_1}^2 - 4H_1\lambda_2 - 2H_1\lambda_1) \\ &+ \left[3\beta - 4\alpha + 2\frac{(\lambda_1 + \lambda_2 - 2H_1)\beta - (\lambda_1 - \lambda_2)\alpha}{\lambda_1 - 2H_1}\right] w_1(H_2). \end{aligned} \quad (19)$$

Combining (13) and (19), we get

$$(P_{1,2}\alpha + P_{2,2}\beta)w_1(H_2) = P_{3,6}, \quad (20)$$

where, the degree of polynomials $P_{1,2}$, $P_{2,2}$ and $P_{3,6}$ in terms of $\lambda_1$ and $\lambda_2$ are (respectively) 2, 2 and 6. Similarly, differentiating (20) along $w_1$ and using equalities (11), (12)-(i) and (20), we get the following equality

$$P_{4,8}\alpha + P_{5,8}\beta = P_{6,5}w_1(H_2), \quad (21)$$

where, the degree of polynomials $P_{4,8}$, $P_{5,8}$ and $P_{6,5}$ are (respectively) 8, 8 and 5. From (18) and (21), we get

$$\begin{aligned} &\left(P_{4,8} + \frac{4}{3}P_{6,5}(\lambda_1 - \lambda_2)(\lambda_1 - 2H_1)\right)\alpha \\ &+ \left(P_{5,8} - \frac{4}{3}P_{6,5}(\lambda_1 + \lambda_2 - 2H_1)(\lambda_1 - 2H_1)\right)\beta = 0. \end{aligned} \quad (22)$$

On the other hand, from (18) with (20) and using Theorem 3.3, we get

$$P_{2,2}(\lambda_1+\lambda_2-2H_1)(\lambda_1-2H_1)\beta^2-P_{1,2}(\lambda_1-\lambda_2)(\lambda_1-2H_1)\alpha^2=\zeta, \quad (23)$$

where
$\zeta=\lambda_2(4H_1-\lambda_1-2\lambda_2)(\lambda_1-2H_1)\left(P_{2,2}(\lambda_1-\lambda_2)-P_{1,2}(\lambda_1+\lambda_2-2H_1)\right)+\frac{3}{4}P_{3,6}$.

Using Theorem 3.3 and equality (22), we get

$$\begin{aligned}\alpha^2&=\frac{\frac{2}{3}P_{6,5}(\lambda_1-2H_1)(\lambda_1-\lambda_4)+P_{5,8}}{P_{4,8}+\frac{4}{3}P_{6,5}(\lambda_1-2H_1)(\lambda_1-\lambda_2)}\lambda_2\lambda_4,\\ \beta^2&=\frac{\frac{4}{3}P_{6,5}(\lambda_1-2H_1)(\lambda_1-\lambda_2)-P_{4,8}}{P_{5,8}-\frac{2}{3}P_{6,5}(\lambda_1-2H_1)(\lambda_1-\lambda_4)}\lambda_2\lambda_4.\end{aligned} \quad (24)$$

From (23), we eliminate $\alpha^2$ and $\beta^2$, so we have

$$\begin{aligned}&-\lambda_2\lambda_4(\lambda_1+2H_1)(\lambda_2-\lambda_1)P_{1,2}\left(P_{5,8}-\frac{2}{3}P_{6,5}(\lambda_1-2H_1)(\lambda_1-\lambda_4)\right)^2\\ &-\frac{1}{2}\lambda_2\lambda_4(\lambda_1+2H_1)(\lambda_1-\lambda_4)P_{2,2}\left(P_{4,8}+\frac{4}{3}P_{6,5}(\lambda_1-2H_1)(\lambda_1-\lambda_2)\right)^2\\ &=\zeta\left(P_{5,8}-\frac{2}{3}P_{6,5}(\lambda_1-2H_1)(\lambda_1-\lambda_4)\right)\left(P_{4,8}+\frac{4}{3}P_{6,5}(\lambda_1-2H_1)(\lambda_1-\lambda_2)\right),\end{aligned} \quad (25)$$

which is a degree 22 polynomial equation.

Let $\gamma(t)$, $(t\in I)$ be an integral curve associated to $w_1$ through point $p_0=\gamma(t_0)$. Since $w_i(\lambda_1)=w_i(\lambda_2)=0$ for $i=2,3,4$ and $w_1(\lambda_1)$, $w_1(\lambda_2)\neq 0$, we can assume $\lambda_2=\lambda_2(t)$ and $\lambda_1=\lambda_1(\lambda_2)$ in some neighborhood of $\lambda_0=\lambda_2(t_0)$. Using (22), we have

$$\begin{aligned}\frac{d\lambda_1}{d\lambda_2}&=\frac{d\lambda_1}{dt}\frac{dt}{d\lambda_2}=\frac{w_1(\lambda_1)}{w_1(\lambda_2)}\\ &=2\frac{(\lambda_1+\lambda_2-2H_1)\beta-(\lambda_1-\lambda_2)\alpha}{(\lambda_1-\lambda_2)\alpha}\\ &=\frac{2\left(P_{4,8}+\frac{4}{3}P_{6,5}(\lambda_1-\lambda_2)(\lambda_1-2H_1)\right)(\lambda_1+\lambda_2-2H_1)}{\left(\frac{4}{3}P_{6,5}(\lambda_1+\lambda_2-2H_1)(\lambda_1-2H_1)-P_{5,8}\right)(\lambda_1-\lambda_2)}-2.\end{aligned} \quad (26)$$

We differentiate (25) with respect to $\lambda_2$ and use equality (26) to get

$$f(\lambda_1, \lambda_2) = 0, \tag{27}$$

as is a degree 30 polynomial equation.

Comparing (25) and (27), we have two polynomial equations as follow:

$$\sum_{j=0}^{22} f_j(\lambda_1)\lambda_2^j = 0, \qquad \sum_{k=0}^{30} g_k(\lambda_1)\lambda_2^k = 0, \tag{28}$$

for each $0 \leq j \leq 22$ and $0 \leq k \leq 30$, $f_i(\lambda_1)$ and $g_j(\lambda_1)$ are polynomials in terms of $\lambda_1$. Eliminating $\lambda_2^{30}$ between two polynomials in (28), we get a new polynomial equation of degree 29 in terms of $\lambda_2$. By substituting this equation in the first one, we get a degree 28 polynomial equation. So, we are able to eliminate $\lambda_2$ by continuing the similar method by using the first equation of (28) and its consequences. Finally, we get a non-trivial polynomial equation with constant coefficients in terms of $\lambda_1$, which gives that $\lambda_1$ has constant value. So, by (4), we get the constancy of $H_2$.

Now, we show the nullity of $H_2$. If $H_2 \neq 0$, then by (3)(i), we get the constancy of $H_3$. Therefore, $M_1^4$ is isoparametric which has not more than one nonzero principal curvature ([9]). This result contradicts with the assumptions. So, $H_2 \equiv 0$. $\square$

## 4 $\mathcal{D}_2$, $\mathcal{D}_3$ and $\mathcal{D}_3$ Types of Shape Operator

In the rest, we study some $\mathbf{H}_2$-proper hypersurfaces with non-diagonal shape operator of matrix forms $\mathcal{D}_2$, $\mathcal{D}_3$ and $\mathcal{D}_3$.

**Theorem 4.1.** *Suppose that $x : M_1^4 \to \mathbb{S}_1^5$ is a $\boldsymbol{H}_2$-proper $\mathcal{D}_2$-hypersurface whose mean curvature is constant. If one of real principal curvatures of $M_1^4$ has constant value, then its scalar curvature is constant. Furthermore, it is 1-minimal or 3-minimal.*

**Proof.** First, we prove the constancy of $H_2$ by giving reason for emptiness of $\mathcal{U} = \{p \in M : \nabla H_2^2(p) \neq 0\}$. We assume $\mathcal{U} \neq \emptyset$. According to a suitable (local) orthonormal basis $\{w_1, \cdots, w_4\}$ on $M_1^4$, we have $Sw_1 = \kappa w_1 - \lambda w_2$, $Sw_2 = \lambda w_1 + \kappa w_2$, $Sw_3 = \eta_1 w_3$, $Sw_4 = \eta_2 w_4$ and then, we have $\mathrm{N}_2 w_1 = [\kappa(\eta_1+\eta_2)+\eta_1\eta_2]w_1 + \lambda(\eta_1+\eta_2)w_2$, $\mathrm{N}_2 w_2 = -\lambda(\eta_1+\eta_2)w_1 + [\kappa(\eta_1+\eta_2)+\eta_1\eta_2]w_2$, $\mathrm{N}_2 w_3 = (\kappa^2+\lambda^2+2\kappa\eta_2)w_3$ and $\mathrm{N}_2 w_4 = (\kappa^2+\lambda^2+2\kappa\eta_1)w_4$.

Condition (3)(ii), by using the equality $\nabla H_2 = \sum_{i=1}^{4} \epsilon_i w_i(H_2) w_i$, gives

$$\begin{aligned}
&\text{(i) } (\kappa\mu_{1,2;1}+\mu_{1,2;2}-9H_2)\epsilon_1 w_1(H_2) - \lambda\mu_{1,2;1}\epsilon_2 w_2(H_2) = 0\\
&\text{(ii) } \lambda\mu_{1,2;1}\epsilon_1 w_1(H_2) + (\kappa\mu_{1,2;1}+\mu_{1,2;2}-3(4-k)H_2)\epsilon_2 w_2(H_2) = 0\\
&\text{(iii) } (\mu_{3;2}-9H_2)\epsilon_3 w_3(H_2) = 0,\\
&\text{(iv) } (\mu_{4;2}-9H_2)\epsilon_4 w_4(H_2) = 0.
\end{aligned} \tag{29}$$

We claim that $w_m(H_2) = 0$ for $m = 1, \ldots, 4$. First, we prove it for $m = 1$. If $w_1(H_2) \neq 0$, then taking $u := \frac{\epsilon_2 w_2(H_2)}{\epsilon_1 w_1(H_2)}$, from (29)(i, ii) we get two equalities

$$\begin{aligned}
&\text{(i) } \ \kappa\mu_{1,2;1}+\mu_{1,2;2}-9H_2 = \lambda\mu_{1,2;1}u,\\
&\text{(ii) } \ (\kappa\mu_{1,2;1}+\mu_{1,2;2}-9H_2)u = -\lambda\mu_{1,2;1},
\end{aligned} \tag{30}$$

which give $\lambda\mu_{1,2;1}(1+u^2) = 0$, then $\lambda\mu_{1,2;1} = 0$. Since $\lambda \neq 0$ (by definition), we get $\mu_{1,2;1} = 0$. So, by (30)(i), we obtain

$$\mu_{1,2;2} = 9H_2. \tag{31}$$

From $\mu_{1,2;1} = 0$ we have $\eta_1 + \eta_2 = 0$. Since $\eta_1$ is assumed to be constant, from 31 we get that $9H_2 = -\eta_1^2 = -\eta_1^2$ is constant which contradicts with assumption $w_1(H_2) \neq 0$. So, we have $w_1(H_2) = 0$. Hence, our claim is affirmed in case $m = 1$.

For $m = 2$ the claim is $w_2(H_2) = 0$. If $w_2(H_2) \neq 0$, then taking $v := \frac{\epsilon_1 w_1(H_2)}{\epsilon_2 w_2(H_2)}$, from (29)(i, ii) we get $\lambda\mu_{1,2;1}(1+v^2) = 0$, which gives $\lambda\mu_{1,2;1} = 0$ and then $\mu_{1,2;1} = 0$. Similar to case $m = 1$, we get contradiction of constancy of $9H_2$, which affirm the claim $w_2(H_2) = 0$.

The proof of the claim in cases $m = 3, 4$ is different from cases $m = 1, 2$.

If $w_3(H_2) \neq 0$, the we get $\mu_{3;2} = 9H_2$ (from (29)(iii)), which gives $-3\kappa^2 + (2\kappa+3\eta_1)(4H_1-\eta_1) = -\lambda^2 < 0$, then, $-2[2\kappa^2 + (\eta_1 - 4H_1)\kappa + 2\eta_1(\eta_1 - 3H_1)] = -(\lambda^2+\kappa^2+\eta_1^2) < 0$.

The last phrase has negative value if and only if $\delta < 0$ where

$$\delta = (\eta_1 - 4H_1)^2 - 16\eta_1(\eta_1 - 3H_1) = -15\eta_1^2 + 40\eta_1 H_1 + 16H_1^2.$$

Clearly, $\delta < 0$ is equivalent to $\bar{\delta} < 0$ where

$$\bar{\delta} = (40H_1)^2 + (4 \times 15 \times 16)H_1^2 = 2560H_1^2,$$

which is impossible. So, $w_3(H_2) = 0$ is proven.

The reason of claim in case $m = 4$ is similar to case $m = 3$. If $w_4(H_2) \neq 0$, then we have $\mu_{4;2} = 9H_2$ from equality (29)(iv). So,

$$-11\kappa^2 + (24H_1 - 10\eta_1)\kappa + 12\eta_1 H_1 - 3\eta_1^2 = -\lambda^2 < 0,$$

and then

$$-2[6\kappa^2 + (5\eta_1 - 12H_1)\kappa + 2\eta_1(\eta_1 - 3H_1)] = -(\lambda^2 + \kappa^2 + \eta_1^2) < 0.$$

which occurs if and only if we have $\varrho < 0$ where

$$\varrho = (5\eta_1 - 12H_1)^2 - 48\eta_1(\eta_1 - 3H_1) = -23\eta_1^2 + 24\eta_1 H_1 + 144H_1^2.$$

which occurs if and only if we have $\bar{\varrho} < 0$ where

$$\bar{\varrho} = (24H_1)^2 + (4 \times 23 \times 144)H_1^2 = 13824H_1^2.$$

But, this is impossible. So, $w_4(H_2) = 0$.

In the next stage, we prove that $H_2 = 0$ or $H_4 = 0$. since $H_2$ is constant, we have $\mathrm{C}H_2 = 0$. Then, by (3)(i), we have $H_2(4H_1H_2 - 2H_3) = 0$. Assuming $H_2 \neq 0$ we get $4H_1H_2 = 2H_3$, which implies the constancy of $H_3$. Hence, $M_1^4$ is isoparametric, which cannot have more than one non-zero real principal curvature ([9]). So, $\eta_1\eta_2 = 0$ which gives $H_4 = (\kappa^2 + \lambda^2)\eta_1\eta_2 = 0$. Therefore, $M_1^4$ is 3-minimal. □

**Theorem 4.2.** *Suppose that $x : M_1^4 \to \mathbb{S}_1^5(c)$ is a $\boldsymbol{H}_2$-proper $\mathcal{D}_3$-hypersurface whose mean curvature and one of whose principal curvatures have constant values. If its principal curvatures are mutually distinct, then it has to be 1-minimal. Furthermore, $M_1^4$ is isoparametric.*

**Proof.** We show the constancy of $H_2$ by proving the emptiness of $\mathcal{U} = \{p \in M : \nabla H_2^2(p) \neq 0\}$. We assume that $\mathcal{U} \neq \emptyset$. According to a (local) orthonormal basis $\{w_1, \cdots, w_4\}$ on $M$, we have $Sw_1 = (\kappa + \frac{1}{2})w_1 - \frac{1}{2}w_2$, $Sw_2 = \frac{1}{2}w_1 + (\kappa - \frac{1}{2})w_2$, $Sw_3 = \lambda_1 w_3$ and $Sw_4 = \lambda_2 w_4$, and for $k = 1, 2, 3$ we have $\mathrm{N}_k w_1 = [\mu_{1,2;k} + (\kappa - \frac{1}{2})\mu_{1,2;k-1}]w_1 + \frac{1}{2}\mu_{1,2;k-1}w_2$, $\mathrm{N}_k w_2 = -\frac{1}{2}\mu_{1,2;k-1}w_1 + [\mu_{1,2;k} + (\kappa - \frac{1}{2})\mu_{1,2;k-1}]w_2$, and $\mathrm{N}_k w_3 = \mu_{3;k}w_3$ and $\mathrm{N}_k w_4 = \mu_{4;k}w_4$.

From condition (3)(ii), using $\nabla H_2 = \sum_{i=1}^{4} \epsilon_i w_i(H_2)w_i$, we get

$$\begin{aligned}
&(i)\ [\lambda_1\lambda_2 + (\kappa - \frac{1}{2})(\lambda_1 + \lambda_2) - 9H_2]\epsilon_1 w_1(H_2) = \frac{1}{2}(\lambda_1 + \lambda_2)\epsilon_2 w_2(H_2),\\
&(ii)\ [\lambda_1\lambda_2 + (\kappa + \frac{1}{2})(\lambda_1 + \lambda_2) - 9H_2]\epsilon_2 w_2(H_2) = -\frac{1}{2}(\lambda_1 + \lambda_2)\epsilon_1 w_1(H_2),\\
&(iii)\ (\kappa^2 + 2\kappa\lambda_2 - 9H_2)\epsilon_3 w_3(H_2) = 0,\\
&(iv)\ (\kappa^2 + 2\kappa\lambda_1 - 9H_2)\epsilon_3 w_4(H_2) = 0.
\end{aligned} \tag{32}$$

We claim that $w_m(H_2) = 0$ for $m = 1, \ldots, 4$. First, we prove it for $m = 1$. If $w_1(H_2) \neq 0$, then taking $u := \frac{\epsilon_2 w_2(H_2)}{\epsilon_1 w_1(H_2)}$, from (32)(i, ii) we get

$$\begin{aligned}
&(i)\ \lambda_1\lambda_2 + (\kappa - \frac{1}{2})(\lambda_1 + \lambda_2) - 9H_2 = \frac{1}{2}(\lambda_1 + \lambda_2)u,\\
&(ii)\ [\lambda_1\lambda_2 + (\kappa + \frac{1}{2})(\lambda_1 + \lambda_2) - 9H_2]u = -\frac{1}{2}(\lambda_1 + \lambda_2),
\end{aligned} \tag{33}$$

which give $(\lambda_1 + \lambda_2)(1 + u)^2 = 0$, then $u = -1$ or $\lambda_1 + \lambda_2 = 0$.
If $\lambda_1 + \lambda_2 = 0$, then, from (33)$(i)$ we get $9H_2 = -\lambda_1^2$, which gives $3\kappa^2 = -\lambda_1^2$. Constancy of $H_1$ and $\kappa = 2H_1$ gives that $\lambda_2$ and $\lambda_1$ have constant value on $M_1^4$. As an isoparametric hypersurface with real principal curvatures, $M_1^4$ satisfies the condition of Corollary 2.7 in [9], so it dos'nt have more than one nonzero principal curvature. This contradiction gives that $\lambda_1 + \lambda_2 \neq 0$ and then $u = -1$.

From $u = -1$, using (33)$(i)$ and $4H_1 = 2\kappa + \lambda_1 + \lambda_2$, gives $5\kappa^2 - 16\kappa H_1 - \lambda_1(4H_1 - 2\kappa - \lambda_1) = 0$. Without loss of generality, we assume that $\lambda_1$ is constant on $M$. So, from the last equation we get that $\kappa$, $\lambda_2$ and $H_2$ are constant on $\mathcal{U}$, which is a contradiction. Therefore, the first claim is proved.

For cases $m = 2, 3, 4$, in similar ways, by assuming $w_m(H_2) \neq 0$ we get $\lambda^2 + 2\kappa\lambda = 9H_2$, which gives a contradiction that $H_2$ is constant on $M$. So, we get the affirmation of claim for $m = 2, 3, 4$.

Now, we prove the nullity of $H_2 = 0$. By constancy of $H_1$ and $H_2$, the condition (3)(i) gives the constancy of $H_3$. Hence, $M_1^4$ as an isoparametric $\mathcal{D}_3$-hypersurface has at most one nonzero principal curvature (by Corollary 2.7 in [9]). So, $\lambda = 0$. Then $H_1 = \frac{1}{2}\kappa$, $H_2 = \frac{1}{6}\kappa^2$ and $H_3 = 0$. Hence, by (3)(i), we get $\kappa = 0$ and then $H_2 = 0$. □

**Theorem 4.3.** *Suppose that $x : M_1^4 \to \mathbb{S}_1^5(c)$ is a $\boldsymbol{H}_2$-proper $\mathcal{D}_4$-hypersurface whose mean curvature has constant value. Then it is 1-minimal. Furthermore, $M_1^4$ is isoparametric.*

**Proof.** We show the constancy of $H_2$ by proving the emptiness of $\mathcal{U} = \{p \in M : \nabla H_2^2(p) \neq 0\}$. We assume that $\mathcal{U} \neq \emptyset$. According to a (local) orthonormal basis $\{w_1, \cdots, w_4\}$ on $M$, we have $Sw_1 = \kappa w_1 - \frac{\sqrt{2}}{2}w_3$, $Sw_2 = \kappa w_2 - \frac{\sqrt{2}}{2}w_3$, $Sw_3 = \frac{\sqrt{2}}{2}w_1 - \frac{\sqrt{2}}{2}w_2 + \kappa w_3$ and $Sw_4 = \lambda w_4$ and then, we have $\mathrm{N}_2 w_1 = (\kappa^2 + 2\kappa\lambda - \frac{1}{2})w_1 + \frac{1}{2}w_2 + \frac{\sqrt{2}}{2}(\kappa + \lambda)w_3$, $\mathrm{N}_2 w_2 = \frac{-1}{2}w_1 + (\kappa^2 + 2\kappa\lambda + \frac{1}{2})w_2 + \frac{\sqrt{2}}{2}(\kappa + \lambda)w_3$, $\mathrm{N}_2 w_3 = \frac{-\sqrt{2}}{2}(\kappa + \lambda)w_1 + \frac{\sqrt{2}}{2}(\kappa + \lambda)w_2 + (\kappa^2 + 2\kappa\lambda)w_3$ and $\mathrm{N}_2 w_4 = 3\kappa^2 w_4$.

From condition (3)(ii), using $\nabla H_2 = \sum_{i=1}^{4} \epsilon_i w_i(H_2) w_i$, we get

$$
\begin{aligned}
&(i)\ (\kappa^2 + 2\kappa\lambda - \frac{1}{2} - 9H_2)\epsilon_1 w_1(H_2) - \frac{1}{2}\epsilon_2 w_2(H_2) = \frac{\sqrt{2}}{2}(\kappa + \lambda)\epsilon_3 w_3(H_2),\\
&(ii)\ \frac{1}{2}\epsilon_1 w_1(H_2) + (\kappa^2 + 2\kappa\lambda + \frac{1}{2} - 9H_2)\epsilon_2 w_2(H_2) = -\frac{\sqrt{2}}{2}(\kappa + \lambda)\epsilon_3 w_3(H_2),\\
&(iii)\ \frac{\sqrt{2}}{2}(\kappa + \lambda)(\epsilon_1 w_1(H_2) + \epsilon_2 w_2(H_2)) = -(\kappa^2 + 2\kappa\lambda - 9H_2)\epsilon_3 w_3(H_2),\\
&(iv)\ \ (3\kappa^2 - 9H_2)\epsilon_4 w_4(H_2) = 0.
\end{aligned}
\tag{34}
$$

We claim that $w_m(H_2) = 0$ for $m = 1, \ldots, 4$. First, we prove it for $m = 1$. If $w_1(H_2) \neq 0$, then taking $u_1 := \frac{\epsilon_2 w_2(H_2)}{\epsilon_1 w_1(H_2)}$ and $u_2 := \frac{\epsilon_3 w_3(H_2)}{\epsilon_1 w_1(H_2)}$ and using the identity $2H_2 = \kappa^2 + \kappa\lambda$, from equalities (34)$(i, ii, iii)$ we get

$$
\begin{aligned}
&(i) \quad -\frac{1}{2}-\frac{7}{2}\kappa^2-\frac{5}{2}\kappa\lambda-\frac{1}{2}u_1-\frac{\sqrt{2}}{2}(\kappa+\lambda)u_2=0\\
&(ii) \quad \frac{1}{2}+(\frac{1}{2}-\frac{7}{2}\kappa^2-\frac{5}{2}\kappa\lambda)u_1+\frac{\sqrt{2}}{2}(\kappa+\lambda)u_2=0\\
&(iii) \quad \frac{-\sqrt{2}}{2}(\kappa+\lambda)(1+u_1)-(\frac{7}{2}\kappa^2+\frac{5}{2}\kappa\lambda)u_2=0.
\end{aligned}
\tag{35}
$$

From (35)$(i,ii)$, we get $\frac{-1}{2}\kappa(5\lambda+7\kappa)(1+u_1)=0$. If $\kappa=0$, then $H_2=0$. If $\kappa\neq 0$, then we get $u_1=-1$ or $\lambda=-\frac{7}{5}\kappa$. If $u_1\neq -1$ then $\lambda=-\frac{7}{5}\kappa$ and then by (35)$(iii)$ we obtain $u_1=-1$, which is a contradiction. So $u_1=-1$, which by (35)$(i,iii)$ gives $u_2=0$.

We check two cases $\lambda=-\frac{7}{5}\kappa$ and $\lambda\neq-\frac{7}{5}\kappa$. If $\lambda=-\frac{7}{5}\kappa$, then, $\kappa=\frac{5}{2}H_1$, $H_2=\frac{-1}{5}\kappa^2$, $H_3=\frac{-4}{5}\kappa^3$ and $H_4=\frac{-7}{5}\kappa^4$ are all constants on U. Also, the case $\lambda\neq-\frac{7}{5}\kappa$ is in contradiction with (35)$(ii)$.

Hence, the first claim $w_1(H_2)\equiv 0$ is affirmed. Similarly, the second claim (i.e. $w_2(H_2)=0$) can be proved.

Applying the results $w_1(H_2)=w_2(H_2)=0$, from (35)$(ii,iii)$ we get $w_3(H_2)=0$.

The final claim (i.e. $w_4(H_2)=0$), can be proved using (35)$(iv)$, in a straightforward manner.

Now, we have $\mathrm{C}H_2=9H_1H_2^2-3H_2H_3=0$ from (3)(i) . If $H_2\neq 0$, we get $3H_1H_2=H_3$, which gives $\kappa(\kappa^2-3H_1\kappa+3H_1^2)=0$, where $\kappa^2-3H_1\kappa+3H_1^2>0$, Hence, $\kappa=0$. Therefore, $H_2=H_3=H_4=0$. □

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