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Up and down Steenrod powers

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Abstract. First we review Steenrod powers, referred to in this article as the up Steenrod powers, and prove some more properties of them. Then, the divided p-power algebras are introduced and the down Steenrod powers, the dual of up Steenrod powers, are defined over these algebras. Finding some efficient tools for calculating the evaluations of up and down powers is the next attempt. Finally, harmonic patterns are exhibited for the action of up and down powers. All considerations are performed for one variable as the Cartan formula naturally extends to the multi-variable case.

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1 Introduction

Let p be an odd prime. For $n > 0$ consider the polynomial algebra

$$\mathbf{P}(n) = \mathbb{F}_p[x_1, \ldots, x_n] = \bigoplus_{d \geq 0} \mathbf{P}^d(n),$$

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viewed as a graded left module over the mod p Steenrod algebra $\mathcal{A}_p$ and put $\mathbf{P}(0) = \mathbf{P}^0(n) = \mathbb{F}_p$. The grading is by the homogeneous polynomials $\mathbf{P}^d(n)$ of degree d in n variables $x_1, \ldots, x_n$ of grading 2. The Steenrod algebra $\mathcal{A}_p$ is briefly defined as the graded algebra over the finite field $\mathbb{F}_p$ generated by symbols $\mathcal{P}^k$, defined in Definition 2.1, called the up Steenrod powers of (reduced) grading $k > 0$ and Bockstein homomorphism subject to Adem relations and the condition $\mathcal{P}^0 = 1$ [2]. In classical texts, the word 'up' is not written for the Steenrod powers $\mathcal{P}^k$, however, we use this word since the dual $\mathcal{P}_k$ of up Steenrod powers, called the 'down' Steenrod powers are the main objective of our study. The latter word was first used in [22] for the 'going down' integral Steenrod square (see also [21]).

More works are done on the (up) Steenrod squares. Of them one may cite [3, 10, 13, 14, 15, 22, 23]. However, less are worked on up Steenrod powers. Tanay-Oner [18] extended the second author works [3] on the action of up Steenrod squares for up Steenrod powers. In particular, they exhibited a matrix method for calculations of up powers [11]. In his recent studies, Turgay [19] explored the connections of up Steenrod powers with other algebras, in particular, Leibniz-Hopf algebra.

Down Steenrod squares are deeply considered in the comprehensive book [20]. In a recent work [1], the present authors exhibited a matrix method for the up and down Steenrod squares. In this article, the authors introduce the divided p-power algebras and afterward investigate the down Steenrod powers defined over these algebras. Down Steenrod powers are not so known.

In Section 2, up Steenrod powers $\mathcal{P}^k$ for $k > 0$ is recalled from [2, Section 4L] and some further properties for them are established. In particular, the following basic tool is demonstrated for $t > 0$, $0 < i < p^t$, $0 \leq j < p^t$, and $r \geq 0$, $s \geq r+1$.

$$\mathcal{P}^{i+rp^t}(x^{j+sp^t}) = \binom{j}{i} x^{j+(p-1)i} \binom{s}{r} x^{(s+(p-1)r)p^t}, \qquad (1)$$

where the binomial coefficients are taken modulo p. In Definition 2.13, a harmonic triangular pattern is created for the evaluation of up powers in one variable. This pattern may be applied for computations involved the $\mathcal{P}^k$'s in computer.

Section 3 is dedicated to down Steenrod powers. First, we introduce the divided p-power algebra as the formal sum $\mathbf{DP}(n) = \sum_{d\geq 0} \mathbf{DP}^d(n)$, where $\mathbf{DP}^d(n) = \mathrm{Hom}(\mathbf{P}^d(n), \mathbb{F}_p)$ is the linear dual of the $\mathbb{F}_p$-vector space $\mathbf{P}^d(n)$. Consider the basis $v_1, \ldots, v_n$ of $\mathbf{DP}^2(n)$ dual to the basis $x_1, \ldots, x_n$ of $\mathbf{P}^2(n)$ with the duality property $\langle v_i, x_j\rangle$, which is 1 if $i = j$ and 0 otherwise. A d-monomial $v_1^{(d_1)} \cdots v_n^{(d_n)}$ in $\mathbf{DP}(n)$ is the dual of the monomial $x_1^{d_1} \cdots x_n^{d_n}$ in $\mathbf{P}(n)$. For any $v \in \mathbf{DP}^2(n)$, put $v^{(0)} = 1$, the identity map of $\mathbb{F}_2$ which is also the identity element of $\mathbf{DP}(1)$. Also put $v^{(1)} = v$. Then the linear dual of the up Steenrod power $\mathcal{P}^k : \mathbf{P}^d(n) \to \mathbf{P}^{d+k(p-1)}(n)$, called the down Steenrod power $\mathcal{P}_k : \mathbf{DP}^{d+k(p-1)}(n) \to \mathbf{DP}^d(n)$ is defined by $\mathcal{P}_k(u) = v$ for $u \in \mathbf{DP}^{d+k(p-1)}(n)$ such that

$$v(f) = \big(\mathcal{P}_k(u)\big)(f) = u\big(\mathcal{P}^k(f)\big),$$

for $f \in \mathbf{P}^d(n)$. Some properties of the $\mathcal{P}_k$ are considered. In particular, a dual version of the basic tool (1) is proved for $t > 0$, $0 < i < p^t$, $0 \leq j < p^t$, and $r \geq 0$, $s \geq r + 1$ as followed.

$$\mathcal{P}_{i+rp^t}(v^{(j+sp^t+(p-1)(i+rp^t))}) = \binom{j}{i} v^{(j)} \binom{s}{r} v^{(sp^t)},$$

where here again the binomial coefficients are calculated modulo p. An analogous harmonic triangular pattern with a bit deformation holds for the evaluation of the down powers in one variable.

An application of the up operations is in the modular hit problem. A homogenous element $f \in \mathbf{P}(n)$ of grading d is said to be modular hit (modulo p) in $\mathbf{P}(n)$ if there is a modular hit equation of the form

$$f = \sum_{i>0} \mathcal{P}^i(f_i),$$

where each f_i has grading less than d. We denote by $\mathbf{Q}(n) = \mathbb{F}_p \otimes_{\mathcal{A}_p} \mathbf{P}(n)$, the quotient of the module $\mathbf{P}(n)$ by the modular hit elements, where $\mathbb{F}_p$ is here viewed as a right $\mathcal{A}_p$-module concentrated in grading 0. Then $\mathbf{Q}(n)$ is a graded vector space over $\mathbb{F}_p$ and a basis for $\mathbf{Q}(n)$ lifts to a minimal generating set for $\mathbf{P}(n)$. The modular hit problem is to find minimal generating sets for $\mathbf{P}(n)$ and criteria for elements to be modular hit. The special case of the modular hit problem in modulo 2 is the well known hit problem [4, 9, 12, 14, 17, 22, 23].

Also, down operations are applied in the dual modular hit problem which is to determine $\mathbf{K}(n) = \sum_{d \geq 0} \mathbf{K}^d(n)$, where $\mathbf{K}^d(n)$ is the set of all elements $v \in \mathbf{DP}^d(n)$ such that $\mathcal{P}_k(v) = 0$ for all $k > 0$. The modular hit problem and dual modular hit problem are both open problems. The two aforementioned open problems may be considered over the symmetric polynomials. The symmetric hit problem [5, 6, 7, 8] is the special case $p = 2$ in this circumstance.

Through this article p is considered to be an odd prime. However, most of the concepts and results are true for $p = 2$. Also, all binomial coefficients are understood to be reduced modulo p.

2 Up Steenrod powers

In this section, we recall some fundamental concepts of up Steenrod powers from [2, Section 4L] and establish some further properties.

Definition 2.1. The total Steenrod power $\mathcal{P} : \mathbf{P}(n) \to \mathbf{P}(n)$ is an algebra map defined by $\mathcal{P}(x_i) = x_i + x_i^p$ for $1 \leq i \leq n$. For $k \geq 0$, the up Steenrod power $\mathcal{P}^k$ is the linear map defined by the restriction

$$\mathcal{P}^k : \mathbf{P}^d(n) \to \mathbf{P}^{d+k(p-1)}(n).$$

Therefore, the total Steenrod power is the formal sum

$$\mathcal{P} = \sum_{k \geq 0} \mathcal{P}^k \tag{2}$$

Remark 2.2. In topological point of view, the Steenrod power $\mathcal{P}^k$ is defined for a topological space X as the operation

$$\mathcal{P}^k : H^d(X; \mathbb{F}_p) \to H^{d+2k(p-1)}(X; \mathbb{F}_p),$$

satisfying some properties. This coincides with Definition 2.1 since for the Eilenberg-MacLane space $X = K(\mathbb{F}_p^n; 2)$, we have $H^*(X; \mathbb{F}_p) = \mathbf{P}(n)$ nothing that $\deg(x_i) = 2$ [16]. Generally, $\mathcal{P}^k$ is given the degree $2k(p-1)$, but for simplicity we regrade $\mathcal{A}_p$ by giving $\mathcal{P}^k$ the 'reduced' degree k. Thus when $p = 2$, $\mathcal{P}^k$ will mean Sq^k, and not Sq^{2k}.

The following are some properties of up Steenrod powers.

Proposition 2.3. *For $x \in \mathbf{P}^2(n)$,*

$$\mathcal{P}(x) = x + x^p.$$

Proposition 2.4 (Cartan formula)**.** *For any $f, g \in \mathbf{P}(n)$ and any $k \geq 0$,*

$$\mathcal{P}^k(fg) = \sum_{i+j=k} \mathcal{P}^i(f)\mathcal{P}^j(g),$$

Proposition 2.5. *$\mathcal{P}^0$ is the identity map of $\mathbf{P}(n)$.*

The next result shows why $\mathcal{P}^k$ is called a 'power' operation.

Proposition 2.6. *For the homogenous polynomial $f \in \mathbf{P}^d(n)$, $\mathcal{P}^k(f) = 0$ if $2k > d$ and $\mathcal{P}^k(f) = f^p$ if $2k = d$.*

The following corollary is immediately concluded.

Corollary 2.7. *Let $k = d_1 + \cdots + d_n$. Then,*

$$\mathcal{P}^k(x_1^{d_1} \cdots x_n^{d_n}) = x_1^{pd_1} \cdots x_n^{pd_n}.$$

The next two results show how to evaluate an up Steenrod power on a monomial.

Proposition 2.8. *For any $x \in \mathbf{P}^2(n)$ we have*

$$\mathcal{P}^k(x^d) = \binom{d}{k} x^{d+(p-1)k}.$$

Proof. By the multiplicative property of $\mathcal{P}$ we write

$$\mathcal{P}(x^d) = (\mathcal{P}(x))^d = (x + x^p)^d = x^d(1 + x^{(p-1)})^d = \sum_{k=0}^{d} \binom{d}{k} x^{d+(p-1)k}.$$

Equating terms of degree $d + (p-1)k$ gives the result. □

Proposition 2.9. *Let $f = x_1^{d_1} \cdots x_n^{d_n}$ be a monomial in $\mathbf{P}(n)$. Then*

$$\mathcal{P}^k(f) = \sum_{k_1+\cdots+k_n=k} \mathcal{P}^{k_1}(x_1^{d_1}) \cdots \mathcal{P}^{k_n}(x_n^{d_n}).$$

Proof. *An induction on n applied using Cartan formula 2.4.* □

To continue we need the well-known Lucas Theorem.

Theorem 2.10 (Lucas Theorem)**.** *For any prime p,*

$$\binom{b}{a} \equiv \prod_{i=1}^{h} \binom{b_i}{a_i} \mod p,$$

where

$$b = b_1 p^{e_1} + \cdots + b_h p^{e_h}, \quad a = a_1 p^{e_1} + \cdots + a_h p^{e_h}$$

are the p-adic expansions of b and a, respectively, where $b_i, a_i \in \mathbb{F}_p$.

The following is an efficient tool in manipulating up Steenrod powers.

Theorem 2.11. *For $t > 0$ let $0 < i < p^t$ and $0 \leq j < p^t$. Let also $r \geq 0$ and $s \geq r+1$. Then,*

$$\mathcal{P}^{i+rp^t}(x^{j+sp^t}) = \binom{j}{i} x^{j+(p-1)i} \binom{s}{r} x^{(s+(p-1)r)p^t}.$$

Proof. Since $i, j < p^t$, by Proposition 2.8 and Lucas Theorem 2.10 we have

$$\begin{aligned}\mathcal{P}^{i+rp^t}(x^{j+sp^t}) &= \binom{j}{i} x^{j+(p-1)i} \binom{sp^t}{rp^t} x^{(s+(p-1)r)p^t} \\ &= \binom{j}{i} x^{j+(p-1)i} \binom{s}{r} x^{(s+(p-1)r)p^t}\end{aligned}$$

Note that $\binom{s}{r}$ is always nonzero mod p since $s > r$. □ The following corollary is immediately concluded. It shows that the evaluation of up power operations in Theorem 2.11 vanishes whenever $j < i$.

Corollary 2.12. *Let $t > 0$ and let $r \geq 0$, $s \geq r+1$. Then*

$$\mathcal{P}^{i+rp^t}(x^{j+sp^t}) = 0,$$

where $i = 1, \ldots, p^t - 1$ and $j = 0, \ldots, i-1$.

Proof. In Theorem 2.11, $\binom{j}{i}$ is zero mod p as $j < i$. □

Consider Corollary 2.12 for $r = 0, \ldots, p-2$ and $s = r+1, \ldots, p-1$. There is a template of dots (zeroes) in groups of $p(p-1)/2$ triangles

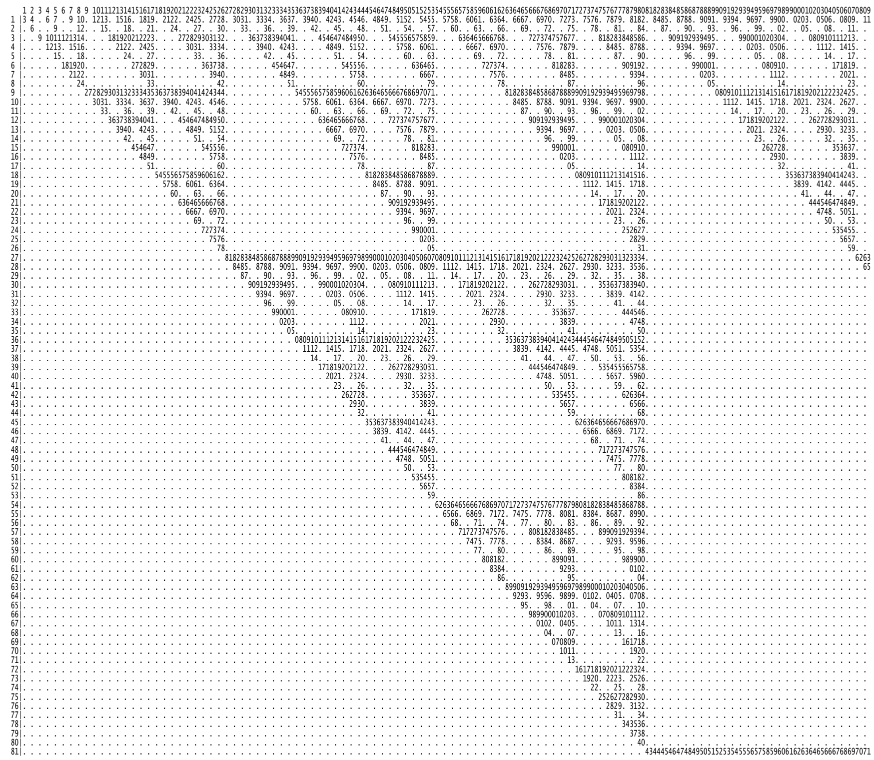

Table 1: Pattern for up Steenrod powers in the case $p = 3$.

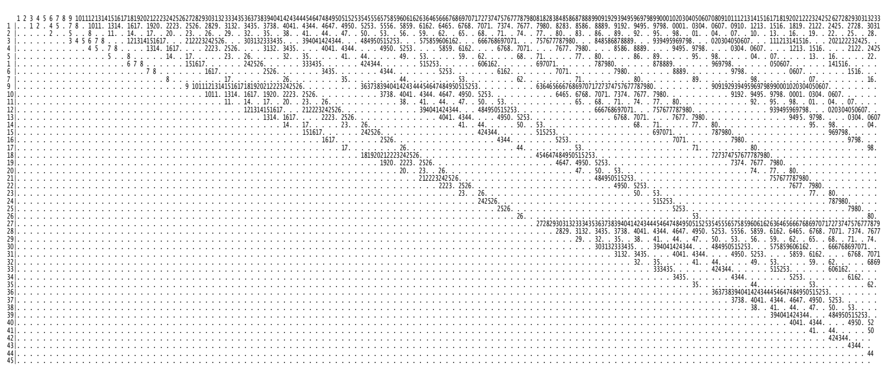

Table 2: Pattern for down Steenrod powers in the case $p = 3$.

which get wider as t gets larger. This template is depicted in Tables 1 and 3 for the cases $p = 3$ and $p = 5$ respectively. In these tables, the header row stands for the power of operands. For example 9 means x^9. Because of place limitation as well as regularity, each number is continuously posited in two columns and for three-digit numbers the rightmost two digit are written in the table. Therefore, after 99, the two-digit number 00 means 100 and each two-digit number after that is added to 100. For example, say, 17 means 117, i.e., x^{117}. The pre-column shows the degree of up power operators. For instance, 12 stand for $\mathcal{P}^{12}$. In these tables, zeros are shown by dots because of shapely. Moreover, coefficients are ignored. In fact, the same template can be arranged for the coefficients which are taken modulo p.

As seen in Table 1, for $t = 1$ we have a group of three triangles each consists of three dots. Also, the case $t = 2$ gives us a group of three 36-dot triangles surrounded by some groups of 3-dots triangles. The latter template may be found from column 9 to 26 in Table 1. We explain this schema from somehow a complement aspect. As seen in Tables 1 and 3, there are patterns of nonzero entries which repeat and grow up harmonically. In the following definition we try to explore the harmony behind the patterns. One may consider this definition as an algorithm.

Definition 2.13 (Triangular algorithm)**.** Fix the odd prime p. Put $\big[\mathcal{U}_p\big]^{(0)} = 0$ and $\big[\mathcal{T}_p\big]^{(0)} = 1$. For $t = 1, 2, \ldots, p$ and i, j from 0 to $p-1$, define inductively the $p^t \times p^t$ block array $\big[\mathcal{U}_p\big]^{(t)}$ by the following block entries.

$$\big[\mathcal{U}_p\big]^{(t)}_{ij} = \begin{cases} \big[\mathcal{O}_p\big]^{(t-1)}, & \text{if } i < j \\ \big[\mathcal{U}_p\big]^{(t-1)} + N\big[\mathcal{T}_p\big]^{(t-1)}, & \text{otherwise} \end{cases}$$

where, $\big[\mathcal{O}_p\big]^{(t-1)}$ is the $p^{t-1} \times p^{t-1}$ zero array, $N = ip^t + (j-i)p^{t-1}$, and the $p^{t-1} \times p^{t-1}$ block array $\big[\mathcal{T}_p\big]^{(t-1)}$ is defined inductively for i, j from 0 to $p-1$ by

$$\big[\mathcal{T}_p\big]^{(t-1)}_{ij} = \begin{cases} \big[\mathcal{O}_p\big]^{(t-2)}, & \text{if } i < j \\ \big[\mathcal{T}_p\big]^{(t-2)}, & \text{otherwise} \end{cases}$$

For example, taking $p = 5$, for $t = 1$ we have the 5×5 array $\big[\mathcal{U}_5\big]^{(1)}$

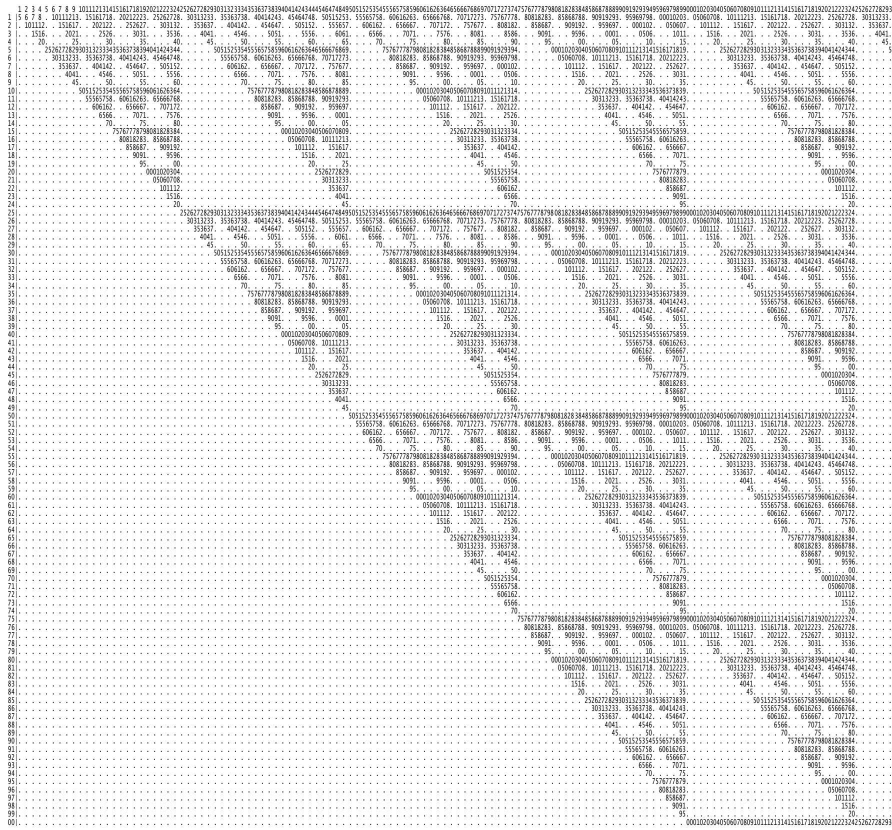

Table 3: Pattern for up Steenrod powers in the case $p = 5$.

by

$$[\mathcal{U}_5]^{(1)}_{ij} = \begin{cases} 0, & \text{if } i < j \\ 0 + 5i + (j - i), & \text{otherwise} \end{cases}$$

and hence,

$$\left[\mathcal{U}_5\right]^{(1)} = \begin{matrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 & 8 \\ 0 & 0 & 10 & 11 & 12 \\ 0 & 0 & 0 & 15 & 16 \\ 0 & 0 & 0 & 0 & 20 \end{matrix}$$

Moreover, for $t = 2$ the following 5×5 block array $\left[\mathcal{U}_5\right]^{(2)}$ is obtained.

$$\left[\mathcal{U}_5\right]^{(2)}_{ij} = \begin{cases} \left[\mathcal{O}_5\right]^{(1)}, & \text{if } i < j \\ \left[\mathcal{U}_5\right]^{(1)} + 25i + 5(j-i)\left[\mathcal{T}_5\right]^{(1)}, & \text{otherwise} \end{cases}$$

where, as in Definition 2.13, $\left[\mathcal{O}_5\right]^{(1)}$ is the 5×5 zero array, and $\left[\mathcal{T}_5\right]^{(1)}$ is the following 5×5 array.

$$\left[\mathcal{T}_5\right]^{(1)}_{ij} = \begin{cases} 0, & \text{if } i < j \\ 1, & \text{otherwise} \end{cases}$$

and hence,

$$\left[\mathcal{T}_5\right]^{(1)} = \begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{matrix}$$

Finally, for the remind $t = 3, 4, 5$, we have the $5^t \times 5^t$ block array $\left[\mathcal{U}_5\right]^{(t)}$ as follows.

$$\left[\mathcal{U}_5\right]^{(t)}_{ij} = \begin{cases} \left[\mathcal{O}_5\right]^{(t-1)}, & \text{if } i < j \\ \left[\mathcal{U}_5\right]^{(t-1)} + N\left[\mathcal{T}_5\right]^{(t-1)}, & \text{otherwise} \end{cases}$$

where $N = 5^t i + 5^{t-1}(j - i)$.

We end this section with a short review on the modular hit problem.

Definition 2.14. A homogenous element $f \in \mathbf{P}(n)$ of grading d is said to be modular hit (in modulo p) in $\mathbf{P}(n)$ if there is a hit equation of the form

$$f = \sum_{i>0} \mathcal{P}^i(f_i),$$

where the pre-images f_i have grading less than d.

Definition 2.15. For the prime p, a monomial of the form $x_1^{c_1p^{t_1}-1}\cdots x_n^{c_np^{t_n}-1}$ where, $t_i \geq 0$ and $1 \leq c_i \leq p-1$, is called a mod p spike.

For example, $2, 5, 8, 17$, in general, x^{c3^t-1} are one-variable mod 3 spikes. Also, $x_1^5x_2^{17}$ is a two-variable mod 3 spike. The mod 2 spikes are the so-called spikes.

The modular hit problem is to find bases for the graded $\mathbb{F}_p$-vector space $\mathbf{Q}(n)$, the quotient of the module $\mathbf{P}(n)$ by the modular hit elements. These bases lift to minimal generating sets for the module $\mathbf{P}(n)$. Finding criteria for elements of $\mathbf{P}(n)$ is also part of the modular hit problem. Since no mod p spike is modular hit in $\mathbf{P}(n)$, they are an inseparable part of any generating set for $\mathbf{P}(n)$. The modular hit problem is not so known nor is in the range of our work in this paper.

3 Down Steenrod powers

To define the down Steenrod powers, we need the notion of the divided p-power algebra. This is analogous to the divided power algebra in modulo 2 and we recall the concerned concepts and results from [1] and [20, Section 9.1].

Definition 3.1. Let n, d be positive integers. Denote the linear dual of the $\mathbb{F}_p$-vector space $\mathbf{P}^d(n)$ by $\mathbf{DP}^d(n) = \mathrm{Hom}(\mathbf{P}^d(n), \mathbb{F}_p)$ and define the divided p-power algebra as the infinite sum $\mathbf{DP}(n) = \sum_{d\geq 0} \mathbf{DP}^d(n)$.

Take the basis $v_1, \ldots, v_n$ of $\mathbf{DP}^2(n)$ dual to the basis $x_1, \ldots, x_n$ of $\mathbf{P}^2(n)$. The duality property is denoted by $\langle v_i, x_j\rangle$, which is 1 if $i = j$ and 0 otherwise. The d-monomial $v_1^{(d_1)}\cdots v_n^{(d_n)}$ in $\mathbf{DP}(n)$ is defined as the dual of the monomial $x_1^{d_1}\cdots x_n^{d_n}$ in $\mathbf{P}(n)$, where the parenthesized exponents are called the divided p-powers. The prefix 'd' in d-monomial is derived from the word 'dual'. For any $v \in \mathbf{DP}^2(n)$, put $v^{(0)} = 1$, the identity map of $\mathbb{F}_2$ which is also the identity element of $\mathbf{DP}(1)$. Also put $v^{(1)} = v$. Define the degree of the d-monomial $v_1^{(d_1)}\cdots v_n^{(d_n)}$ as $d = d_1 + \cdots + d_n$. The degree of a d-polynomial in $\mathbf{DP}(n)$ is defined naturally.

A product on $\mathbf{DP}(n)$ is defined as in mod 2. Starting with one variable $v_1 = v$, the product of $v^{(r)}$ and $v^{(s)}$ in $\mathbf{DP}(1)$ is defined for positive integers r, s by

$$v^{(r)}v^{(s)} = \binom{r+s}{r} v^{(r+s)}. \tag{3}$$

Substitute $\mathbb{F}_p$ by a field of characteristic 0. Since $v^{(r)}v = (r+1)v^{(r+1)}$ for $r \geq 0$, an induction on r in (3) leads to $v^{(r)} = \frac{1}{r!}v^r$ which is the so-called r-th divided power of v. The following definition involves Lucas Theorem 2.10.

Definition 3.2. For $a > 0$, consider the unique p-adic expansion $a = a_1p^{e_1} + \cdots + a_hp^{e_h}$, where the p-powers e_i are distinct and $a_i \in \mathbb{F}_p$. One may take this expansion in the ascending order of p-powers although this condition is not our requirement. Define $p\text{-exp}(a) = \{a_1p^{e_1}, \ldots, a_hp^{e_h}\}$, and put $p\text{-exp}(0) = \emptyset$. The analogous definition in mod 2 is $\text{bin}(a) = \{2^{e_1}, \ldots, 2^{e_h}\}$, where $a = 2^{e_1} + \cdots + 2^{e_h}$ is the binary expansion of a.

Example 3.3. For $p = 3$,

$3\text{-exp}(58) = \{1{\cdot}1, 1{\cdot}3, 2{\cdot}3^3\}, 3\text{-exp}(12) = \{1{\cdot}3, 1{\cdot}3^2\}, 3\text{-exp}(16) = \{1{\cdot}1, 2{\cdot}3, 1{\cdot}3^2\}.$

Definition 3.4. Consider the p-adic expansions

$$a = a_1p^{e_1} + \cdots + a_hp^{e_h}, b = b_1p^{e_1} + \cdots + b_hp^{e_h}.$$

We say that $p\text{-exp}(a) \preceq p\text{-exp}(b)$ if and only if $a_i \leq b_i$ for any $1 \leq i \leq h$. Otherwise we write $p\text{-exp}(a) \npreceq p\text{-exp}(b)$. For example, $3\text{-exp}(12) \preceq 3\text{-exp}(16)$ but, $3\text{-exp}(16) \npreceq 3\text{-exp}(58)$.

Proposition 3.5. *For $a, b \geq 0$, $\binom{b}{a}$ is not zero modulo p if and only if $p\text{-exp}(a) \preceq p\text{-exp}(b)$*

Proof. Put

$$a = a_1p^{e_1} + \cdots + a_hp^{e_h}, b = b_1p^{e_1} + \cdots + b_hp^{e_h}.$$

By Lucas Theorem, in modulo p, $\binom{b}{a}$ is not zero if and only if for any i, $\binom{b_i}{a_i}$ is not zero which is true if and only if $a_i \leq b_i$. This is equivalent to $p\text{-exp}(a) \preceq p\text{-exp}(b)$ by Definition 3.4. $\square$

Now, the product (3) can be written as follows.

$$v^{(r)}v^{(s)} = \begin{cases} \binom{r+s}{r} v^{(r+s)}, & \text{if } p\text{-exp}(r) \preceq p\text{-exp}(r+s), \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

The product in $\mathbf{DP}(n)$ is commutative and it is naturally extended for d-polynomials. Analogous discussion as in mod 2 holds except that here, coefficients exist and that $\deg(x_i) = 2$ for $1 \leq i \leq n$. In particular, $\mathbf{DP}(n)$ is a graded algebra over $\mathbb{F}_p$ for all $n \geq 1$ and is also the Hopf dual of $\mathbf{P}(n)$.

Definition 3.6. The dual of the up Steenrod power $\mathcal{P}^k : \mathbf{P}^d(n) \to \mathbf{P}^{d+k(p-1)}(n)$ is the linear map $\mathcal{P}_k : \mathbf{DP}^{d+k(p-1)}(n) \to \mathbf{DP}^d(n)$, called the down Steenrod power, so that if $\mathcal{P}_k(u) = v$ for $u \in \mathbf{DP}^{d+k(p-1)}(n)$ then

$$v(f) = \big(\mathcal{P}_k(u)\big)(f) = u\big(\mathcal{P}^k(f)\big),$$

for $f \in \mathbf{P}^d(n)$. In bilinear notation, this can be written as

$$\langle \mathcal{P}_k(u), f\rangle = \langle u, \mathcal{P}^k(f)\rangle.$$

We call $\mathcal{P}_k$ the 'down' power operation as it lowers degree by $k(p-1)$, versus the 'up' power operation $\mathcal{P}^k$ which ups degree by the same amount. From Definition 3.6 it follows that $\mathcal{P}_0$ is the identity homomorphism.

Proposition 3.7. *$\mathcal{P}_0$ is the identity map of $\mathbf{DP}(n)$.*

Definition 3.8. The total down power $\mathcal{P}_* : \mathbf{DP}(n) \to \mathbf{DP}(n)$ is the algebra map $\mathcal{P}_*(u) = \sum_{k\geq 0} \mathcal{P}_k$ for $u \in \mathbf{DP}(n)$.

The proof of algebra map property of $\mathcal{P}_*$ is goes the same lines as that of the total down square [20, Proposition 9.3.3].

Thus $\langle \mathcal{P}_*(u), f\rangle = \langle u, \mathcal{P}(f)\rangle$ for $u \in \mathbf{DP}(n)$ and $f \in \mathbf{P}(n)$, so that $\mathcal{P}_*$ for $\mathbf{DP}(n)$ is the graded dual for $\mathbf{P}(n)$.

Proposition 3.9 (Cartan formula for $\mathbf{DP}(n)$)**.** *For $k \geq 0$ and $u, v \in \mathbf{DP}(n)$,*

$$\mathcal{P}_k(uv) = \sum_{i+j=k} \mathcal{P}_i(u)\mathcal{P}_j(v).$$

Proof. Appealing the algebra property of $\mathcal{P}_*$ and equating graded parts leads to the result. □ The down powering operation commute with linear substitution. The demonstrations is as for the down squares [20, Propositions 9.3.5].

Proposition 3.10. *$\mathcal{P}_k : \mathbf{DP}^{d+k(p-1)}(n) \to \mathbf{DP}^d(n)$ is a right $\mathbb{F}_pM(n)$-module map.*

The next result enables us to manipulate down Steenrod powers. The proof goes the same lines as that of down Steenrod squares [20, Proposition 9.3.6]

Proposition 3.11. *For all $v \in \mathbf{DP}^2(n)$,*

$$\mathcal{P}_k(v^{(d)}) = \binom{d-(p-1)k}{k} v^{(d-(p-1)k)}.$$

The following results is immediately concluded.

Corollary 3.12. *For any $v \in \mathbf{DP}(n)$, $\mathcal{P}_k(v^{(d)}) = 0$ if $pk > d$ and $\mathcal{P}_k(v^{(pk)}) = v^{(k)}$.*

The last part of Corollary 3.12 may be extended.

Corollary 3.13. *Let $k = d_1 + \cdots + d_n$. Then,*

$$\mathcal{P}_k(v_1^{(pd_1)} \cdots v_n^{(pd_n)}) = v_1^{(d_1)} \cdots v_n^{(d_n)}$$

The next result enables us to calculate the action of $\mathcal{P}_k$ for d-polynomials. The proof is by induction on n using Cartan formula 3.9.

Proposition 3.14. *Given a d-monomial $u = v_1^{(d_1)} \cdots v_n^{(d_n)}$ we have*

$$\mathcal{P}_k(u) = \sum_{k_1+\cdots k_n=k} \mathcal{P}_{k_1} v_1^{(d_1)} \cdots \mathcal{P}_{k_n} v_1^{(d_n)}.$$

The dual version of Theorem 2.11 provides an efficient tool for manipulating down Steenrod powers.

Theorem 3.15. *Take $t > 0$ and let $0 < i < p^t$ and $0 \leq j < p^t$. Also, let $r \geq 0$ and $s \geq r+1$. Then,*

$$\mathcal{P}_{i+rp^t}(v^{(j+sp^t+(p-1)(i+rp^t))}) = \binom{j}{i} v^{(j)} \binom{s}{r} v^{(sp^t)}.$$

Proof. By Proposition 3.11 we have

$$\begin{aligned}\mathcal{P}_{i+rp^t}(v^{(j+sp^t+(p-1)(i+rp^t))}) &= \binom{j+sp^t}{i+rp^t} v^{(j+sp^t)} \\ &= \binom{j}{i} v^{(j)} \binom{sp^t}{rp^t} v^{(sp^t)} \quad (\text{since } i,j<p^t) \\ &= \binom{j}{i} v^{(j)} \binom{s}{r} v^{(sp^t)}.\end{aligned}$$

In the second and third equalities Lucas Theorem 2.10 is applied. Note that $\binom{s}{r}$ is always nonzero mod p since $s > r$. □ The following corollary is the dual version of Corollary 2.12. The proof is the same.

Corollary 3.16. *Let $t > 0$ and let $r \geq 0$, $s \geq r+1$. Then*

$$\mathcal{P}_{i+rp^t}(v^{(j+sp^t+(p-1)(i+rp^t))}) = 0,$$

where $i = 1, \ldots, p^t - 1$ and $j = 0, \ldots, i-1$.

Remark 3.17. Similar triangular pattern as in Definition 2.13 and the preamble also holds for down powers. The difference is that each row λ in the up-case moves $(p-1)\lambda$ columns forward in the down-case. This dual aspect of movement may be seen in Table 2 for the cases $p = 3$ which is the dual of Table 1 and that is why we have depicted them together. The same explanations in the paragraph after Corollary 2.12 works here. Due to this dual nature, matrix methods for the action of up powers [11] can be utilized for down powers.

The algebra generated by the down powers $\mathcal{P}_k$ is isomorphic to the opposite algebra $\mathcal{A}_p^{\text{op}}$. The observation is the same as in mod 2.

Definition 3.18. For the prime p, a mod p d-spike is a d-monomial of the form $v_1^{(c_1p^{e_1}-1)} \cdots v_n^{(c_np^{e_n}-1)}$, where $c_i = 1, \ldots, p-1$ and $e_i \geq 0$. Mod 2 d-spikes are the so-called d-spikes.

For example, the mod 3 d-spikes in one variable are $v^{(2)}, v^{(5)}, v^{(8)}, v^{(17)}, v^{(26)}, v^{(53)}$ and so on. Down powers annihilates the mod p d-spikes.

Proposition 3.19. *For any positive integers k, $\mathcal{P}_k(v^{(cp^e-1)}) = 0$, where $e > 0$ and $1 \leq c \leq p-1$. In general, for any* mod p *d-spike $u = v_1^{(c_1p^{e_1}-1)} \cdots v_n^{(c_np^{e_n}-1)}$, $\mathcal{P}_k(u) = 0$.*

Proof. We start with one variable. By Proposition 3.11, we must show that

$$\binom{cp^e - 1 - (p-1)k}{k} = 0. \tag{5}$$

Since $p^e - 1 - (p-1)k < p^e$, it suffices by Lucas Theorem to prove (5) for $c = 1$. To do this, we prove

$$p\text{-exp}(k) \npreceq p\text{-exp}(p^e - 1 - (p-1)k).$$

We have

$$p^e - 1 = \sum_{i=0}^{e-1} (p-1)p^{e_i}.$$

Consider $k = \sum_{i<e-1} a_i p^{e_i}$ and suppose that ℓ is the least positive integer such that $a_\ell > 0$. There are two possibilities for a_ℓ. If $a_\ell = 1$ then, in the p-adic expansion of $p^e - 1 - (p-1)k$, the coefficient of p^ℓ disappears and $p\text{-exp}(k) \npreceq p\text{-exp}(p^e - 1 - (p-1)k)$. On the other hand, if $a_\ell > 1$ then in the p-adic expansion of $p^e - 1 - (p-1)k$, the coefficient of p^ℓ gets smaller than a_ℓ and again $p\text{-exp}(k) \npreceq p\text{-exp}(p^e - 1 - (p-1)k)$. Therefore, in each case the result holds. The general case follows from Proposition 3.14. □

Definition 3.20. The mod p Steenrod kernel is defined as the formal sum $\mathbf{K}(n) = \sum_{d\geq 0} \mathbf{K}^d(n)$, where $\mathbf{K}^d(n)$ is the set of all elements $v \in \mathbf{DP}^d(n)$ such that $\mathcal{P}_k(v) = 0$ for all $k > 0$.

We know that mod p spikes cannot appear in the image of any $\mathcal{P}^k$ for $k > 0$. However, due to the duality nature, mod p d-spikes are always in the image of some $\mathcal{P}_k$. One may find the powers of one-variable mod 3 d-spikes, that is, 2, 5, 8, 17, 26, 53 in Table 2. All these d-spikes lie in $\mathbf{K}(1)$. In fact, Proposition 3.19 states that mod p Steenrod kernel contains all mod p d-spikes. The dual modular hit problem is to determine $\mathbf{K}(n)$ and is not so known.

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