Journal of Mathematical Extension
Vol. 19, No. 4 (2025) (5) 1-25
ISSN: 1735-8299
URL: http://doi.org/10.30495/JME.2025.3191
Original Research Paper

An Application of Orthonormal Bernoulli Polynomials for Solving Generalized Abel's Integral Equations

A. Seifi
Hamedan Branch, Islamic Azad University

M. Darabi
Hamedan Branch, Islamic Azad University

N. Parandin*
Kermanshah Branch, Islamic Azad University

Abstract. Integral equations, and in particular Generalized Abel's Integral equations (GAIEs), have been widely used to model various phenomena in applied science. Several numerical methods have been proposed to solve GAIEs, many of which require significant computational effort to achieve convergence. In this paper, we develop a stable method for solving GAIEs using Bernoulli orthogonal polynomials constructed via the Gram-Schmidt orthogonalization algorithm. Since our method does not rely on collocation points, the computational time is significantly reduced. Moreover, the proposed method demonstrates several advantages over existing approaches in terms of both accuracy and performance. Under certain conditions, we also establish the error bounds and provide a convergence analysis. To evaluate the effectiveness of the method, several numerical examples are presented. The results indicate that, on average, our method yields lower absolute errors compared to other techniques.

Received: January 2024; Accepted: February 2025
*Corresponding Author

AMS Subject Classification: 30E20; 45E05; 65R10; 65R20
Keywords and Phrases: Operational matrix method; Generalized Abel's integral equation; Bernoulli polynomial

1 Introduction

The theory of integral equations is one of the most important branches of applied mathematics and plays a vital role in various scientific fields, such as engineering, chemistry, and biology. Many phenomena in applied sciences can be modeled using integral equations, and solving these equations can provide valuable insights into the underlying processes. However, in practice, only a limited number of integral equations have explicit solutions; most require numerical methods. Therefore, the accuracy of numerical solutions to integral equations is critically important.

To solve integral equations numerically, a variety of methods have been introduced [7–9, 12, 18, 30, 33, 34]. A well-known class of integral equations involves singular kernels, which arise in various applications. For example, such equations arise in heat transfer [13], crystal growth [19], and fluid mechanics [27], attracting special attention due to their wide range of applications.

As mentioned earlier, obtaining analytical solutions for integral equations with singular kernels is often complex and, in many cases, impossible. Therefore, researchers have focused on developing numerical methods to solve such equations. Two notable examples of these are weakly singular Fredholm integral equations and Abel's integral equations. In recent years, a variety of numerical approaches and polynomial-based methods have been proposed to address these problems, as outlined below.

To solve weakly singular Fredholm integral equations, several numerical methods have been proposed in the literature. Cao and Xu [3] introduced technique based on singularity-preserving Galerkin methods. Pedas and Vainikko [28] developed an approach using smoothing transformations combined with piecewise polynomial projection methods. Cao et al. [4] later proposed hybrid collocation methods, while

Okayama et al. [24] employed Sinc-collocation techniques.

In addition, Lakestani et al. [17] applied Legendre multiwavelets to numerically solve weakly singular Fredholm integral-differential equations. Maleknejad et al. [21] presented a wavelet Galerkin method for singular integral equations. Subsequently, Assari et al. [1] developed the meshless product integration (MPI) method to address weakly singular integral equations.

One of the most well-known singular integral equations is the Abel integral equation, which was first introduced by Niels Henrik Abel in 1823 [33,34]. Abel formulated this equation while addressing a mechanical problem [2,11,14,29]. A notable example of solving Abel's integral equation was presented by Saadatmandi and Dehghan [31], who employed the collocation method in their approach. In [20], Majidi presents a numerical method based on a change of variable to solve Abel integral equations by removing the singularity and approximating the solution using orthogonal polynomials. Numerical results confirm the method's high accuracy, good conditioning, and efficiency compared to similar approaches. In [23], integral equations with singular or weakly singular kernels are solved using Bernoulli polynomials to provide a numerical solution. An improved method for solving Volterra integral equations of the second kind using Bernoulli polynomials and dividing the interval into subintervals is presented. By increasing the number of subintervals without changing the degree of the polynomial, a suitable result was obtained [15].

Consider the generalized Abel integral equation as follows [5]:

$$\lambda_1(s)\int_a^s \frac{x(t)}{(s-t)^{\alpha}}dt+\lambda_2(s)\int_s^b \frac{x(t)}{(t-s)^{\alpha}}dt=\zeta(s), \qquad s\in(a,b), \quad 0<\alpha<1, \tag{1}$$

where $\zeta(s)$ is a given function. Several methods have been proposed to solve the generalized Abel integral equation as follows.

Chakrabarti and George [5] introduced a formula for the solution of general Abel integral equation. Later, Chakrabarti [6] used a direct function-theoretic method to solve the same class of equations. Dixit et al. [10] employed an almost operational matrix approach to find a solution to the generalized Abel integral equation. Furthermore, Pandey and

Mandal [25] applied a numerical method using Bernstein polynomials to solve a system of generalized Abel integral equations. Reference [26] introduces a collocation-based approach for approximating solutions to generalized Abel's integral equations. The study shows that employing different orthogonal polynomials can yield highly accurate results, even when only a limited number of polynomials are used.

In this paper, we use Bernoulli orthogonal polynomials to solve generalized Abel integral equations. The proposed method offers several advantages over previous approaches, notably by avoiding the use of collocation points, which helps to reduce computational cost. Furthermore, by exploiting the orthogonality properties of the polynomials, the method achieves improved accuracy.

The remainder of this paper is organized as follows: Section 2 introduces the Bernoulli basis polynomials and their orthogonalization using the Gram-Schmidt process. In Section 3, we present the computational matrix-based approach for solving GAIEs. Section 4 provides the error estimation and convergence analysis. Numerical experiments are presented in Section 5 to demonstrate the effectiveness of the proposed method. Finally, Section 6 concludes the paper with a brief summary.

2 Preliminaries

Bernoulli polynomials, despite not being orthogonal, are frequently employed in solving integral equations because of their unique analytical properties.

Definition 2.1 (Bernoulli polynomials)**.** According to Samadyar [32], the Bernoulli polynomials of degree k can be defined using the following two main formulas

$$\beta_k(s) = \sum_{i=0}^{k} \binom{k}{i} \alpha_{k-i} s^i, \qquad s \in [0,1], \tag{2}$$

for $k \geq 0$, where α_k are Bernoulli numbers. Alternatively, the Bernoulli polynomials can also be defined by the following formula:

$$\sum_{i=0}^{k} \binom{k+1}{i} \beta_i(s) = (k+1)s^k, \qquad k = 0, 1, \dots \ .$$

Definition 2.2 (Orthogonality of Bernoulli polynomials)**.** [22] By applying the Gram-Schmidt orthogonalization process to the linearly independent set $\{\beta_0(s), \dots, \beta_n(s)\}$, we obtain a set of Orthonormal Bernoulli basis functions $\{B_0(s), \dots, B_n(s)\}$, where

$$B_0(s) = \frac{\beta_0(s)}{\|\beta_0(s)\|}, \qquad B_k(s) = \frac{\phi_k(s)}{\|\phi_k(s)\|}, \tag{3}$$

and

$$\phi_k(s) = \beta_k(s) - \sum_{j=0}^{k-1} \int_a^b \beta_k(s) B_j(s) ds B_j(s), \qquad k = 1, \dots, n.$$

The functions $B_0(s), \dots, B_n(s)$ are defined on the interval $[0, 1]$ and serve as the Orthonormal Bernoulli basis functions (OBBFs). The explicit formula for the OBBF of n is given by

$$B_n(s) = \sqrt{2n+1} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{2n-i}{n-i} s^{n-i}, \qquad n = 0, 1, \dots . \tag{4}$$

Definition 2.3 (Construction of an approximate series based on OBBFs [22])**.** Let $f(s) \in L^2[a, b]$ and let $\{B_k(s)\}$ denote the orthonormal Bernoulli basis functions (OBBFs) defined by Eqs. (3)-(4), then the function $f(s)$ can be represented as an infinite series

$$f(s) = \sum_{k=0}^{\infty} f_k B_k(s), \tag{5}$$

where the coefficients f_k are defined by the inner product. To approximate the function $f(s)$, we can truncate the series in Eq. (5) at $k = n$. The resulting finite approximation is given by

$$f(s) \approx \sum_{k=0}^{n} f_k B_k(s) = \sum_{k=0}^{n} \int_a^b f(s) B_k(s) ds B_k(s). \tag{6}$$

To approximate the integral terms appearing in Eq. (1) using orthonormal Bernoulli polynomials, we use the definition of inner product for the coefficients in the finite series. Specifically, we have

$$\int_a^s B_i(t)dt \approx \sum_{k=0}^{n} \int_a^b \int_a^s B_i(t)dtB_k(s)dsB_k(s), \qquad i = 0, \ldots, n,$$

$$\int_a^s \frac{B_i(t)}{(s-t)^\alpha}dt \approx \sum_{k=0}^{n} \int_a^b \int_a^s \frac{B_i(t)}{(s-t)^\alpha}dtB_k(s)dsB_k(s), \qquad i = 0, \ldots, n. \tag{7}$$

Similarly, for the integral terms from s to b, we have

$$\int_s^b B_i(t)dt \approx \sum_{k=0}^{n} \int_a^b \int_s^b B_i(t)dtB_k(s)dsB_k(s), \qquad i = 0, \ldots, n,$$

$$\int_s^b \frac{B_i(t)}{(t-s)^\alpha}dt \approx \sum_{k=0}^{n} \int_a^b \int_s^b \frac{B_i(t)}{(t-s)^\alpha}dtB_k(s)dsB_k(s), \qquad i = 0, \ldots, n. \tag{8}$$

Definition 2.4 (Construction of Operational Matrices for OBBFs)**.** Following the approach in [22], the remainder of this section is devoted to constructing the operational matrix of Eqs. (6), (7) and (8).The operational matrix form of a function $f(s) \in L^2[a,b]$ can be expressed as

$$f(s) = \left([F]^B\right)^T B(s), \tag{9}$$

where $[F]^B$ and $B(s)$ are column vectors of size $(n+1) \times 1$, defined as follows

$$[F]^B = \begin{bmatrix} \langle f(s), B_0(s)\rangle \\ \vdots \\ \langle f(s), B_n(s)\rangle \end{bmatrix}, \qquad B(s) = \begin{bmatrix} B_0(s) \\ \vdots \\ B_n(s) \end{bmatrix}. \tag{10}$$

Based on the relations (7) and (9), the operational matrix representation of the term $\int_a^s \frac{B_i(t)}{(s-t)^\alpha}dt$ can be written as

$$\int_a^s \frac{B_i(t)}{(s-t)^\alpha}dt = \begin{bmatrix} \langle \int_a^s \frac{B_i(t)}{(s-t)^\alpha}dt, B_0(s)\rangle \\ \vdots \\ \langle \int_a^s \frac{B_i(t)}{(s-t)^\alpha}dt, B_n(s)\rangle \end{bmatrix}^T \begin{bmatrix} B_0(s) \\ \vdots \\ B_n(s) \end{bmatrix}.$$

Therefore, using Eq. (10), the operational matrix form of the vector-valued function $\int_a^s \frac{B(t)}{(s-t)^\alpha}dt$ becomes

$$\int_a^s \frac{B(t)}{(s-t)^\alpha}dt = \begin{bmatrix} \int_a^s \frac{B_0(t)}{(s-t)^\alpha}dt \\ \vdots \\ \int_a^s \frac{B_i(t)}{(s-t)^\alpha}dt \\ \vdots \\ \int_a^s \frac{B_n(t)}{(s-t)^\alpha}dt \end{bmatrix}$$

$$= \begin{bmatrix} \langle \int_a^s \frac{B_0(t)}{(s-t)^\alpha}dt, B_0(s)\rangle & \dots & \langle \int_a^s \frac{B_0(t)}{(s-t)^\alpha}dt, B_n(s)\rangle \\ \vdots & \ddots & \vdots \\ \langle \int_a^s \frac{B_i(t)}{(s-t)^\alpha}dt, B_0(s)\rangle & \dots & \langle \int_a^s \frac{B_i(t)}{(s-t)^\alpha}dt, B_n(s)\rangle \\ \vdots & \ddots & \vdots \\ \langle \int_a^s \frac{B_n(t)}{(s-t)^\alpha}dt, B_0(s)\rangle & \dots & \langle \int_a^s \frac{B_n(t)}{(s-t)^\alpha}dt, B_n(s)\rangle \end{bmatrix}_{(n+1)\times(n+1)}$$

$$\times \begin{bmatrix} B_0(s) \\ \vdots \\ B_i(s) \\ \vdots \\ B_n(s) \end{bmatrix}_{(n+1)\times 1} .$$

Thus, we summarize this operation as

$$\int_a^s \frac{B(t)}{(s-t)^\alpha}dt = [V_1]B(s).$$

Similarly, from Eqs. (8) and (9), the operational matrix representation for the term $\int_s^b \frac{B_i(t)}{(t-s)^\alpha}dt$ is given by

$$\int_s^b \frac{B_i(t)}{(t-s)^\alpha}dt = \begin{bmatrix} \langle \int_s^b \frac{B_i(t)}{(t-s)^\alpha}dt, B_0(s)\rangle \\ \vdots \\ \langle \int_s^b \frac{B_i(t)}{(t-s)^\alpha}dt B_n(s)\rangle \end{bmatrix}^T \begin{bmatrix} B_0(s) \\ \vdots \\ B_n(s) \end{bmatrix}.$$

Let $B(s)$ denote the orthonormal Bernoulli basis functions as in Eq. (10), then the operational matrix form of $\int_s^b \frac{B(t)}{(t-s)^\alpha}dt$ is obtained as

$$\int_s^b \frac{B(t)}{(t-s)^\alpha}dt = \begin{bmatrix} \int_s^b \frac{B_0(t)}{(t-s)^\alpha}dt \\ \vdots \\ \int_s^b \frac{B_i(t)}{(t-s)^\alpha}dt \\ \vdots \\ \int_s^b \frac{B_n(t)}{(t-s)^\alpha}dt \end{bmatrix}$$

$$= \begin{bmatrix} \langle \int_s^b \frac{B_0(t)}{(t-s)^\alpha}dt, B_0(s)\rangle & \dots & \langle \int_s^b \frac{B_0(t)}{(t-s)^\alpha}dt, B_n(s)\rangle \\ \vdots & \ddots & \vdots \\ \langle \int_s^b \frac{B_i(t)}{(t-s)^\alpha}dt, B_0(s)\rangle & \dots & \langle \int_s^b \frac{B_i(t)}{(t-s)^\alpha}dt, B_n(s)\rangle \\ \vdots & \ddots & \vdots \\ \langle \int_s^b \frac{B_n(t)}{(t-s)^\alpha}dt, B_0(s)\rangle & \dots & \langle \int_s^b \frac{B_n(t)}{(t-s)^\alpha}dt, B_n(s)\rangle \end{bmatrix}_{(n+1)\times(n+1)}$$

$$\times \begin{bmatrix} B_0(s) \\ \vdots \\ B_i(s) \\ \vdots \\ B_n(s) \end{bmatrix}_{(n+1)\times 1}.$$

Finally, the above result can be compactly represented as

$$\int_s^b \frac{B(t)}{(t-s)^\alpha}dt = [V_2]B(s).$$

3 Description of the Numerical Technique

3.1 Construction the operational matrices of $\zeta(s)$ and $x(s)$

In this section, we transform the various components of Eq. (1) into the matrix forms using the orthonormal Bernoulli basis functions (OBBFs). The known function $\zeta(s)$ can be approximated by finite truncated series.

The known coefficients ζ_k are obtained by using the inner multiplication $\langle \zeta(s), B_k(s) \rangle = \int_a^b \zeta(s) B_k(s) ds$ in this series

$$\zeta(s) \approx \sum_{k=0}^{n} \zeta_k B_k(s) = \sum_{k=0}^{n} \int_a^b \zeta(s) B_k(s) ds B_k(s),$$

Hence, the matrix representation of $\zeta(s)$ is obtained by the inner product of the row coefficient vector and the basis vector

$$\zeta(s) = \begin{bmatrix} \langle \zeta(s), B_0(s) \rangle & \ldots & \langle \zeta(s), B_n(s) \rangle \end{bmatrix} \begin{bmatrix} B_0(s) \\ \vdots \\ B_n(s) \end{bmatrix},$$

or more compactly

$$\zeta(s) = \left([Z]^B \right)^T B(s), \tag{11}$$

where $[Z]^B$ denotes the coefficient vector of $\zeta(s)$ in the Bernoulli basis. To solve GAIE (1), the unknown function $x(s)$ can similarly be approximated using OBBFs $B(s)$ as follow

$$x(s) \approx \sum_{k=0}^{n} x_k B_k(s), \tag{12}$$

and its matrix form is

$$x(s) = \begin{bmatrix} x_0 & \ldots & x_n \end{bmatrix} \begin{bmatrix} B_0(s) \\ \vdots \\ B_n(s) \end{bmatrix},$$

or equivalently

$$x(s) = \left([X]^B \right)^T B(s). \tag{13}$$

3.2 Construction of the operational matrices of Eq. (1)

By substituting the operating matrices from Eqs. (11) and (13) into Eq. (1), we obtain the following matrix form of the generalized Abel integral equation

$$\lambda_1(s)\left([X]^B\right)^T\int_a^s\frac{B(t)}{(s-t)^\alpha}dt+\lambda_2(s)\left([X]^B\right)^T\int_s^b\frac{B(t)}{(t-s)^\alpha}dt \\ =\left([Z]^B\right)^T B(s). \tag{14}$$

Similarly, Eq. (14) can be rewritten in the expanded matrix form as follows

$$\begin{aligned}
&\lambda_1(s)\begin{bmatrix}x_0 & \dots & x_i & \dots & x_n\end{bmatrix}\\
&\times\begin{bmatrix}\int_a^b\int_a^s\frac{B_0(t)}{(s-t)^\alpha}dtB_0(s)ds & \dots & \int_a^b\int_a^s\frac{B_0(t)}{(s-t)^\alpha}dtB_n(s)ds\\ \vdots & \ddots & \vdots\\ \int_a^b\int_a^s\frac{B_i(t)}{(s-t)^\alpha}dtB_0(s)ds & \dots & \int_a^b\int_a^s\frac{B_i(t)}{(s-t)^\alpha}dtB_n(s)ds\\ \vdots & \ddots & \vdots\\ \int_a^b\int_a^s\frac{B_n(t)}{(s-t)^\alpha}dtB_0(s)ds & \dots & \int_a^b\int_a^s\frac{B_n(t)}{(s-t)^\alpha}dtB_n(s)ds\end{bmatrix}\begin{bmatrix}B_0(s)\\ \vdots\\ B_i(s)\\ \vdots\\ B_n(s)\end{bmatrix}\\
&+\lambda_2(s)\begin{bmatrix}x_0 & \dots & x_i & \dots & x_n\end{bmatrix}\\
&\times\begin{bmatrix}\int_a^b\int_s^b\frac{B_0(t)}{(t-s)^\alpha}dtB_0(s)ds & \dots & \int_a^b\int_s^b\frac{B_0(t)}{(t-s)^\alpha}dtB_n(s)ds\\ \vdots & \ddots & \vdots\\ \int_a^b\int_s^b\frac{B_i(t)}{(t-s)^\alpha}dtB_0(s)ds & \dots & \int_a^b\int_s^b\frac{B_i(t)}{(t-s)^\alpha}dtB_n(s)ds\\ \vdots & \ddots & \vdots\\ \int_a^b\int_s^b\frac{B_n(t)}{(t-s)^\alpha}dtB_0(s)ds & \dots & \int_a^b\int_s^b\frac{B_n(t)}{(t-s)^\alpha}dtB_n(s)ds\end{bmatrix}\begin{bmatrix}B_0(s)\\ \vdots\\ B_i(s)\\ \vdots\\ B_n(s)\end{bmatrix}\\
&=\begin{bmatrix}\int_a^b\zeta(s)B_0(s)ds & \dots & \int_a^b\zeta(s)B_i(s)ds & \dots & \int_a^b\zeta(s)B_n(s)ds\end{bmatrix}\begin{bmatrix}B_0(s)\\ \vdots\\ B_i(s)\\ \vdots\\ B_n(s)\end{bmatrix}.
\end{aligned}\tag{15}$$

Equation (15) can be written in a simpler and more compact form

$$\lambda_1(s)\left([X]^B\right)^T[V_1]^B B(s)+\lambda_2(s)\left([X]^B\right)^T[V_2]^B B(s)=\left([Z]^B\right)^T B(s). \tag{16}$$

After removing the common basis function vector $B(s)$ from both sides of Eq. (16), the equation reduces to the following matrix form

$$\lambda_1(s)\left([X]^B\right)^T[V_1]^B+\lambda_2(s)\left([X]^B\right)^T[V_2]^B=\left([Z]^B\right)^T,$$

Therefore, solving for $[X]^B$, we get

$$\left([X]^B\right)^T=\left([Z]^B\right)^T\left(\lambda_1(s)[V_1]^B+\lambda_2(s)[V_2]^B\right)^{-1}.$$

Finally, the approximate solution $x(s)$ can be obtained by substituting the coefficients stored in the matrix $[X]^B$ into the truncated orthonormal Bernoulli series defined in Eq. (12).

$$x(s)=\left([X]^B\right)^T B(s).$$

4 Error Estimation and Convergence Analysis

Here, we explain the convergence analysis and error bound of the approximate solutions of the presented scheme in Section 3 for solving Eq. (1). We present and prove the error estimation theorem by referring to two well-known theorems.

Theorem 4.1. *[16] Let f be a function in $C^{n+1}[a,b]$, and let p be the polynomial of degree $\leq n$ that interpolates the function f at $n+1$ distinct points $s_0, s_1, ..., s_n$ in the interval $[a,b]$. For each $s\in[a,b]$, there exists a corresponding point $\xi_s\in(a,b)$ such that*

$$f(s)-p(s)=\frac{1}{(n+1)!}f^{(n+1)}(\xi_s)\prod_{i=0}^{n}(s-s_i),$$

Theorem 4.2 (The Weierstrass Approximation Theorem)**.** *[16] If f is continuous on $[a,b]$ and $\epsilon>0$, then there exists a polynomial p such that $\left|f(x)-p(x)\right|\leq\epsilon$ on the interval $[a,b]$.*

Theorem 4.3. *Suppose f is a Bernoulli polynomial basic function in $C^{n+1}[0,1]$, and f_n is its approximation. Let $s_0, s_1, ..., s_n$ be randomly selected points satisfying*

$$0 = s_0 < s_1 < \cdots < s_n = 1.$$

Then for every $s \in [0,1]$, there exists a $\xi_s \in (0,1)$ such that

$$f(s) - f_n(s) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_s) \prod_{i=0}^{n} (s - s_i),$$

where

$$R_n(s) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (s - s_i),$$

and

$$\left|R_n(s)\right| \leqslant \frac{\Omega_1 \Omega_2}{(n+1)!},$$

with

$$\Omega_1 = \max_{0 \le s \le 1} \left|(s - s_0)(s - s_1)...(s - s_n)\right|,$$
$$\Omega_2 = \max_{0 \le \xi_s \le 1} \left|f^{(n+1)}(\xi_s)\right|.$$

Proof. If s is one of the selected points, then both sides of the equation are zero, so the claim holds trivially. For other points, using Eqs. (2), (3), and the series expansion

$$f(s) = \sum_{k=0}^{\infty} f_k B_k(s),$$

and truncating the series at degree n, we have the interpolating polynomial

$$f(s) = \sum_{i=0}^{n} f_i B_i(s) + \frac{f^{(n+1)}(\xi_s)}{(n+1)!}(s - s_0)(s - s_1)...(s - s_n).$$

Letting

$$f_n(s) = \sum_{i=0}^{n} f_i B_i(s),$$

we get

$$f(s) = f_n(s) + \frac{f^{(n+1)}(\xi_s)}{(n+1)!} \prod_{i=0}^{n} (s - s_i),$$

and hence

$$f(s) - f_n(s) = \frac{f^{(n+1)}(\xi_s)}{(n+1)!} \prod_{i=0}^{n} (s - s_i),$$

therefore

$$\left| f(s) - f_n(s) \right| = \left| \frac{f^{n+1}(\xi_s)}{(n+1)!} \prod_{i=0}^{n} (s - s_i) \right|,$$

which implies

$$\left| R_n(s) \right| \leq \frac{\Omega_1 \Omega_2}{(n+1)!}.$$

□

4.1 Convergence analysis

Theorem 4.4. *Assume that the known function $\zeta(s) \in C^{n+1}[0,1]$, and $x_n(s)$ is the orthonormal Bernoulli basis approximation of the exact solution $x(s)$ of GAIE (1). Let $\bar{x}_n(s) = \sum_{i=0}^{n} \bar{x}_i B_i(s)$ be the approximate solution obtained using the present method in Section 3, then there exist constants M_1, M_2 such that*

$$\left\| x(s) - \bar{x}_n(s) \right\|_2 \leq \epsilon_1 M_1 M_2 \frac{\Omega_1 \Omega_2}{(n+1)!} + \epsilon_2 \left\| X - \overline{X} \right\|_2,$$

where

$$\Omega_1 = \max_{0 \leq s \leq 1} \left| (s - s_0)(s - s_1)...(s - s_n) \right|,$$
$$\Omega_2 = \max_{0 \leq \xi_s \leq 1} \left| f^{(n+1)}(\xi_s) \right|.$$

Proof. Let $x_n(s) = \sum_{i=0}^{n} x_i B_i(s)$ be the orthonormal Bernoulli basis approximations of the exact solution $x(s)$ of Eq. (1), then we can write

$$\lambda_1(s) \int_a^s \frac{x(t)}{(s-t)^\alpha} dt + \lambda_2(s) \int_s^b \frac{x(t)}{(t-s)^\alpha} dt = \zeta(s).$$

Defining linear integral operators

$$L_1(x(s)) = \int_a^s \frac{x(t)}{(s-t)^\alpha} dt,$$
$$L_2(x(s)) = \int_s^b \frac{x(t)}{(t-s)^\alpha} dt,$$

we can rewrite

$$\lambda_1(s) L_1\big(x(s)\big) + \lambda_2(s) L_2\big(x(s)\big) = \zeta(s).$$

Since $x_n(s)$ is an approximation of the analytical solution $x(s)$, we also have

$$\lambda_1(s) L_1(x_n(s)) + \lambda_2(s) L_2(x_n(s)) = \zeta(s).$$

Subtracting yields

$$\lambda_1(s) L_1(x(s) - x_n(s)) + \lambda_2(s) L_2(x(s) - x_n(s)) = 0.$$

Therefore

$$e_n(s) = x(s) - x_n(s) = -\frac{\lambda_2(s)}{\lambda_1(s)} L_1^{-1}\Big(L_2\big(x(s) - x_n(s)\big)\Big).$$

Taking the 2-norm, we get

$$\big\|e_n(s)\big\|_2 = \left|\frac{\lambda_2(s)}{\lambda_1(s)}\right| M_1 M_2 \big\|x(s) - x_n(s)\big\|_2.$$

Now, we conclude that the approximation error satisfies the following inequality

$$\begin{aligned}
\big\|x(s) - \bar{x}_n(s)\big\| =& \big\|x(s) - x_n(s)\big\|_2 + \big\|x_n(s) - \bar{x}_n(s)\big\|_2 \\
\leq& \left|\frac{\lambda_2(s)}{\lambda_1(s)}\right| \big\|L_1^{-1}\big\|_2 \|L_2\|_2 \left(\int_0^1 \left(\frac{\Omega_1 \Omega_2}{(n+1)!}\right)^2 ds\right)^{\frac{1}{2}} \\
&+ \left(\int_0^1 \left(\sum_{i=0}^{n} (x_i - \bar{x}_i) B_i(s)\right)^2 ds\right)^{\frac{1}{2}} \\
\leq& \epsilon_1 M_1 M_2 \frac{\Omega_1 \Omega_2}{(n+1)!} + \epsilon_2 \big\|X - \bar{X}_2\big\|
\end{aligned}$$

□

5 Numerical Experiments

In this section, to demonstrate the accuracy and efficiency of the OBBF matrix method, three numerical examples are presented. In all computational experiments, absolute errors for different values of n are provided, supported by graphs and tables. These numerical results are illustrated using figures generated by a program written in Mathematica 11.0. In the first three examples, the results obtained using the current technique are presented along with the absolute errors in Tables 1, 2 and 3.

Example 5.1. [26] Consider the Generalized Abel's integral equation Eq. (1) with $\lambda_1(s) = \lambda_2(s) = 1$, $\alpha = \frac{1}{2}$ and $\zeta(s) = \frac{4s^{3/2}}{3} - \frac{32s^{7/2}}{35} - \frac{32}{35}\sqrt{1-s}s^3 - \frac{16}{35}\sqrt{1-s}s^2 + \frac{104}{105}\sqrt{1-s}s + \frac{8\sqrt{1-s}}{21}$. The exact solution is $x(s) = s - s^3$.

The accuracy of the approximate solution is clearly shown in Figure 1. Figures 2 and Table 1 illustrate the behavior of absolute errors of the OBBF scheme for $n = 3$ and $n = 5$. In Table 1, a comparison between the errors in the current technique for various n is presented. Since the exact solution is a polynomial of degree 3, then it is sufficient to consider the value of $n = 3, \ldots$ to approximate $x_n(s)$. This example demonstrates that the operational matrix scheme for the generalized Abel's integral equation performs well when the exact solution is a polynomial function.

Example 5.2. Consider the GAIE

$$\lambda_1(s)\int_a^s \frac{x(t)}{(s-t)^\alpha}dt + \lambda_2(s)\int_s^b \frac{x(t)}{(t-s)^\alpha}dt = \zeta(s), \quad s \in (0,1),$$

with $\lambda_1(s) = \lambda_2(s) = 1$, $\alpha = \frac{1}{5}$, $\zeta(s) = \frac{e^s(1-s)^{4/5}\Gamma\left(\frac{4}{5}\right)}{(s-1)^{4/5}} + e^s\Gamma\left(\frac{4}{5}\right) - \frac{e^s(1-s)^{4/5}\Gamma\left(\frac{4}{5},s-1\right)}{(s-1)^{4/5}} - e^s\Gamma\left(\frac{4}{5},s\right)$ and the exact solution is $x(s) = e^s$.

Here, we evaluate the performance and accuracy of the proposed method for a problem whose exact solution is an exponential function. Figure 3 contains a graphs of absolute errors for $n = 3$ and $n = 5$. Also in Table 2, the absolute error values for $n = 3$ and $n = 5$ are compared. It is well observed that by increasing the value of n, the absolute error decreases.

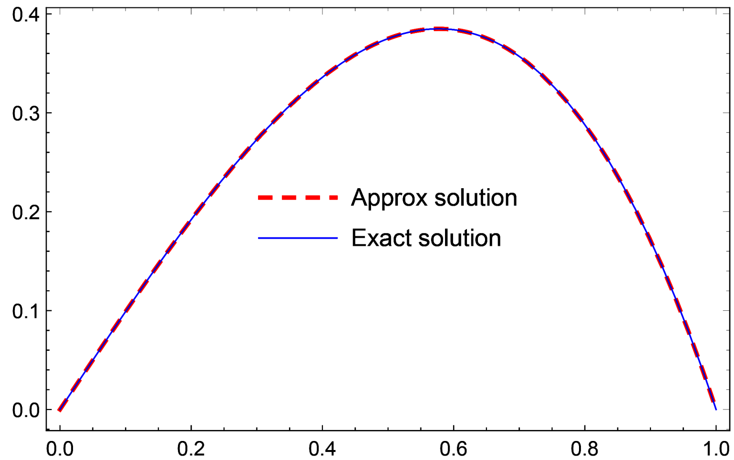
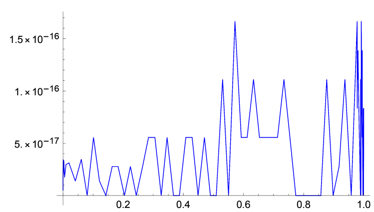
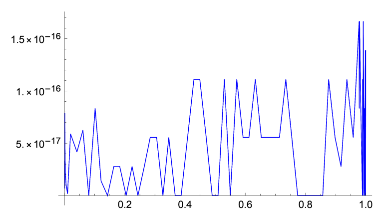


Figure 1: Comparison of solutions for Example 5.1 with $n = 5$

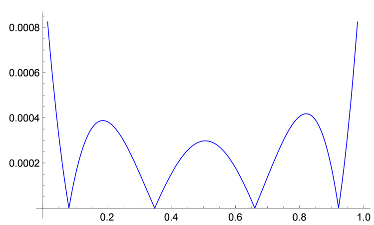


(a) $n = 3$

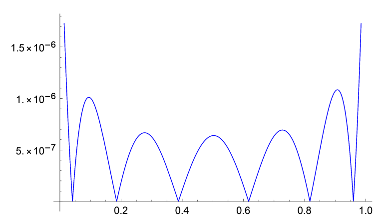


(b) $n = 5$

Figure 2: $\left|x_n(s) - x(s)\right|$ of Example 5.1 with $n = 3$ and $n = 5$



(a) $n = 3$



(b) $n = 5$

Figure 3: $\left|x_n(s) - x(s)\right|$ of Example 5.2 with $n = 3$ and $n = 5$

Table 1: The absolute errors of Example 5.1

Node	x(s)	$\lvert x(s) - x_3(s) \rvert$	$\lvert x(s) - x_5(s) \rvert$
0.0	0.000	5.55112×10^{-17}	0
0.1	0.099	1.38778×10^{-17}	2.77556×10^{-17}
0.2	0.192	5.55112×10^{-17}	5.55112×10^{-17}
0.3	0.273	1.11022×10^{-16}	5.55112×10^{-17}
0.4	0.336	0	0
0.5	0.375	5.55112×10^{-17}	5.55112×10^{-17}
0.6	0.384	5.55112×10^{-17}	5.55112×10^{-17}
0.7	0.357	0	0
0.8	0.288	1.11022×10^{-16}	1.11022×10^{-16}
0.9	0.171	2.77556×10^{-17}	2.77556×10^{-17}
1.0	0.000	0	5.55112×10^{-17}

Table 2: The absolute errors of Example 5.2

Node	x(s)	$\lvert x(s) - x_3(s) \rvert$	$\lvert x(s) - x_5(s) \rvert$
0.0	1.00000	1.08629×10^{-3}	3.08271×10^{-6}
0.1	1.10517	1.37385×10^{-4}	1.00427×10^{-6}
0.2	1.2214	3.83427×10^{-4}	1.86423×10^{-7}
0.3	1.34986	1.58981×10^{-4}	6.27842×10^{-7}
0.4	1.49182	1.51157×10^{-4}	1.1191×10^{-7}
0.5	1.64872	2.97398×10^{-4}	6.39618×10^{-7}
0.6	1.82212	1.79589×10^{-4}	1.53654×10^{-7}
0.7	2.01375	1.37281×10^{-4}	6.34939×10^{-7}
0.8	2.22554	4.05703×10^{-4}	2.34141×10^{-7}
0.9	2.4596	1.76459×10^{-4}	1.07061×10^{-6}
1.0	2.71828	1.22260×10^{-3}	3.36075×10^{-6}

Example 5.3. Consider the GAIE Eq. (1) with

$$\zeta(s) = \frac{6}{5}\, {}_2F_1\left(-\frac{5}{6}, \frac{1}{2}; \frac{1}{6}; s\right) + \frac{\sqrt{\pi} s^{5/6} \Gamma\left(-\frac{5}{6}\right)}{\Gamma\left(-\frac{1}{3}\right)} + \frac{\sqrt{\pi} s^{5/6} \Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{11}{6}\right)},$$

$\lambda_1(s) = \lambda_2(s) = 1$, $\alpha = \frac{1}{2}$ and the exact solution is $x(s) = s^{\frac{1}{3}}$.

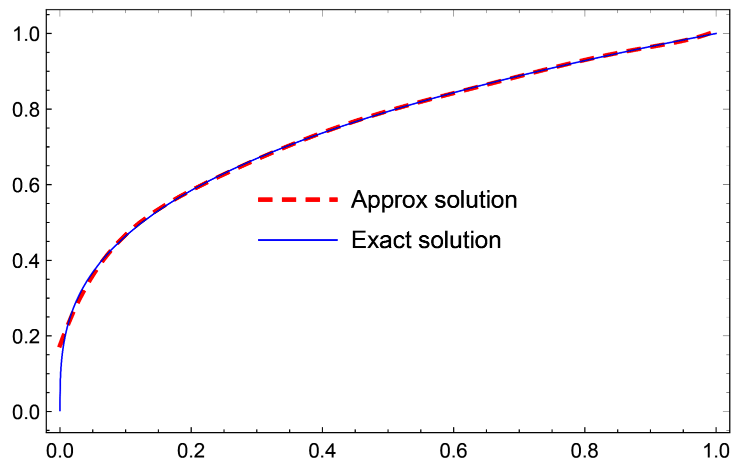
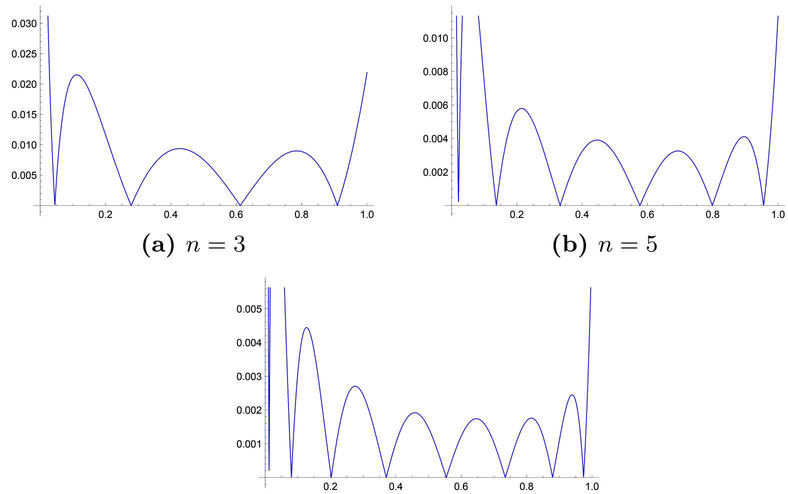


Figure 4: Comparison of solutions for Example 5.3 with $n = 7$

Table 3: The absolute errors of Example 5.3

Node	x(s)	$\lvert x(s) - x_3(s)\rvert$	$\lvert x(s) - x_5(s)\rvert$
0.1	0.464159	2.11370×10^{-2}	7.34728×10^{-3}
0.2	0.584804	1.16665×10^{-2}	5.62435×10^{-3}
0.3	0.669433	2.54156×10^{-3}	2.11392×10^{-3}
0.4	0.736806	9.08339×10^{-3}	3.23251×10^{-3}
0.5	0.793701	7.53728×10^{-3}	3.06711×10^{-3}
0.6	0.843433	9.4146×10^{-4}	1.00374×10^{-3}
0.7	0.887904	6.2501×10^{-3}	3.24168×10^{-3}
0.8	0.928318	8.88536×10^{-3}	7.50257×10^{-5}
0.9	0.965489	1.42448×10^{-3}	4.09531×10^{-3}

Now, we apply the proposed method from Section 3 to equation (1), where the exact solution is given as a fractional power of s. Table 3 shows the absolute errors for $n = 3$ and $n = 5$. In Figure 4, the exact and approximate solutions are plotted for $n = 7$. Also, the errors for



(a) $n = 3$ **(b)** $n = 5$

(c) $n = 7$

Figure 5: The absolute error $\left|x_n(s) - x(s)\right|$ of Example 5.3 with $n = 3, 5$ and $n = 7$

$n = 3, 5$ and $n = 7$ are plotted through graphs in Figure 5.

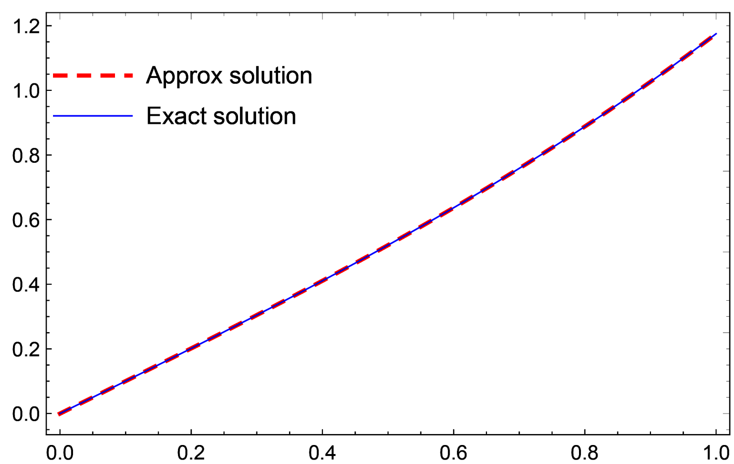
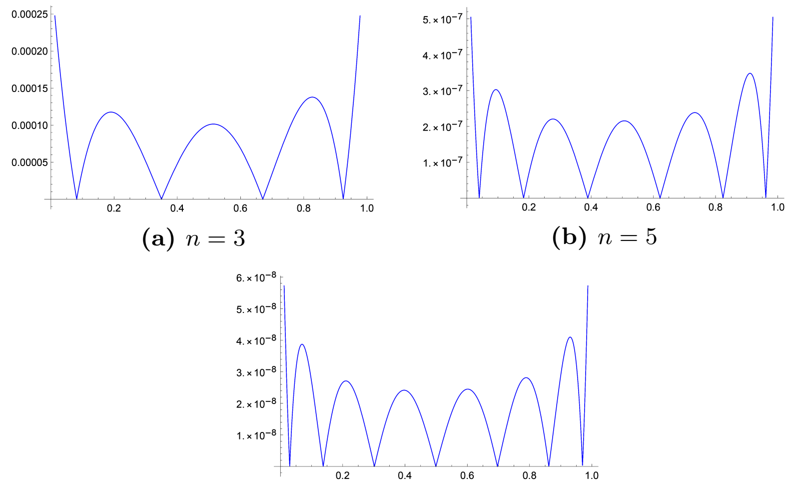


Figure 6: Comparison of solutions for Example 5.4 with $n = 3$



(a) $n = 3$ **(b)** $n = 5$

(c) $n = 7$

Figure 7: The absolute error $\left|x_n(s) - x(s)\right|$ of Example 5.4 with $n = 3, 5$ and $n = 7$

Example 5.4. Consider the Generalized Abel's integral equation (1) with

$$\zeta(s) = -\frac{1}{2}\sqrt{\pi}e^{-s}\text{erf}\left(\sqrt{1-s}\right) + \frac{1}{2}\sqrt{\pi}e^{s}\text{erf}\left(\sqrt{s}\right) + \frac{1}{2}\sqrt{\pi}e^{s}\text{erfi}\left(\sqrt{1-s}\right) - \frac{1}{2}\sqrt{\pi}e^{-s}\text{erfi}\left(\sqrt{s}\right),$$

$\lambda_1(s) = \lambda_2(s) = 1$, $\alpha = \frac{1}{2}$ and the exact solution is $x(s) = \sinh(s)$.

In Figure 6, the exact and approximate solutions are shown for $n = 3$. Additionally, the errors for $n = 3, 5,$ and 7 are illustrated in the graphs presented in Figure 7.

6 Conclusion

This paper presented a numerical approximation method for solving Generalized Abel's integral equations using simple operational matrices. The functional matrix form of equation (1) leads to the formation of a system of integral equations based on the OBBF. In addition, error analysis and the convergence of the proposed technique were established. This method provides a stable approach without requiring collocation points, relying solely on the orthogonality of the Bernoulli polynomials to enhance the accuracy of the scheme. Testing several types of exact solution functions for GAIEs demonstrated the high efficiency and accuracy of the method.

Acknowledgements
The authors would like to show their gratitude to the reviewers for their beneficial remarks.

References

[1] P. Assari, H. Adibi, M. Dehghan, The numerical solution of weakly singular integral equations based on the meshless product integration (MPI) method with error analysis, *Appl. Numer. Math.* 81 (2014) 76–93. doi:10.1016/j.apnum.2014.02.013.

[2] K.E. Atkinson, *The Numerical Solution of Integral Equations of Second Kind*, Cambridge University Press, Cambridge, UK, 1997.

[3] Y. Cao, Y. Xu, Singularity preserving Galerkin methods for weakly singular Fredholm integral equations, *J. Integr. Equations Appl.* 6 (1994) 303–334. doi:10.1216/jiea/1181075816.

[4] Y. Cao, M. Huang, L. Liu, Y. Xu, Hybrid collocation methods for Fredholm integral equations with weakly singular kernels, *Appl. Numer. Math.* 57 (2007) 549–561. doi:10.1016/j.apnum.2006.07.007.

[5] A. Chakrabarti, A.J. George, A formula for the solution of general Abel integral equation, *Appl. Math. Lett.* 7 (1994) 87–90. doi:10.1016/0893-9659(94)90037-X.

[6] A. Chakrabarti, Solution of the generalized Abel integral equation, *J. Integr. Equations Appl.* 20 (2008) 1–11. doi:10.1216/JIE-2008-20-1-1.

[7] T. Diogo, N.J. Ford, P.M. Lima, S.M. Thomas, Solution of a singular integral equation by a split-interval method, *Int. J. Numer. Anal. Model.* 4 (2007) 63–73.

[8] T. Diogo, P. Lima, Collocation solutions of a weakly singular Volterra integral equation, *TEMA – Tendências em Matemática Aplicada e Computacional* 8 (2007) 229–238. doi:10.5540/tema.2007.08.02.0229.

[9] T. Diogo, N.J. Ford, P. Lima, S. Valtchev, Numerical methods for a Volterra integral equation with non-smooth solutions, *J. Comput. Appl. Math.* 189 (2006) 412–423. doi:10.1016/j.cam.2005.10.019.

[10] S. Dixit, R.K. Pandey, S. Kumar, O.P. Singh, Solution of the generalized Abel integral equation by using almost Bernstein operational matrix, *Am. J. Comput. Math.* 1 (2011) 226–234. doi:10.4236/ajcm.2011.14026.

[11] F.D. Gakhov, *Boundary Value Problems*, Addison-Wesley Publishing Company, 1996.

[12] W. Han, Existence, uniqueness and smoothness results for second-kind Volterra equations with weakly singular kernels, *J. Integr. Equations Appl.* 6 (1994) 365–384. doi:10.1216/jiea/1181075819.

[13] E. Hopf, *Mathematical Problems of Radiative Equilibrium*, Stechert-Hafner Service Agency, New York, 1964.

[14] R.P. Kanwal, *Linear Integral Equations: Theory and Techniques*, Academic Press, New York, 1971.

[15] A. Khansari, S. Khezerloo, M. Nouri, M. Arghand, An improved Bernoulli collocation method for solving Volterra integral equations, 2024.

[16] D. Kincaid, W. Cheney, *Numerical Analysis*, Pacific Grove, California, 1990.

[17] M. Lakestani, B.N. Saray, M. Dehghan, Numerical solution for the weakly singular Fredholm integro-differential equations using Legendre multiwavelets, *J. Comput. Appl. Math.* 235 (2011) 3291–3303. doi:10.1016/j.cam.2011.01.043.

[18] W. Lamb, A spectral approach to an integral equation, *Glas. Math. J.* 26 (1985) 83–89. doi:10.1017/S0017089500005802.

[19] J.S. Lew, R. Ghez, An integral equation for crystal growth, *SIAM J. Appl. Math.* 17 (1975) 687–690.

[20] R. Majidi, F. Dastmalchi, M. Jahangirirad, Solving system of Abel's integral equations by using change of variable with orthogonal polynomials, *Int. J. Appl. Comput. Math.* 9 (2023) 141.

[21] K. Maleknejad, M. Nosrati, E. Najafi, Wavelet Galerkin method for solving singular integral equations, *Comput. Appl. Math.* 31 (2012) 373–390. doi:10.1590/S1807-03022012000200009.

[22] R.K. Maurya, V. Devi, N. Srivastava, V.K. Singh, An efficient and stable Lagrangian matrix approach to Abel integral and integro-differential equations, *Appl. Math. Comput.* 374 (2020) 1–30. doi:10.1016/j.amc.2019.125005.

[23] M.M. Mustafa, H.A. Abd-Alrazak, Bernoulli polynomials method for solving integral equations with singular kernel, *Baghdad Science Journal* 21 (2024) 16.

[24] T. Okayama, T. Matsuo, M. Sugihara, Sinc-collocation methods for weakly singular Fredholm integral equations of the second kind, *J. Comput. Appl. Math.* 234 (2010) 1211–1227. doi:10.1016/j.cam.2009.07.049.

[25] R.K. Pandey, B.N. Mandal, Numerical solution of a system of generalized Abel integral equations using Bernstein polynomials, *J. Adv. Res. Sci. Comput.* 2 (2010) 44–53.

[26] R.K. Pandey, S. Sharma, K. Kumar, Collocation method for generalized Abel's integral equations, *J. Comput. Appl. Math.* 302 (2016) 118–128. doi:10.1016/j.cam.2016.01.036.

[27] H.J. Park, E. Mitsoulis, Numerical simulation of circular entry flows of fluid M1 using an integral constitutive equation, *J. Non-Newtonian Fluid Mech.* 42 (1992) 301–314. doi:10.1016/0377-0257(92)87015-4.

[28] A. Pedas, G. Vainikko, Smoothing transformation and piecewise polynomial projection methods for weakly singular Fredholm integral equations, *Commun. Pure Appl. Anal.* 5 (2006) 395–413. doi:10.3934/cpaa.2006.5.395.

[29] A.C. Pipkin, *A Course on Integral Equations*, Springer, New York, 1991. doi:10.1007/978-1-4612-4446-2.

[30] A. Saadatmandi, M. Dehghan, Variational iteration method for solving a generalized pantograph equation, *Comput. Math. Appl.* 58 (2009) 2190–2196. doi:10.1016/j.camwa.2009.03.017.

[31] A. Saadatmandi, M. Dehghan, A collocation method for solving Abel's integral equations of first and second kinds, *Z. Naturforsch. A* 63 (2008) 752–756. doi:10.1515/zna-2008-1202.

[32] N. Samadyar, F. Mirzaee, Orthonormal Bernoulli polynomials collocation approach for solving stochastic Itô–Volterra integral equations of Abel type, *Int. J. Numer. Model. Electron. Netw. Devices Fields* 33 (2020) 1–14. doi:10.1002/jnm.2688.

[33] A.M. Wazwaz, *Linear and Nonlinear Integral Equations*, Springer, Berlin, 2011. doi:10.1007/978-3-642-21449-3.

[34] A.M. Wazwaz, *A First Course in Integral Equations*, World Scientific, Singapore, 1997.

Ali Seifi
PhD in Applied Mathematics
Department of Mathematics
Hamedan Branch, Islamic Azad University
Hamedan, Iran
E-mail: aliseifi.math@gmail.com

Mohsen Darabi

PhD in Applied Mathematics
Department of Mathematics
Hamedan Branch, Islamic Azad University
Hamedan, Iran
E-mail: darabi.math@gmail.com

Nouredin Parandin
Associate Professor of Applied Mathematics
Department of Mathematics
Kermanshah Branch, Islamic Azad University
Kermanshah, Iran
E-mail: nparandin54@gmail.com