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A New Class of Weighted Lindley Distributions

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Abstract. In this paper, a new class of distributions called the weighted Lindley (WL) distribution is proposed. Different properties of the distribution are investigated. This new family has two unknown parameters. It is observed that the maximum likelihood estimators of the unknown parameters cannot be obtained in explicit forms and they have to be obtained by solving some numerical methods. One real data application illustrates the performance of the distribution.

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1. Introduction

The Lindley distribution was originally introduced by Lindley [6] in the context of Bayesian statistics. Its probability density function (pdf) is given by

$$f(x) = \frac{\alpha^2}{\alpha+1}(1+x)e^{-\alpha x}, \ \ x > 0, \alpha > 0. \tag{1}$$

Due to the popularity of the exponential distribution in statistics especially in lifetime analysis, Lindley distribution has been overlooked in the literature. In 2008 Ghitany et al. studied some mathematical

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properties of (1) and showed that it is a better than the exponential distribution, [2]. Afterwards, introducing the new classes of distributions based on modifications of the Lindley distribution has received special attentions. Among the introduced distributions, generalized Lindley introduced by Zakerzadeh and Dolati, generalized Lindley introduced by Nadarajah and et al, two-parameter weighted Lindley and power Lindley are the most popular parametric models. In 2009 Zakerzadeh and Dolati introduced a three-parameter generalization of Lindley distribution (GL1), based on certain mixtures of the gamma distributions [8]. They studied various properties of the proposed distribution and, provided real data example to show the flexibility of the model. Nadarajah and et al [7] introduced another generalization for Lindley distribution (GL2). They generalized the Lindley distribution by considering the exponentiation of its distribution function. Ghitany et al. (2011) proposed the two-parameter weighted Lindley distribution by considering a normalizing constant and a suitable weighted function. Here, we denote two-parameter weighted Lindley by TWL. The TWL distribution has pdf

$$f(x) = \frac{\alpha^{\theta+1}}{(\alpha+\theta)\Gamma(\theta)} x^{\theta-1}(1+x)e^{-\alpha x}, \;\; x > 0, \alpha > 0, \theta > 0.$$

where $\Gamma(\theta)$ is the complete gamma function, for more insights see [4]. Recently, to obtain a more flexible family of distributions, a new extension of the Lindley distribution called the power Lindley (PL) distribution is introduced by Ghitany et al. [3].

A random variable is said to be skew-normal with parameter λ, if density function is

$$f(x) = 2\phi(x)\Phi(\lambda x),$$

where $\phi(x)$ and $\Phi(x)$ are the standard normal density and distribution functions, respectively, which is introduced by Azzalini [1]. Afterwards, several authors have done an extensive work on introducing shape/skewness parameters for other symmetric distributions such as skew-t, skew-Cauchy, skew-Laplace, and skew-logistic. However, it seems that no attempts were made to implement Azzalini's idea for non-symmetric distributions. Gupta and Kundu, [5] have followed a similar

approach as of Azzalini for introducing shape parameters to the exponential distribution. They showed that by applying Azzalini's method to the exponential distribution, a new class of weighted exponential distribution can be obtained.
In this paper, a weighted Lindley (WL) distribution obtained by implementing Azzalini's method to the Lindley distribution. Some important properties of the WL have been established. The proposed distribution provides a better fit than the some existing distributions such as TWL, GL1, GL2 and PL.

The paper is organized as follows. In Section 2 we introduce the WL distribution. Different properties are discussed in Section 3. Maximum likelihood estimation of the model parameters are investigated in Section 4. Finally, one application to real data sets is presented in Section 5 for illustrative purposes.

2. Definitions and Basic Properties

In this section, we construct the weighted Lindley distribution exactly the same way Azzalini [1] obtained the skew-normal distribution from two independent and identical normal distributions.

Definition 2.1. *A random variable X is said to have weighted Lindley distribution, $WL(\alpha_1, \alpha_2)$ with shape parameters $\alpha_1 > 0$ and $\alpha_2 > 0$, if the pdf of X is given as following*

$$f(x) = K(\alpha_1, \alpha_2)(1 + x)e^{-\alpha_1 x}(1 - (1 + \frac{\alpha_1\alpha_2 x}{\alpha_1 + 1})e^{-\alpha_1\alpha_2 x}), \;\; x > 0, \quad (2)$$

and 0 otherwise, where $K(\alpha_1, \alpha_2) = \frac{(\alpha_1^3+\alpha_1^2)(\alpha_2+1)^3}{(\alpha_2+1)^2(\alpha_1+1)^2\alpha_2+\alpha_2^2-\alpha_2}$.

Interpretation: suppose X_1 and X_2 are two independent identically distributed random variables, with the pdf $f(.)$ and cumulative distribution function (cdf)$F(.)$, then for any $\alpha_2 > 0$, consider a new random variable $X = X_1$ given that $\alpha_2 X_1 > X_2$. The pdf of new random variable$X$ is

$$f(x) = \frac{1}{P(\alpha_2 X_1 > X_2)} f_Y(x) F_Y(\alpha_2 x), \quad x > 0. \tag{3}$$

Now equation (2) can be obtained from equation (3) by replacing $f_Y(x) = \frac{\alpha_1^2}{\alpha_1+1}(1+x)e^{-\alpha_1 x}$ *and* $F_Y(x) = 1 - (1 + \frac{\alpha_1 x}{\alpha_1+1})e^{-\alpha_1 x}$.

As $\alpha_2 \to \infty$, *WL* (α_1, α_2) *converges to Lindley distribution with parameter* α_1 *and as* $\alpha_2 \to 0$, *it converges to* $TWL(\alpha_1, 2)$. *Suppose* X_1, X_2 *are independent random variables of Lindley distribution with parameter* α_1 *and represent the failure times of the components of a series system, assumed to be independent. Then, failure distribution of the system becomes* $WL(\alpha_1, 1)$.

The distribution function of WL can be written as

$$F_X(x) = 1 - K(\alpha_1, \alpha_2)e^{-\alpha_1 x}(\frac{1+\alpha_1}{\alpha_1^2} + \frac{x}{\alpha_1} - C_1(\alpha_1, \alpha_2)e^{-\alpha_1\alpha_2 x}$$

$$-C_2(\alpha_1, \alpha_2)xe^{-\alpha_1\alpha_2 x} - \frac{\alpha_2}{(\alpha_1+1)(\alpha_2+1)}x^2 e^{-\alpha_1\alpha_2 x}),$$

where, $C_1(\alpha_1, \alpha_2) = \frac{(\alpha_1^2+2\alpha_1)(\alpha_2+1)^2+3\alpha_2+1}{(\alpha_1^3+\alpha_1^2)(\alpha_2+1)^3}$, $C_2(\alpha_1, \alpha_2) = \frac{\alpha_1(\alpha_2+1)^2+3\alpha_2+1}{(\alpha_1^2+\alpha_1)(\alpha_2+1)^2}$.

Figure 1 shows the graph of (2) for different values of α_1, α_2.

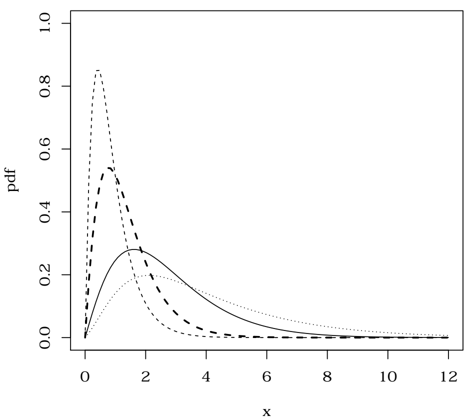

Figure 1: Examples of the WL density for $(\alpha_1, \alpha_2) = (2, 2)(- - -)$, $(\alpha_1, \alpha_2) = (0.5, 3)(...)$, $(\alpha_1, \alpha_2) = (1, 0.05)(-)$ and $(\alpha_1, \alpha_2) = (1.5, 1)$(**---**).

3. Statistical and Reliability Properties

Several statistical and reliability properties of WL, such as failure rate (or hazard rate) function, moment generating function, mean residual life time, kth moment, median and mode are investigated in this section.

3.1 Failure rate function

The failure rate is a key notion in reliability and survival analysis for measuring the ageing process. Failure rate function of WL can be written as

$$h(x) = \frac{(1+x)(1-(1+\frac{\alpha_1\alpha_2 x}{\alpha_1+1})e^{-\alpha_1\alpha_2 x})}{\frac{1+\alpha_1+\alpha_1 x}{\alpha_1^2} - (C_1(\alpha_1,\alpha_2) + C_2(\alpha_1,\alpha_2)x + \frac{\alpha_2}{(\alpha_1+1)(\alpha_2+1)}x^2)e^{-\alpha_1\alpha_2 x}}.$$

It is obvious that $h(0) = 0$ and $\lim_{x\to\infty} h(x) = \alpha_1$. Since $f(x)$ is always log-concave, therefore h will be an increasing function of x . Figure 2 shows some shapes of the failure rate function with some different values of α_1 and α_2.

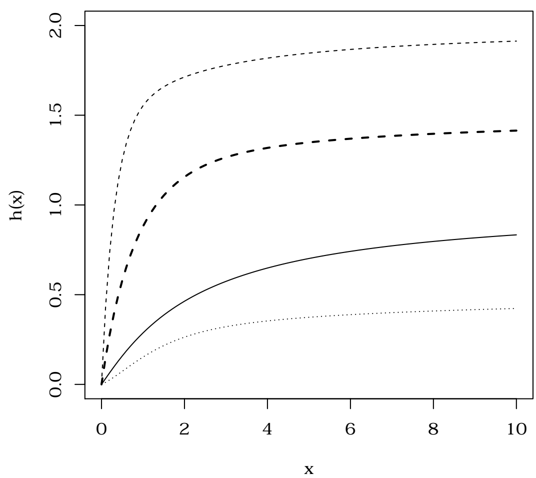

Figure 2: Hazard function of the WL density for $(\alpha_1,\alpha_2) = (2,2)(---)$, $(\alpha_1,\alpha_2) = (0.5,3)(...)$, $(\alpha_1,\alpha_2) = (1,0.05)(-)$ and $(\alpha_1,\alpha_2) = (1.5,1)$(**---**).

3.2 Moment generating function and mean residual life time

Many of the interesting characteristics and features of a distribution can be studied through its moments. The moment generating function of WL is immediately written as

$$M_X(t) = K(\alpha_1,\alpha_2)(\frac{\alpha_1 - t + 1}{(\alpha_1 - t)^2} - \frac{\alpha_1\alpha_2 + \alpha_1 - t + (1 + \frac{\alpha_1\alpha_2}{1+\alpha_1})}{(\alpha_1\alpha_2 + \alpha_1 - t)^2} - \frac{2\alpha_1\alpha_2}{(\alpha_1 + 1)(\alpha_1\alpha_2 + \alpha_1 - t)^3}).$$

and the kth moment of WL distribution can derived as

$$E(X^k) = K(\alpha_1,\alpha_2)\Gamma(k+1)(\frac{\alpha_1 + k + 1}{\alpha_1^{k+2}} - \frac{\alpha_1\alpha_2 + \alpha_1 + (1 + \frac{\alpha_1\alpha_2}{1+\alpha_1})(k+1)}{(\alpha_1\alpha_2 + \alpha_1)^{k+2}} - \frac{\alpha_1\alpha_2(k+2)(k+1)}{(\alpha_1 + 1)(\alpha_1\alpha_2 + \alpha_1)^{k+3}}).$$

Also, mean residual life time can be written as

$$m(t) = \frac{\frac{\alpha_1+2}{\alpha_1^3} + \frac{t}{\alpha_1^2} - (\frac{c_1}{\alpha_1\alpha_2+\alpha_1} + \frac{c_2}{(\alpha_1\alpha_2+\alpha_1)^2} + \frac{2\alpha_2}{(\alpha_1^4+\alpha_1^3)(\alpha_2+1)^4})e^{-\alpha_1\alpha_2 t}}{\frac{\alpha_1+1}{\alpha_1^2} + \frac{t}{\alpha_1} - (C_1(\alpha_1,\alpha_2) + C_2(\alpha_1,\alpha_2)t + \frac{\alpha_2}{(\alpha_1+1)(\alpha_2+1)}t^2)e^{-\alpha_1\alpha_2 t}} - \frac{(\frac{c_2}{\alpha_1\alpha_2+\alpha_1} + \frac{2\alpha_2}{(\alpha_1^3+\alpha_1^2)(\alpha_2+1)^3})te^{-\alpha_1\alpha_2 t} + \frac{\alpha_2}{(\alpha_1^2+\alpha_1)(\alpha_2+1)^2}t^2 e^{-\alpha_1\alpha_2 t}}{\frac{\alpha_1+1}{\alpha_1^2} + \frac{t}{\alpha_1} - (C_1(\alpha_1,\alpha_2) + C_2(\alpha_1,\alpha_2)t + \frac{\alpha_2}{(\alpha_1+1)(\alpha_2+1)}t^2)e^{-\alpha_1\alpha_2 t}}.$$

3.3 Median and mode

Evaluating explicit expressions for the median and mode of WL is not easy. So, we can find the median of WL by solving the following relation

$$K(\alpha_1,\alpha_2)e^{-\alpha_1 x}(\frac{\alpha_1 + 1}{\alpha_1^2} + \frac{x}{\alpha_1} - C_1(\alpha_1,\alpha_2)e^{-\alpha_1\alpha_2 x} - C_2(\alpha_1,\alpha_2)xe^{-\alpha_1\alpha_2 x} - \frac{\alpha_2}{(\alpha_1 + 1)(\alpha_2 + 1)}x^2 e^{-\alpha_1\alpha_2 x}) - \frac{1}{2} = 0.$$

The mode of WL will be the solution of the following equation

$$(1-\alpha_1(1+x))e^{-\alpha_1 x}+(\alpha_1\alpha_2+\alpha_1\alpha_2 x+\frac{(\alpha_1\alpha_2)^2}{\alpha_1+1}x+\frac{(\alpha_1\alpha_2)^2}{\alpha_1+1}x^2$$
$$-1-\frac{\alpha_1\alpha_2}{\alpha_1+1}-\frac{2\alpha_1\alpha_2}{\alpha_1+1}x)e^{-\alpha_1\alpha_2 x}=0.$$

4. Maximum Likelihood Estimation

The estimation of the parameters of the WL distribution using the maximum likelihood estimation is studied in this section. Let $X_1, X_2, ..., X_n$ be a random sample with observed values $x_1, x_2, ..., x_n$ from WL distribution with parameters α_1 and α_2. The log-likelihood function is given by

$$L=n\ln(K(\alpha_1,\alpha_2))+\sum_{i=1}^{n}\ln(1+x_i)-\alpha_1\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\ln(1-(1+\frac{\alpha_1\alpha_2 x_i}{\alpha_1+1})e^{-\alpha_1\alpha_2 x_i})$$

To find MLEs of unknown parameters, we differentiate the log-likelihood function with respect to α_1 and α_2 as follows;

$$\frac{dL}{d\alpha_1}=\frac{n(3\alpha_1+2)}{\alpha_1^2+\alpha_1}-\frac{2n(\alpha_2+1)^2(\alpha_1+1)\alpha_2}{(\alpha_2+1)^2(\alpha_1+1)^2\alpha_2+\alpha_2^2-\alpha_2}-\sum_{i=1}^{n}x_i$$
$$+\sum_{i=1}^{n}\frac{\frac{\alpha_2 x_i}{(\alpha_1+1)^2}e^{-\alpha_1\alpha_2 x_i}-\alpha_2 x_i(1+\frac{\alpha_1\alpha_2 x_i}{\alpha_1+1})e^{-\alpha_1\alpha_2 x_i}}{1-(1+\frac{\alpha_1\alpha_2 x_i}{\alpha_1+1})e^{-\alpha_1\alpha_2 x_i}}=0,$$

$$\frac{dL}{d\alpha_2}=\frac{3n}{\alpha_2+1}-\frac{n((\alpha_1+1)^2(3\alpha_2^2+4\alpha_2+1)+2\alpha_2-1)}{((\alpha_1+1)^2(\alpha_2+1)^2\alpha_2)+\alpha_2^2-\alpha_2}$$
$$+\sum_{i=1}^{n}\frac{\frac{\alpha_1 x_i}{\alpha_1+1}e^{-\alpha_1\alpha_2 x_i}-\alpha_1 x_i(1+\frac{\alpha_1\alpha_2 x_i}{\alpha_1+1})e^{-\alpha_1\alpha_2 x_i}}{1-(1+\frac{\alpha_1\alpha_2 x_i}{\alpha_1+1})e^{-\alpha_1\alpha_2 x_i}}=0.$$

The MLEs of the unknown parameters cannot be obtained explicitly. They have to be obtained by solving some numerical methods.

5. Data Analysis

To investigate the advantage of proposed distribution we consider one real data set. The following data represent the marks of forty eight slow space students in Mathematics in the final examination of the Indian Institute of Technology, Kanpur in year 2003, studied by Gupta and Kundu, [5];

29 25 50 15 13 27 15 18 7 7 8 19 12 18 5 21 15 86 21 15 14 39 15 14 70 44 6 23 58 19 50 23 11 6 34 18 28 34 12 37 4 60 20 23 40 65 19 31.

For these data, we fit the WL distribution and compare its fitting with TWL, PL with pdf given by $\frac{\theta\alpha^2}{\alpha+1}(1+x^\theta)x^{\theta-1}e^{-\alpha x^\theta}$ where $\alpha, \theta > 0$, GL1 with pdf given by $\frac{\alpha^2(\alpha x)^{\theta-1}(\theta+\lambda x)e^{-\alpha x}}{(\lambda+\alpha)\Gamma(\theta+1)}$ where $\lambda > 0, \alpha > 0, \theta \geqslant 0$, and GL2 with pdf given $\frac{\theta\alpha^2}{\alpha+1}(1+x)(1-(1+\frac{\alpha x}{\alpha+1})e^{-\alpha x})^{\theta-1}e^{-\alpha x}, \theta > 0, \alpha > 0$.

Table 1: Summary of fitted distributions

Model	MLEs	Log- Lik	K-S test	P-value
TWL	α=0.09, θ=1.391	-197.308	0.107	0.639
PL	α=0.057, θ=1.078	-197.548	0.107	0.638
GL1	α=0.086, θ=1.23, λ=13.72	-197.127	0.103	0.683
GL2	α=0.084, θ=1.275	-197.276	0.106	0.660
WL	α=0.079, θ=5.828	-196.582	0.093	0.791

Table 1 gives the MLEs of the parameters of the considered models, their estimated log-likelihood functions, the Kolmogrov-Smirnov (K-S) statistics and the corresponding p-values of these models. A close examination of Table 1 reveals that the WL model provides the best fit for the given data among all these models.

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