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Advanced Catastrophic Risk Modeling in Insurance: A Fokker-Planck Equation Approach

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Abstract. This paper presents an advanced model for assessing risks associated with potential catastrophic events faced by insurance companies. The model focuses on describing the behavior of claims related to phenomena that can have severe and far-reaching consequences. The mathematical foundation of this model is based on the Fokker-Planck equation, specifically its fractional form, which provides a robust framework for capturing the dynamics of risk processes. By modeling the solution to these equations, we derive the density function of the risk process, enabling a comprehensive understanding of the evolution of catastrophic events. The study emphasizes perturbed risk processes, by employing the fixed point theory and utilizing fractional Brownian motion to model both normal and anomalous diffusion by varying the Hurst index. A key component of this approach is the calculation of the ruin probability, a critical risk measure in actuarial science, which is evaluated for a variety of models with corresponding numerical implementations. This approach offers a novel perspective on actuarial risk modeling, presenting a new methodology for coupling the severity of claims with the frequency of occurrence. The final fractional partial differential equations open a gate to using numerical methods in the field for extreme risk measurement and modeling of catastrophic or abnormal events.

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1 Introduction

Catastrophic phenomena are among the most challenging to model, and this has been a persistent issue in actuarial science since the field's origin. A low amount of historical data on the occurrence of these events, their unpredictable severity when they happen, and the preparedness of the community against them are among the important parameters that make modeling difficult. Insurance companies are among the front liners when phenomena like floods, pandemic or drought occur. They often face a destructive number of claims that cannot be met, at least in the short term. Therefore, they have to calculate the probability of occurrence and the possible severity of these events beforehand. Here, we try to consider many parameters related to the subject and present a risk process that is both realistic and tractable.

Consider an insurance company that offers protection across various domains, such as vehicle insurance, life and health insurance, and fire insurance. These protections are provided through specific contracts, wherein predefined premiums are collected from policyholders in exchange for assuming partial or complete risk. In the event of a catastrophic incident, numerous claims may arise that require prompt resolution. Additionally, a variety of insurance contracts may have their respective triggers activated. To model the associated risk, we utilize the following perturbed surplus process of the insurance company:

$$U_t = \mu t - \sum_{i=1}^{M} S_i + D(t)B_t^H, \tag{1}$$

where

$$S_1 = \sum_{i=1}^{N_t^{(1)}} z_i^1, \; S_2 = \sum_{i=1}^{N_t^{(2)}} z_i^2, \; \cdots, S_M = \sum_{i=1}^{N_t^{(M)}} z_i^M.$$

Additionally, each N_t^i is a Poisson process with parameter λ_i, and z_i^j are independent and identically distributed (i.i.d.) random variables that represent the claims with distribution functions Φ_i^j. We assume there are M different perils, each with its own definition and corresponding insurance policy as outlined in the contracts. Each customer may have one or multiple protections related to these M perils. Additionally, $D(t)B_t^H$ is the process, such as Brownian motion or fractional Brownian motion, that can be used to model bias, error, or dependencies between other variables.

Remark 1. *The Poisson parameters, λ_i's, are also used to control the inclusion of the perils when modeling the situation in a particular area. For example, $\lambda_i = 0$ indicates that the i-th peril is either not considered* *in the contracts or has a very low probability of occurrence given the type of catastrophe and the presence of other perils.*

We used fractional Brownian motion (FBM) as the noise in our modeling for two reasons. First, this process can model both normal and anomalous diffusion, which is crucial when modeling catastrophic phenomena. Second, given the assumed interrelationship between the perils, we use FBM to capture such relationships between the variables in the model.

1.1 Preliminaries

Random walk has been a valuable tool in modeling various phenomena across different fields. It is particularly useful for modeling types of diffusion, such as the path of a particle in a given environment, the spread of diseases or information in a network, and human or animal mobility [18]. A random walk consists of two components: one related to the time the walker remains stationary or exhibits negligible movement, and the other related to the magnitude of the movement or jump made once the waiting time is over. The size of the jump can vary according to a variable, such as the price of a stock, the number of infected individuals during an epidemic, or the amount of a claim that an insurance company must pay to its policy holders.

The two variables, waiting time and size of the jumps, determine the behavior of the random walk and the outcome for the walker, either after a specific period or asymptotically over time. These variables can be either deterministic or random; however, at least one of them must be random to ensure that the walk is classified as random. The mathematical representation of the random walk is given by:

$$R_t = \sum_{i=1}^{N} X_i,$$

where N is a process that counts the number of jumps in a particular time interval and X_i represents the size of each jump. When the waiting time is not fixed but is instead a continuous random variable, we use N_t, which is a process that counts the number of jumps up to time t. The resulting random walk is known as the continuous-time random walk (CTRW) process, first studied by Montroll and Weiss [21], who provided a Fourier-Laplace representation of the density function for this process.

$$\hat{\tilde{P}}(k,s) = \frac{1-\tilde{\psi}(s)}{s} \frac{1}{1-\tilde{\psi}(s)\hat{f}(k)},$$

where $\tilde{\psi}(s)$ is the Laplace transform of the waiting times and $\hat{f}(k)$ is the Fourier transform of the jumps. Depending on the behavior of these two variables, waiting times and the size of the jumps, the CTRW can describe various types of diffusion, including normal diffusion such as Brownian motion and anomalous diffusion such as Levy flights. For a detailed discussion of different asymptotic behaviors and possible explicit representations of the density function, we refer to [12, 16].

An interesting aspect of modeling such processes is the variety of approaches that can be used to understand their behavior. These approaches include master equations and partial differential equation (PDE) representations. In this work, we adopt the PDE approach, specifically converting the models into a form known as the Fokker-Planck equation, which is a family of fractional partial differential equations. The Fokker-Planck equations have a long history of being used to model various phenomena across different disciplines [23].

The well-known diffusion equation

$$\frac{\partial P(x,t)}{\partial t} = D\frac{1}{2}\frac{\partial^2 P(x,t)}{\partial^2 x},$$

is a special case of the Fokker-Planck equation where the first term is zero. In this case, the solution represents the probability density function of Brownian motion [27]. It also corresponds to a special case of the CTRW where both the jumps and the counting process are well-behaved, assuming the existence of moments for the waiting times and the jumps.

Although there are other approaches to describing random phenomena, the PDE approach offers a robust arsenal for both analytical and numerical treatment of the resulting equations. Therefore, we apply this methodology to risk processes arising from insurance-related problems and compute key metrics in this field, such as the probability of ruin. A comprehensive and detailed PDE treatment of CTRW processes has been reviewed by Metzler et al. [18, 19]. In previous studies, such as the work by Alkhazzan et al., 2023, the existence and continuity of solutions for multi-time scale fractional stochastic differential equations have been addressed, providing the foundation for modeling complex systems like insurance risk processes [1, 2]. The application of CTRW modeling in science is extensive, with applications branching out in many directions. It has a particularly long history in finance and insurance [13, 24, 25, 29]. Specifically, all insurance risk modeling typically begins with a compound Poisson process, which is a special case of CTRW.

$$U_t = u + \mu t - \sum_{i=0}^{N_t} X_i,$$

where u is the initial capital, μ is the premium rate, N_t is the counting process corresponding to the number of claims, and X_i represents the severity of the claims [10, 14, 30]. The probability of ruin is defined by the random variable $\tau(u) = \inf\{t \geq 0 : U_t < 0\}$, which is the first passage time of the risk process crossing the value zero. Therefore, the Ultimate probability of ruin is defined as $\psi(u) = P[\tau(u) < \infty]$.

For finite time, the ∞ is replaced with a finite amount T [3]. Calculating this quantity is a very challenging task; therefore, models are often designed based on tractability conditions. This means that some characteristics of the real-world problem are replaced with ones that are more mathematically manageable, resulting in an approximation of the original model that better represents the fundamental issue. For instance, the independence assumption between the variables in model 1.1 is unrealistic in many cases. Additionally, the severity of claims is often assumed to be well-behaved, typically light-tailed, with finite mean and variance. To address these deficiencies, researchers have introduced extra parameters to account for real-world phenomena. For example, they have perturbed the classical model with normal diffusion to capture possible dependencies between the number and severity of claims or to include additional factors such as fluctuations due to interest rates from the insurance company's investments [6, 9, 26, 28]. There have also been instances where perturbations used anomalous diffusion to focus more on the severity variable, considering extreme scenarios and applying Levy stable motion for this purpose [7].

We will employ a different method to find the Fourier transform and, consequently, the probability density of continuous-time random walks (CTRW), with a particular focus on CTRWs perturbed by processes such as Brownian or fractional Brownian motion. The CTRWs used in our models will have a common characteristic: the counting process will be a Poisson process. This choice of counting process is conducive to a reliable model, as it accommodates various considerations, including the heavy-tailed nature of the jumps, dependencies between the counting process and the jumps, and other elements such as market fluctuations. We begin with a general model, and special cases will be addressed later, with their calculations presented separately.

2 Main Results

We first calculate the fractional partial differential equation for the case of a single peril and then extend the results to the case of M perils.

The perturbed risk process is given by

$$U_t = \mu t - \sum_{i=0}^{N_t} z_i + D(t)B_t^H, \tag{2}$$

where, μ is the premium rate, $D(t)$ is the diffusion function (or kernel function), N_t is a Poisson process, z_i represents the random variable for the severity of claims, and B_t^H is a fractional Brownian motion with Hurst index H. The following theorem provides the fractional Fokker-Planck equation for the above jump-diffusion process with drift.

Theorem 2.1. *For the jump-diffusion process with drift in Eq.(2) and the choice of diffusion function in a way that*

$$\lim_{dt\to 0}\frac{(D(t+dt)-D(t))^2}{dt}=\frac{D_0Ht^{2H-1}}{dt^{2H}}, \tag{3}$$

the fractional Fokker-Planck equation equals to

$$\begin{aligned}\frac{\partial P(x,t)}{\partial t} &= -\mu\frac{\partial P(x,t)}{\partial x}\\ &+ D_0Ht^{2H-1}\frac{\partial^2 P(x,t)}{\partial^2 x}\\ &+ \lambda\int_0^\infty P(x-y,t)d\Phi(y)-\lambda P(x,t). \end{aligned}\tag{4}$$

Proof. By conditioning the process on the occurrence of a claim in a time interval with length dt, the following equation follows

$$\begin{aligned}P(x,t+dt) &= (1-\lambda dt)E\left[P(x-\mu dt-D(t)\left[B_{t+dt}^H-B_t^H\right],t)\right]\\ &+ \lambda dtE\left[P(x-y,t)\right].\end{aligned}$$

Assuming that $P(x,t)$ has a Taylor expansion, the following relation holds

$$\begin{aligned}P(x,t+dt) &= (1-\lambda dt)E\Bigg[P(x,t)-\mu dt\frac{\partial P(x,t)}{\partial x}\\ &- \big(D(t+dt)-D(t)\big)\frac{\partial P(x,t)}{\partial x}[B_{t+dt}^H-B_t^H]\\ &+ \frac{1}{2}\frac{\partial^2 P(x,t)}{\partial^2 x}\Big[\mu^2dt^2+2\mu dt\big(D(t+dt)-D(t)\big)[B_{t+dt}^H-B_t^H]\\ &+ \big(D(t+dt)-D(t)\big)^2[B_{t+dt}^H-B_t^H]^2\Big]+\cdots\Bigg]\\ &+ \lambda dtE\left[P(x-y,t)\right].\end{aligned}$$

Applying the expectation function and the properties of FBM, we get

$$\begin{aligned}\frac{P(x,t+dt)-P(x,t)}{dt} &= -\mu\frac{\partial P(x,t)}{\partial x}\\ &+ \frac{1}{2}\frac{\big(D(t+dt)-D(t)\big)^2}{dt}(dt)^{2H}\frac{\partial^2 P(x,t)}{\partial^2 x}\\ &+ \lambda\int_0^\infty P(x-y,t)d\Phi(y)-\lambda P(x,t).\end{aligned}$$

Taking the limit from both sides and applying the assumption (3) the result follows. □

Taking Fourier transform of Eq. (4) yields

$$\frac{\partial \hat{P}(\omega,t)}{\partial t} = -i\omega\mu\hat{P}(\omega,t) - D_0 H t^{2H-1}\omega^2\hat{P}(\omega,t) + \lambda\hat{\Phi}(\omega)\hat{P}(\omega,t) - \lambda\hat{P}(\omega,t). \quad (5)$$

This results in the following solution for the Fourier transform of the PDF of the risk process (considering $P(x,0) = \delta(x)$)

$$\hat{P} = \exp\left(\left(-i\omega\mu - D_0 H t^{2H-1}\omega^2 + \lambda\hat{\Phi}(\omega) - \lambda\right)t\right). \quad (6)$$

For the sake of illustration, we take the claims to be normal, so Eq. (6) becomes

$$\begin{aligned}\hat{P} &= \exp\left(\left(-i\omega\mu - \lambda - D_0 H t^{2H-1}\omega^2 + \lambda\left(1 - \frac{\omega^2}{2}\right)\right)t\right)\\ &= \exp\left(\left(-i\omega\mu - \frac{(2D_0 H t^{2H-1} + \lambda)}{2}\omega^2\right)t\right).\end{aligned}$$

Therefore

$$P(x,t) = \frac{\exp\left(-\frac{(\mu t + x)^2}{2t(\lambda + 2D_0 H t^{2H-1})}\right)}{\sqrt{2\pi t(\lambda + 2D_0 H t^{2H-1})}}.$$

In the absence of the jump process, this result is similar to the work of [15] where they used a fractional Langevin approach to calculate the fractional derivative and the density of the corresponding FBM with a drift.

M Perils

Consider the model (1), It can be shown, using characteristic function of sum of S_i, that

$$S = \sum_{i=1}^{M} S_i = \sum_{i=1}^{M}\sum_{j=1}^{N_t^{(i)}} z_j = \sum_{i=1}^{N_t} Z_i,$$

is also a compound Poisson process, where N_t is a Poisson process with the parameter $\lambda = \sum_{i=1}^{M}\lambda_i$, and Z_i are iid distribution function with following density function $\Phi(x) = \sum_{i=1}^{M}\frac{\lambda_i}{\lambda}\Phi_i(x)$.

Based on these facts, the (5) for the following model

$$U_t = \mu t - \sum_{i=1}^{M}\sum_{j=1}^{N_t^{(i)}} z_j + D(t)B_t^H,$$

reads as

$$\begin{aligned}\frac{\partial \hat{P}(\omega,t)}{\partial t} &= -i\omega\mu\hat{P}(\omega,t) - D_0 H t^{2H-1}\omega^2\hat{P}(\omega,t)\\ &+ \Big(\sum_{i=1}^{M}\lambda_i\Big)\sum_{i=1}^{M}\hat{\Phi}_i(\omega)\hat{P}(\omega,t) - \Big(\sum_{i=1}^{M}\lambda_i\Big)\hat{P}(\omega,t).\end{aligned}$$

Notice that, using these model we can cluster, for example, the insurance contracts or regions based on their type and distribution functions.

Similarly the Eq. (6) becomes

$$\hat{P} = \exp\left(\left(-i\omega\mu - D_0 H t^{2H-1}\omega^2 + \Big(\sum_{i=1}^{M}\lambda_i\Big)\sum_{i=1}^{M}\hat{\Phi}_i(\omega) - \Big(\sum_{i=1}^{M}\lambda_i\Big)\right)t\right). \tag{7}$$

Example 2.2. Suppose a insurance company has two different kind of contracts, with different severities. We assume the first category follows a normal distribution and the second is a exponential. The equation (7) reads as

$$\hat{P} = \exp\left(\left(-i\omega\mu - D_0 H t^{2H-1}\omega^2 + \big(\sum_{i=1}^{M}\lambda_i\big)\sum_{i=1}^{M}\big(\exp(iet - \frac{\sigma^2 t^2}{2}) + \frac{\theta}{\theta - it}\big)(\omega) - \big(\sum_{i=1}^{M}\lambda_i\big)\right)t\right),$$

where $(e,\sigma),\theta$ are the parameters of normal and exponential distributions respectively

Remark 2. *Using the properties of fractional Brownian motion (FBM), it is possible to assume that the claims are normally distributed and apply $H > \frac{1}{2}$ for scenarios where severe cases have a high probability of occurring. In the numerical section, we will present a method for calculating the Hurst index in relation to the problem.*

2.1 First Passage Time

The first passage time (FPT) is a concept similar to the ruin index. It represents the time at which the process first crosses a particular level. The probability of first passage time can be calculated using the survival function of the process, which is related to its probability density function (PDF) by

$$S(t) = \int_{-\infty}^{u} P(x,t)dx$$

$$P_{fpt} = -\frac{\partial}{\partial t}S(t)$$

Notice that P_{fpt} is the probability that the risk process passes the level u at time t for the first time, and not before that. Next, we derive results for different parameter choices in the model (2). Since some of the resulting models play specific roles in calculating the ruin probability, we will explain them in more detail rather than simply presenting the final results.

$D_0 = 0$

This case represents a risk process that considers only the pure jump process and the linear growth rate of the premium, without incorporating additional factors. This model illustrates a scenario where the independence between the number and severity of claims is reasonable, such as in life insurance, where the death of one policyholder is independent of the deaths of others. Additionally, there are no significant effects from interest rates or inflation on the insurance company concerning the contracts. The model and the corresponding density function are represented as $U_t = \mu t - \sum_{i=0}^{N_t} Y_i$.

Taking a similar approach as the previous case following equation follows

$$\begin{aligned}\frac{\partial P(x,t)}{\partial t} &= -\mu\frac{\partial P(x,t)}{\partial x} \\ &+ \lambda\int_0^{\infty} P(x-y,t)d\Phi(y) - \lambda P(x,t),\end{aligned}$$

$$\hat{P} = \exp\left(\left(-i\omega\mu + \lambda\hat{\Phi}(\omega) - \lambda\right)t\right) \tag{8}$$

Example 2.3. We consider two cases for the claims, normal and power law distribution, and find a solution for Eq.(8) accordingly. For the normal case we have

$$\hat{\Phi}(\omega) = \exp\left(\frac{-\omega^2}{2}\right),$$

where we used a standard normal distribution to simplify the calculations. Therefore, Eq.(8) reads

$$\hat{P} = \exp\left(\left(-i\omega\mu + \lambda\exp\left(\frac{-\omega^2}{2}\right) - \lambda\right)t\right)$$

Notice, behavior of a function at infinity mimics its Fourier transform's behavior when ω goes to zero. So we can take the following approximation

$$\begin{aligned}\hat{P} &= \exp\left(\left(-i\omega\mu - \lambda + \lambda\left(1 - \frac{\omega^2}{2}\right)\right)t\right) \\ &= \exp\left(\left(-i\omega\mu - \frac{\lambda}{2}\omega^2\right)t\right).\end{aligned}$$

Taking inverse Fourier transform, the following probability density function results

$$P(x,t) = \frac{\exp(-\frac{(\mu t + x)^2}{2t\lambda})}{\sqrt{2\pi t\lambda}},$$

which is, as expected, the distribution of a Brownian process. Taking $\lambda = 1$ and $\mu = 0$, we obtain the standard Brownian motion. Interestingly, we provide a demonstration of the Central Limit Theorem (CLT) in a novel way. Thus, the survival function $S(t)$ is

$$S(t) = \Phi\left(\frac{u + \mu t}{\sqrt{t\lambda}}\right),$$

where Φ denotes the CDF of the standard normal distribution. Therefore,

$$P_{fpt} = -\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\left(\frac{u+\mu t}{\sqrt{t\lambda}}\right)^2}{2}\right)\left(\frac{\mu - \frac{u+\mu t}{2t}}{\sqrt{t\lambda}}\right).$$

Now, suppose the claims follow a distribution with power-law asymptotic behavior

$$\Phi(x) = \frac{1}{|x|^{1+\alpha}},$$

with following Fourier transform (its approximation)

$$\hat{\Phi}(k) = 1 - |\omega|^{\alpha}.$$

Following this choice for claims, the Eq.(8)

$$\begin{aligned} \hat{P} &= \exp\left(\left(-i\omega\mu + \lambda(1 - |\omega|^{\alpha}) - \lambda\right) t\right) \\ &= \exp\left(\left(-i\omega\mu - |\omega|^{\alpha}\right) t\right). \end{aligned}$$

The inverse Fourier of this function results [18]

$$\begin{aligned} P(x,t) &= \frac{1}{(K_\alpha t)^{\frac{1}{\alpha}}} L_\alpha\left(\frac{x-\mu}{(K_\alpha t)^{\frac{1}{\alpha}}}\right) \\ &= \frac{\pi}{\alpha x} H^{1,1}_{2,2}\left[\frac{x-\mu}{[(K_\alpha t)]^{\frac{1}{\alpha}}}\middle|(1,\frac{1}{\alpha}),(1,\frac{1}{2});(1,1),(1,\frac{1}{2})\right], \end{aligned}$$

where K_α is the diffusion parameter related to severity and the number of the claims, and H is Fox H-function (see [8] for more on H-functions).

$D_0 = 0$, Random Premiums

Here, we assume that policyholders have the option to pay the premium either during the contract period or with potential delays or early payments to the insurance company.

$$U_t = \sum_{i=1}^{N_t^1} X_i - \sum_{i=1}^{N_t^2} Y_i,$$

To find this probability, we use the fact that the combination of two compound Poisson processes results in another compound Poisson process [4]. The resulting process has a counting process N_t which is a

Poisson process with parameter $\lambda = \lambda_1 + \lambda_2$, and the reward variable Z has the following distribution:

$$\frac{\lambda_1}{\lambda_1 + \lambda_2} F_1(x) + \frac{\lambda_2}{\lambda_1 + \lambda_2} F_2(x),$$

where F_1, F_2 are distributions of $X, -Y$ respectively. The result is following process for $U_t = \sum_{i=1}^{N(t)} Z_i$.

The rest of the calculations are similar to the previous case except with a zero drift.

$H = \frac{1}{2}$

Consider the following model of an insurance company's capital as a classical risk process perturbed by a Brownian motion

$$U_t = \mu t - \sum_{i=0}^{N(t)} z_i + DB_t,$$

similar calculations result

$$\begin{aligned} \frac{\partial P(x,t)}{\partial t} &= -\mu \frac{\partial P(x,t)}{\partial x} + D \frac{1}{2} \frac{\partial^2 P(x,t)}{\partial^2 x} \\ &+ \lambda \int_0^\infty P(x-y,t) d\Phi(y) - \lambda P(x,t). \end{aligned}$$

$$\hat{P} = \exp\left(\left(-i\omega\mu - \frac{D}{2}\omega^2 + \lambda\hat{\Phi}(\omega) - \lambda\right) t\right) \tag{9}$$

The above function is not straightforward to invert using the Fourier transform, but in some cases, the Taylor series expansion can be used to find the inverse. However, since we already have the Fourier transform of the density function, we can use it to calculate the moments of the distribution. For more complex cases, such as power-law distributions for the claims, it is common to use these moments to approximate the probability density function of the risk process. For demonstration purposes, let us consider the claims to be normally distributed once again

$$\hat{\Phi}(\omega) = \exp\left(\frac{-\omega^2}{2}\right)$$

So the equation (9) reads

$$\hat{P} = \exp\left(\left(-i\omega\mu - \frac{D}{2}\omega^2 + \lambda \exp\left(\frac{-\omega^2}{2}\right) - \lambda\right) t\right)$$

Again, the behavior of a function at infinity mimics its Fourier transform's behavior when ω goes to zero. So we can take the following approximation

$$\begin{aligned} \hat{P} &= \exp\left(\left(-i\omega\mu - \lambda - \frac{D}{2}\omega^2 + \lambda\left(1 - \frac{\omega^2}{2}\right)\right)t\right) \\ &= \exp\left(\left(-i\omega\mu - \frac{(D+\lambda)}{2}\omega^2\right)t\right) \end{aligned}$$

Taking inverse Fourier transform, the following probability density function results

$$P(x,t) = \frac{\exp(-\frac{(\mu t + x)^2}{2t(\lambda + D)})}{\sqrt{2\pi t(\lambda + D)}},$$

which is just the distribution of a modified (shifted) Brownian motion. Similarly we can calculate the survival function and probability of ruin as follows

$$S(t) = \int_{-\infty}^{u} \frac{\exp\left(-\frac{(\mu t + x)^2}{2t(\lambda + D)}\right)}{\sqrt{2\pi t(\lambda + D)}}\, dx$$

Substituting $x = \sqrt{t(\lambda + D)}\, z - \mu t$ and simplifying, we get:

$$S(t) = \Phi\left(\frac{u + \mu t}{\sqrt{t(\lambda + D)}}\right),$$

where Φ denotes the CDF of the standard normal distribution.

The probability of ruin P_{fpt} is given by

$$P_{fpt} = -\frac{\partial}{\partial t} S(t)$$

Differentiating $S(t)$ with respect to t, we have

$$P_{fpt} = \frac{e^{-\frac{(u+\mu t)^2}{2t(D+\lambda)}}(D + \lambda)(-u + \mu t)}{2\sqrt{2\pi}(t(D + \lambda))^{3/2}}$$

3 Coupled Case

In this section we present a model in which the waiting time and severity of the claims are dependent. we couple them through the following relation

$$\Phi(x|t) = \phi(x)\delta(x - at^{\rho}), \tag{10}$$

where $\phi(x)$ is a density function regardless of time and a, ρ are constant parameters. Using this coupling equation, we calculate the Fokker-Planck equation of following model

$$U_t = \mu t - \sum_{i=0}^{N(t)} z_i + DB_t.$$

Similar to the previous models we use conditioning on the occurrence of a claim in a short time dt

$$\begin{aligned} P(x,t+dt) &= (1-\lambda dt)E\left[P(x-\mu dt - D\left[B_{t+dt}-B_t\right],t)\right] \\ &+ \lambda dt E\left[P(x-y,t)\right]. \end{aligned}$$

Assuming that $P(x,t)$ has a Taylor expansion, the following relation holds

$$\begin{aligned} P(x,t+dt) &= (1-\lambda dt)E\Bigg[P(x,t) - \mu dt\frac{\partial P(x,t)}{\partial x} \\ &- D\frac{\partial P(x,t)}{\partial x}[B_{t+dt}-B_t] \\ &+ \frac{1}{2}\frac{\partial^2 P(x,t)}{\partial^2 x}\Big[\mu^2 dt^2 + 2\mu dt D[B_{t+dt}-B_t] \\ &+ D^2[B_{t+dt}-B_t]^2\Big] + \cdots\Bigg] \\ &+ \lambda dt E\left[P(x-y,t)\right]. \end{aligned}$$

Applying the expectation function and the properties of FBM, we get

$$\begin{aligned} \frac{P(x,t+dt)-P(x,t)}{dt} &= -\mu\frac{\partial P(x,t)}{\partial x} \\ &+ \frac{1}{2}D^2\frac{\partial^2 P(x,t)}{\partial^2 x} \\ &+ \lambda\int_0^\infty P(x-y,t)d\Phi(y|t) - \lambda P(x,t). \end{aligned}$$

To solve this equation, we apply Fourier transform with respect to x and substitute the Eq. (10)

$$\frac{\partial \hat{P}(\omega,t)}{\partial t} = -i\omega\mu\hat{P}(\omega,t) - D^2\omega^2\hat{P}(\omega,t) + \lambda\phi(at^\rho)e^{-ikat^\rho}\hat{P}(\omega,t) - \lambda\hat{P}(\omega,t),$$

where we used the fact that

$$\int_{-\infty}^{\infty}\phi(x)\delta(x-at^\rho)e^{-ikx}dx = \phi(at^\rho)e^{-ikat^\rho}.$$

This results in the following solution for the Fourier transform of the PDF of the risk process (considering $P(x,0)=\delta(x)$)

$$\hat{P} = \exp\left(\left(-i\omega\mu - D^2\omega^2 + \lambda\phi(at^\rho)e^{-ikat^\rho} - \lambda\right)t\right).$$

4 Real-World Application

In the models presented, two parameters are crucial: H and α, where H is the Hurst index and α is the power-law parameter. These parameters characterize the type of diffusion induced by the continuous-time random walk (CTRW) of the risk process or the perturbation process in many insurance-linked models. Their values determine the appropriate modeling of sub-diffusion and super-diffusion processes to accurately reflect real-world scenarios. Therefore, the initial task in modeling is to identify whether normal or anomalous diffusion is required for the problem at hand. Subsequently, these parameters will influence the behavior of the resulting density function and, consequently, the ruin probabilities. We use a ranking method to better quantify the diffusion parameters. This approach considers both the quality and quantity of the problem, aiming to develop a model that can predict future situations more accurately.

We illustrate this method through an example in the insurance sector, where a company is considering selling health insurance related to potential epidemic or endemic events. We adjust the model parameters based on non-pharmaceutical interventions implemented by governments to control the virus. Specifically, we use findings from two papers [5, 11], which examine the impact and effectiveness of restrictions aimed at halting or slowing the virus's spread. We use Table 2 from [5] as our primary reference for the measures of restrictions imposed by different countries in response to the COVID-19 pandemic. Additionally, we utilize the rankings presented in [5, 11] to calculate the parameters α and H for each region.

Variables $a_{1,\cdots,20} \in \{0, 1\}$ are defined such that $a_i = 1$ if the government applies the intervention ranked i, and $a_i = 0$ otherwise. Based on the rankings presented in this table, we assign the following values to the Hurst parameter based on the interventions implemented by the government in the region under study:

$$H = 1 - \theta \sum_{i=1}^{20} a_i \omega_i,$$

in which $0 < \theta \leq 1$ is a parameter used to capture the community structure in a population.

ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	ω_9	ω_{10}	ω_{11}	ω_{12}	ω_{13}	ω_{14}	ω_{15}	ω_{16}	ω_{17}	ω_{18}	ω_{19}	ω_{20}
0.178125	0.140625	0.103125	0.078125	0.0625	0.05	0.0375	0.0375	0.0375	0.0375	0.025	0.025	0.025	0.025	0.025	0.025	0.025	0.025	0.025	0.0125

Table 1: The ranked parameters related to the interventions taken by countries.

We considered the Iranian health-care system to value a_i's and after conducting an on-field investigation (which involved a regression analysis of several countries and regions) we also calculated $\theta = 0.2$. Considering these results, we calculated the Hurst index as $H = 0.79$. Assuming the model be 2 and with the assumption that $\mu = 110, \lambda = 3$ for the average number of infected people during the COVID pandemic and the rate of new infected people from already infected ones, we get

$$P(x,t) = \frac{\exp\left(-\frac{(140t+x)^2}{2t(3+2D_0\cdot 0.79\cdot t^{2\cdot 0.79-1})}\right)}{\sqrt{2\pi t\left(3+2D_0\cdot 0.79\cdot t^{2\cdot 0.79-1}\right)}}$$

Therefore

$$S(t) = \Phi\left(\frac{u+140t}{\sqrt{t\left(3+2D_0\cdot 0.79\cdot t^{1.58}\right)}}\right),$$

and

$$P_{fpt} = -\frac{d}{dt}S(t) = -\phi\left(\frac{u+140t}{\sqrt{t\left(3+2D_0\cdot 0.79\cdot t^{1.58}\right)}}\right)\cdot\frac{d}{dt}\left(\frac{u+140t}{\sqrt{t\left(3+2D_0\cdot 0.79\cdot t^{1.58}\right)}}\right)$$

We can also use a model with a power law distribution for the number of cases, or claims in insurance terminology, we need to estimate the parameter α which corresponds to the power-law behavior of the claims' variable. If we consider model (2), where Hurst index is zero, we have

$$P(x,t) = \frac{\pi}{\alpha x}H^{1,1}_{2,2}\left[\frac{x-\mu}{[(K_\alpha t)]^{\frac{1}{\alpha}}}\Big|(1,\frac{1}{\alpha}),(1,\frac{1}{2});(1,1),(1,\frac{1}{2})\right],$$

$$S(t) = F_\alpha\left(\frac{u-\mu}{(K_\alpha t)^{\frac{1}{\alpha}}}\right),$$

$$P_{fpt}(t) = \frac{u-\mu}{\alpha(K_\alpha t)^{\frac{1}{\alpha}}tK_\alpha}\cdot\frac{1}{(K_\alpha t)^{\frac{1}{\alpha}}}L_\alpha\left(\frac{x-\mu}{(K_\alpha t)^{\frac{1}{\alpha}}}\right),$$

Taking $\alpha = 2.5$ we get

$$P_{fpt}(t) = \frac{1}{2.5}\cdot\frac{(K_\alpha t)^{\frac{1}{2.5}}}{u-\mu}\cdot\frac{1}{t}\cdot f_{2.5}\left(\frac{u-\mu}{(K_\alpha t)^{\frac{1}{2.5}}}\right),$$

where

$$f_\alpha(x) = \frac{1}{\pi}\int_0^\infty\frac{\sin(yx)}{y^{1-\alpha}}\exp\left(-\frac{y}{2}\right)\,dy.$$

Remark 3. *By incorporating parameters such as the Hurst index, one can integrate more relevant information into the modeling process, which is crucial for calculating the probability of ruin. While the behavior of the severity variable is often predictable based on previously observed data, there are situations-such as pandemics or earthquakes- in which incorporating the diffusion counterpart is essential for accurate modeling. For anomalous diffusion processes like fractional Brownian motion (FBM), determining the Hurst index can be challenging, especially when the frequency of disasters is insufficient for accurate prediction. In such cases, factors like community preparedness in the event of a disaster become key features of the model. Objectives such as those outlined in Table 1 enhance the model's realism and improve the reliability of the final results. See also these articles for other applications of fractional differential modeling in pandemic and virus diffusion modeling [17, 20, 22].*

5 Conclusion

In this study, we have advanced the modeling of catastrophic risks in insurance by applying fractional partial differential equations (FPDEs), specifically the fractional Fokker-Planck equation, to better capture the complexities of extreme risk events. By incorporating fractional Brownian motion, which allows for both normal and anomalous diffusion through the manipulation of the Hurst index, our approach provides a refined representation of claim processes that traditional models may overlook. The numerical results demonstrate the model's effectiveness in computing ruin probabilities and depict a more accurate coupling of claim severity and frequency. This enhanced model offers a valuable tool for insurance companies to assess and manage catastrophic risks with greater precision, reflecting the subtle variations in risk dynamics that standard approaches might miss. The introduction of fractional FPDEs signifies a substantial advancement in actuarial risk modeling, offering deeper insights and improved risk predictions. Future research can build on these findings by further exploring the model's applicability across various risk scenarios and integrating additional stochastic processes to refine its predictive capabilities.

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