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Original Research Paper

The Investigation of Stability And Optimal Control of The Multigroup Covid-19 Epidemic Model in Iran

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Abstract. The global dynamics of a multigroup COVID-19 model in Iran is presented in this paper. We have stratified the population of each group as susceptible, infective without symptoms, infective with symptoms, and recovered. The dynamic properties of the model, including the reproduction number and the stability of the equilibrium points, have been analyzed. Furthermore, to eliminate Covid-19 in some groups in an area, some suitable control strategies were designed that reduce the number of asymptomatic and symptomatic infected people. Finally, actual limited data of the Isfahan, Fars and Khorasan-Razavi provinces in Iran to was used to investigate the efficiency of the proposed method.

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1 Introduction

Coronaviruses transmit infection to humans and animals and are one of the biggest viruses that have caused epidemiological problems in recent years. For the first time, this complex virus appeared in December 2019 in Wuhan China[31]. The corona virus was transferred through various methods such as sneezing, coughing and talking to an infected person. In this field, several investigations have been performed to illustrate the transmission of the virus through contact with contaminated surfaces [32]. The spread of this virus caused many social, economic and health problems in the world and several million people in the world got infected with this virus or died and researchers were looking for a vaccine for this virus[6, 11].

In mathematical epidemiology, the authors have proposed models for the transmission of this disease, such as the susceptible-infected-recovered model (SIR) and the susceptible-exposed-infected-recovered model (SEIR)[10, 2, 34]. Mathematical modeling of the spread of infectious diseases is a momentous tool for understanding the dynamics of disease spread. It is also very beneficial for policy makers to make timely decisions to reduce and control the disease [26, 9].

Mathematical modeling has been very successful in recommending critical decisions for various diseases, including the flu, corona virus, etc. Also, to check the isolation of a person infected with Covid-19, it is suggested to quarantine the people who have been in contact with them, limit travel and stay at home. Several models have been suggested to reduce the spread of Covid-19 in different regions of the world. Contreras [25] propsed a multi-group SEIRA model. Maier and Brockmann studied the effect of effective control in disease outbreak in China[5]. Crokidakis [20] investigated the spread of COVID-19 in Brazil. The model proposed by Mohsen et al. considers the effects of media coverage [4]. Hataf and Yousefi's model describes interactions within the host [14]. Also, Zine et al. [12], to show the effectiveness of the restrictions of Covid-19 in

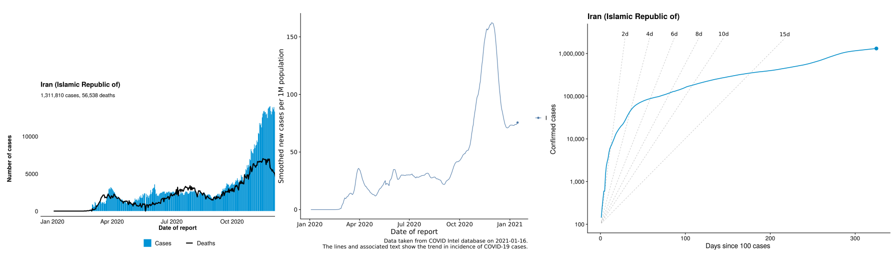
Morocco, a random time delay model has been presented. Khoshnaw et al. [27] suggested that they should pay attention to people's health and quarantine. Serhani and Labbardi [17] stated that staying at home plays an essential role in controlling Covid-19.

The first confirmed cases of Covid-19 in Iran were reported in Qom on February 19, 2020. Then, the Iranian government announced the closure of all schools, universities, Bekaa Motbaraka shopping centers, and markets, and canceled Friday prayers. The government also took economic measures, including helping families and businesses. Heavy traffic between cities continued before Nowruz Eid. The government later announced that despite the ban on travel between cities, the number of infected people increased. The government gradually eased restrictions from April and the number of infected reached its lowest point in May, but with the easing of restrictions, a new wave was reported on June 4 and the number of deaths increased. Figure 1 clearly shows the total cases and deaths of Corona virus, smoothed new cases, and total confirmed cases in Iran, from the actual data. In Iran, from 3 January 2020 to 2:09 pm CEST, 12 October 2023, 7,618,727 confirmed cases of COVID-19 with 146,436 deaths, reported to WHO. On 14 August 2023, a total of 155,445,801 vaccine doses were administered.

Multi-group modeling of infectious diseases is a vital field of applied mathematics, and because the transmission of many infectious diseases such as measles, influenza, AIDS, and hepatitis can be modeled, it has attracted the attention of researchers [3]. For the multigroup model of an infectious disease, the disease-free equilibrium is first checked to see, it is globally asymptotically stable. Therefore, it is necessary to first obtain the basic reproduction number, which means that if the basic reproduction number is less than or equal to one, the disease-free equilibrium is globally asymptotically stable. Otherwise, it is unstable when the reproduction number is greater that one.[13, 15, 24, 25, 18, 7, 8, 28].

In this paper, a new multi-group model of Covid-19 is presented. First, we examine the global stability of the disease-free equilibrium. Then we prove that the asymptotically globally stability of the system is dependent on the basic reproduction number. Furthermore, the existence of optimal controls is proved and their forms are obtained using the Hamiltonian function and maximum principle. Finally, we use the multigroup COVID-19 model to simulate the COVID-19 outbreak in the Isfahan, Fars, and Khorasan-Razavi provinces in Iran. The aim of this research is to study the dynamics of COVID-19 in Iran.

This paper is organized as follows. In Section 1, an introduction is given. In Section 2, we will introduce the model and analyze the stability of the disease-free equilibrium. In Section 3, we analyze the stability of the endemic equilibrium. Section 4 presents the control strategy for the proposed multigroup COVID-19 model. Section 5 provides a simulation of the COVID-19 outbreak in the Isfahan, Fars, and Khorasan-Razavi provinces of Iran. Finally, the conclusion is given in Section 6.



(a) Total Coronavirus cases and deaths in Iran **(b)** Smoothed new cases in Iran **(c)** Total confirmed cases in Iran

Figure 1: Covid-19 in Iran

2 Model Introduction

Let consider the following Covid-19 model [21]:

$$\begin{aligned}
\frac{dS}{dt} &= b - \frac{\alpha}{1+\delta I_u} S I_u - d_1 S, \\
\frac{dI_u}{dt} &= \frac{\alpha}{1+\delta I_u} S I_u - (\beta + d_1) I_u, \\
\frac{dI_k}{dt} &= \beta I_u - (\gamma + d_2) I_k, \\
\frac{dR}{dt} &= \gamma I_k - d_1 R,
\end{aligned} \tag{1}$$

where S, I_u, I_k and R stand for the susceptible, infective without symptom, infective with symptom and the recovered, respectively.

Corresponding to model (1), we make the multi-group model of Covid-19. Each population group is meant to be in a specific geographical location, and the model is such that the total population is distributed among the same 4 subsets mentioned in the model (1). The flow diagram of Multi-group COVID-19 model is given in Figure 2.
Based on model (1), we suggest the following model:

$$
\begin{aligned}
&N_i(t) = S_i(t) + I_{u_i}(t) + I_{k_i}(t) + R_i(t), \quad i = 1, 2, \cdots, N,\\
&\frac{dS_i}{dt} = b_i - \sum_{j=1}^{N} \frac{\alpha_{ij}}{1+\delta_i I_{u_j}} S_i I_{u_j} - d_{1i} S_i + \epsilon_i R_i,\\
&\frac{dI_{u_i}}{dt} = \sum_{j=1}^{N} \frac{\alpha_{ij}}{1+\delta_i I_{u_j}} S_i I_{u_j} - (\beta_i + d_{1i}) I_{u_i},\\
&\frac{dI_{k_i}}{dt} = \beta_i I_{u_i} - (\gamma_i + d_{2i}) I_{k_i},\\
&\frac{dR_i}{dt} = \gamma_i I_{k_i} - d_{1i} R_i - \epsilon_i R_i.
\end{aligned}
\tag{2}
$$

In model (2), We have the following assumptions:

(1) Symptomatic patients are assumed to be those identified and they are being treated in the hospital and have no connection with the society and they are discharged from the hospital after complete recovery and then they enter the improved ones but due to the long-term spread of the disease and due to contact with susceptible people, they can enter the susceptible group again.

(2) It is possible that an asymptomatic patient enters another group unintentionally. For this reason, the second sentence of the first equation is considered as accumulation that in the i-th group of the population in the susceptible section with enough contact with asymptomatic infected people of the j-th group, they enter the population of asymptomatic infected people.

(3)The population in the susceptible section enters the population of asymptomatic infectious individuals with sufficient contact with asymptomatic infectious individuals.

(4) Some of the recovered people are again susceptible to this disease and re-enter the group of susceptible people.

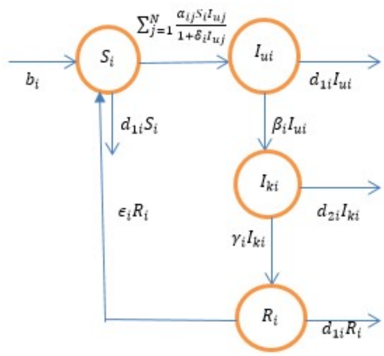


Figure 2: Flow diagram of the multigroup COVID-19 model

All parameters are described in Table 1.

Table 1: Description of the parameters

Parameter	Description
S_i	The susceptible population of i^{th} group
I_{u_i}	asymptomatic infectious population of i^{th} group
I_{k_i}	symptomatic infectious population of i^{th} group
R_i	Recovered population of the i^{th} group
N_i	The total population in i^{th} group
b_i	The rate at which new individuals enter the population of the i^{th} group
α_{ij}	The transmission coefficient from S_i to I_{u_j}
δ_i	The parameter describing the psychological effect of the general public on the infectives.
β_i	The transmission coefficient from an asymptomatic infectious population to an symptomatic infectious population
γ_i	The transmission coefficient from infective population to treatment
d_{1i}	Natural death rate of the i^{th} group
d_{2i}	Death rate due to COVID-19 plus d_{1i} of i^{th} group
ϵ_i	The rate of transmission from the recovered population to the susceptible population

For non-negative and bounded solutions, the following Proposition can be proven.

Proposition 2.1. *The closed set*

$$\Delta = \left\{ (S_1, I_{u_1}, I_{k_1}, R_1, \cdots, S_N, I_{u_N}, I_{k_N}, R_N) \in \mathbb{R}_+^{4N} : 0 \leq N_i \leq \frac{b_i}{d_{1i}} \right\}$$

so that $i = 1, \cdots, N$, *is positively invariant according to model (2).*

proof. Let $(S_i, I_{u_i}, I_{k_i}, R_i)$ be the solution of (2) starting from a point in Δ. Then we have

$$\begin{aligned}\dot{N}_i &= \dot{S}_i + \dot{I}_{u_i} + \dot{I}_{k_i} + \dot{R}_i = b_i - \sum_{j=1}^{N} \frac{\alpha_{ij}}{1+\delta_i I_{u_j}} S_i I_{u_j} - d_{1i} S_i + \epsilon_i R_i \\ &+ \sum_{j=1}^{N} \frac{\alpha_{ij}}{1+\delta_i I_{u_j}} S_i I_{u_j} - (\beta_i + d_{1i}) I_{u_i} + \beta_i I_{u_i} - (\gamma_i + d_{2i}) I_{k_i} \\ &+ \gamma_i I_{k_i} - d_{1i} R_i - \epsilon_i R_i = b_i - (S_i + I_{u_i} + R_i) d_{1i} - d_{2i} I_{k_i}.\end{aligned}$$

Then we have

$$\dot{N}_i + d_{1i} N_i = b_i - (d_{2i} - d_{1i}) I_{k_i}.$$

Therefore

$$\dot{N}_i \leq b_i - d_{1i} N_i.$$

According to the theory of differential equation, we have

$$N_i(t) \leq e^{-d_{1i} t} N_i(0) + \frac{b_i}{d_{1i}} (1 - e^{-d_{1i} t}),$$

and for $t \to \infty$, we have

$$\overline{\lim_{t \to \infty}} N_i(t) \leq \frac{b_i}{d_{1i}}.$$

The second equation of system (2) can be written as:

$$\frac{dS_i}{dt} = b_i + \epsilon_i R_i - \left(\sum_{j=1}^{N} \frac{\alpha_{ij} I_{u_j}}{1+\delta_i I_{u_j}} + d_{1i}\right) S_i$$

Therefore

$$\frac{dS_i}{dt} \geq -(\sum_{j=1}^{N} \frac{\alpha_{ij} I_{u_j}}{1+\delta_i I_{u_j}} + d_{1i}) S_i(t)$$

$$\frac{dS_i}{dt} + (\sum_{j=1}^{N} \frac{\alpha_{ij} I_{u_j}}{1+\delta_i I_{u_j}} + d_{1i}) S_i(t) \geq 0$$

Then multiplying the equation by $\exp(\int_0^t (d_{1i} + \sum_{j=1}^{N} \frac{\alpha_{ij} I_{u_j}(S)}{1+\delta_i I_{u_j}(S)}) dS)$, we have

$$\frac{d}{dt}[S_i(t) \exp(d_{1i} t + \int_0^t \sum_{j=1}^{N} \frac{\alpha_{ij} I_{u_j}(S)}{1+\delta_i I_{u_j}(S)} dS)] \geq 0.$$

Then by integration we have

$$S_i(t) \exp(d_{1i} t + \int_0^t \sum_{j=1}^{N} \frac{\alpha_{ij} I_{u_j}(S)}{1+\delta_i I_{u_j}(S)} dS) - S_i(0) \geq 0.$$

Then

$$S_i(t) \geq S_i(0) \exp[-(d_{1i} t + \int_0^t \sum_{j=1}^{N} \frac{\alpha_{ij} I_{u_j}(S)}{1+\delta_i I_{u_j}(S)} dS)].$$

This implies that if $S_i(0) \geq 0$, then $S_i(t) \geq 0$, $\forall t \geq 0$.
For other variables, it is simply proved that they are nonnegative with nonnegative initial conditions. Therefore, this completes the Proof.

2.1 Stability analysis of disease-free equilibrium of the model

The system of (2) has a disease-free equilibrium $\mathcal{P}_0$

$$\begin{aligned} \mathcal{P}_0 &= \left(S_1^0, I_{k_1}^0, I_{u_1}^0, R_1^0, \cdots, S_N^0, I_{k_N}^0, I_{u_N}^0, R_N^0\right) \\ &= \left(\frac{b_1}{d_{1_1}}, 0, 0, 0, \cdots, \frac{b_N}{d_{1_N}}, 0, 0, 0\right). \end{aligned}$$

For model (2), first we consider the system components that are directly related to contamination, that is, I_{u_i} and I_{k_i}. Let $\mathcal{F}_i$ and $\mathcal{V}_i$ represent the rate of infection and the number of people entering the study area, respectively, in the i^{th} compartment. Then

$$\begin{bmatrix} \frac{dI_{u_i}}{dt} \\ \\ \frac{dI_{k_i}}{dt} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{N} \frac{\alpha_{ij} S_i I_{u_j}}{1+\delta_i I_{u_j}} \\ \\ 0 \end{bmatrix} - \begin{bmatrix} (\beta_i + d_{1i}) I_{u_i} \\ \\ -\beta_i I_{u_i} + (\gamma_i + d_{2i}) I_{k_i} \end{bmatrix} = \mathcal{F}_i - \mathcal{V}_i.$$

According to [29]

$$F = [\frac{\partial \mathcal{F}_i(\mathcal{P}_0)}{\partial x_j}], \quad V = [\frac{\partial \mathcal{V}_i(\mathcal{P}_0)}{\partial x_j}], \quad x_j = (I_{u_j}, I_{k_j}), \quad j = 1, 2, \cdots, n.$$

As pointed out in [30], $F = (f_{ij})_{N\times N}$ and $V = diag\{v_{11}, \cdots, v_{NN}\}$, where

$$f_{ij} = \begin{bmatrix} \alpha_{ij} S_i^0 & 0 \\ \\ 0 & 0 \end{bmatrix},$$

$$v_{ii} = \begin{bmatrix} \beta_i + d_{1i} & 0 \\ \\ -\beta_i & \gamma_i + d_{2i} \end{bmatrix}.$$

Therefore $V^{-1} = diag\left\{v_{11}^{-1}, \cdots, v_{NN}^{-1}\right\}$, where

$$v_{ii}^{-1} = \begin{bmatrix} \frac{1}{\beta_i + d_{1i}} & 0 \\ \\ \frac{\beta_i}{(\beta_i + d_{1i})(\gamma_i + d_{2i})} & \frac{1}{\gamma_i + d_{2i}} \end{bmatrix}.$$

Hence, $FV^{-1} = (f_{ij} v_{ii}^{-1})_{N*N}$, where,

$$f_{ij} v_{ii}^{-1} = \begin{bmatrix} \frac{\alpha_{ij} S_i^0}{\beta_i + d_{1i}} & 0 \\ \\ 0 & 0 \end{bmatrix}.$$

Definition 2.2. Let $R_0 = \rho(FV^{-1})$ is the spectral radius of FV^{-1} matrix. In the epidemic literature it is referred by basic reproduction number.[22]

Theorem 2.3. *For the system (2), define the function* $f_{ij}(S_i, I_{u_j}) = \frac{S_i I_{u_j}}{1+\delta_i I_{u_j}}$ *with the basic assumptions. Let consider the following [23]: (H1)* $0 < \lim_{I_{u_j}\to 0^+} \frac{f_{ij}(S_i,I_{u_j})}{I_{u_j}} = C_{ij} \leq +\infty$, $0 < S_i \leq S_i^0$ *;*
(H2) $f_{ij}(S_i, I_{u_j}) \leq C_{ij}(Si)I_{u_j}$ *for all* $I_{u_j} > 0$ *and* $0 < S_i \leq S_i^0$ *;*
(H3) $C_{ij}(S_i) < C_{ij}(S_i^0)$, *for all* $0 < S_i \leq S_i^0$.
Then, the disease-free equilibrium $\mathcal{P}_0$ *is globally asymptotically stable if* $R_0 \leq 1$ *, and the disease-free equilibrium* $\mathcal{P}_0$ *is unstable if* $R_0 > 1$.

proof. Since

$$V^{-1}F = V^{-1}FV^{-1}V,$$

Then $V^{-1}F$ is similar to FV^{-1} and the similar matrices have the same eigenvalues. Therefore,

$$\rho(V^{-1}F) = \rho(FV^{-1}) = R_0.$$

On the other hand $V^{-1}F$ is a nonnegative matrix, because we have

$$v_{ii}^{-1} f_{ij} = \begin{bmatrix} \frac{\alpha_{ij} S_i^0}{\beta_i + d_{1i}} & 0 \\ \\ \frac{\beta_i \alpha_{ij} S_i^0}{(\beta_i+d_{1i})(\gamma_i+d_{2i})} & 0 \end{bmatrix}.$$

It is easy to see that all its terms are nonnegative and by Perron–Frobenius Theorem [1], $V^{-1}F$ has a corresponding left eigenvector x such that is positive. Therefore, we have

$$x(V^{-1}F) = \rho(FV^{-1})x.$$

Let $\S = (v_1, v_2, \cdots, v_n, w_1, w_2, \cdots, w_n)$; such that $v_k, w_k > 0, k = 1, \cdots, n$. Then

$$(v_1, v_2, \cdots, v_n, w_1, w_2, \cdots, w_n)V^{-1}F = R_0(v_1, v_2, \cdots, v_n, w_1, w_2, \cdots, w_n).$$

Let $\mathcal{W} = \sum_{i=1}^{N} k_i I_{u_i} + \sum_{i=1}^{N} l_i I_{k_i}$ be a candidate Lyapunov function, where

$$(k_1, l_1, k_2, l_2, \cdots, k_n, l_n) = (v_1, v_2, \cdots, v_n, w_1, w_2, \cdots, w_n)V^{-1}.$$

If the derivative of W are calculated along the solution of (2), we obtain

$$\begin{aligned}
\dot{\mathcal{W}} &= \sum_{i=1}^{N}[k_i \frac{dI_{u_i}}{dt} + l_i \frac{dI_{k_i}}{dt}] \\
&= \sum_{i=1}^{N} k_i[\sum_{j=1}^{N} \frac{\alpha_{ij} S_i I_{u_j}}{1+\delta_i I_{u_j}} - \beta_i I_{u_i} - d_{1i} I_{u_i}] + \sum_{i=1}^{N} l_i[\beta_i I_{u_i} - \gamma_i I_{k_i} - d_{2i} I_{k_i}] \\
&\leq \sum_{i=1}^{N} k_i[\sum_{j=1}^{N} \frac{\alpha_{ij} S_i^0 I_{u_j}}{1+\delta_i I_{u_j}} - \beta_i I_{u_i} - d_{1i} I_{u_i}] + \sum_{i=1}^{N} l_i[\beta_i I_{u_i} - \gamma_i I_{k_i} - d_{2i} I_{k_i}] \\
&\leq \sum_{i=1}^{N} k_i[\sum_{j=1}^{N} (\alpha_{ij} S_i^0 I_{u_j}) - \beta_i I_{u_i} - d_{1i} I_{u_i}] + \sum_{i=1}^{N} l_i[\beta_i I_{u_i} - \gamma_i I_{k_i} - d_{2i} I_{k_i}]
\end{aligned}$$

$$\begin{aligned}
&= (k_1, k_2, \cdots, k_n)[\sum_{j=1}^{N} \frac{\alpha_{ij} b_i I_{u_j}}{d_{1i}} - (\beta_i + d_{1i}) I_{u_i}] \\
&+(l_1, l_2, \cdots, l_n)[\beta_i I_{u_i} - (\gamma_i + d_{2i}) I_{k_i}] \\
&= (k_1, l_1, k_2, l_2, \cdots k_n, l_n)(F - V)X \\
&= (v_1, v_2, \cdots, v_n, w_1, w_2, \cdots, w_n)V^{-1}(F - V)X \\
&= (v_1, v_2, \cdots, v_n, w_1, w_2, \cdots, w_n)(V^{-1}F - I)X \\
&= (v_1, v_2, \cdots, v_n, w_1, w_2, \cdots, w_n)(R_0 - 1)X,
\end{aligned}$$

where $X = (I_{u_1}, I_{k_1}, I_{u_2}, I_{k_2}, \cdots, I_{u_n}, I_{k_n})$. So we derive the following results:
(i) When the basic reproduction number is less than unity, we obtain $\dot{\mathcal{W}} \leq 0$.
(ii) When $R_0 < 1$ then $\dot{\mathcal{W}} = 0$ if and only if $X = 0$.
(iii) When $R_0 = 1$ then $\dot{\mathcal{W}} = 0$ implies that $I_{u_i} = 0$, $I_{k_i} = 0$ or $S_i = S_i^0$, $i = 1, 2, \cdots, n$. It can be verified that the only compact invariant subset of the set where $\dot{\mathcal{W}} = 0$ is the singleton $\mathcal{P}_0$.

Therefore, LaSalle's principle guarantees that $\mathcal{P}_0$ is globally asymptotically stable.
If $R_0 > 1$ and $X \neq 0$ then $\dot{\mathcal{W}} > 0$ in a neighborhood of $\mathcal{P}_0$. Hence, the disease-free equilibrium $\mathcal{P}_0$ is unstable when $R_0 > 1$. The proof is complete.

3 Global Stability of Endemic Equilibrium

The $\mathcal{P}^* = (S_1^*, I_{u_1}^*, I_{k_1}^*, R_1^*, \cdots, S_n^*, I_{u_n}^*, I_{k_n}^*, R_n^*)$ in the interior of Δ is the endemic equilibrium, where

$$\begin{aligned}
&b_i - \sum_{j=1}^{N} \frac{\alpha_{ij}}{1+\delta_i I_{u_j}^*} S_i^* I_{u_j}^* - d_{1i} S_i^* + \epsilon_i R_i^* = 0,\\
&\sum_{j=1}^{N} \frac{\alpha_{ij}}{1+\delta_i I_{u_j}^*} S_i^* I_{u_j}^* - (\beta_i + d_{1i}) I_{u_i}^* = 0,\\
&\beta_i I_{u_i}^* - (\gamma_i + d_{2i}) I_{k_i}^* = 0,\\
&\gamma_i I_{k_i}^* - d_{1i} R_i^* - \epsilon_i R_i^* = 0, \quad i = 1, 2, \cdots, N.
\end{aligned} \tag{3}$$

Let $I_{k_i}^* > 0$ Therefore

$$\begin{aligned}
R_i^* &= \frac{\gamma_i}{(d_{i1} - \epsilon_i)} I_{k_i}^*\\
I_{u_i}^* &= \frac{(\gamma_i + d_{2i})}{\beta_i} I_{k_i}^*\\
S_i^* &= \frac{b_i}{d_{1i}} - [\frac{(\beta_i + d_{1i})(\gamma_i + d_{2i})}{d_{1i}\beta_i} - \frac{\epsilon_i \gamma_i}{(d_{1i} - \epsilon_i) d_{1i}}] I_{k_i}^* \quad i = 1, 2, \cdots, N.
\end{aligned}$$

And if $I_{k_i}^* = 0$ then $\mathcal{P}^* = (\frac{b_i}{d_{1i}}, 0, 0, 0)$; $i = 1, 2, \cdots, N$. To establish the global stability of the endemic equilibrium, the following theorems are proven.

Theorem 3.1. *[19] Let us assume that:*
(1) There exist functions $V_i(t, u_i)$, $F_{ij}(t, u_i, u_j)$, and constants $a_{ij} \geq 0$

such that

$$\dot{V}_i(t,u_i) \leq \sum_{j=1}^{n} a_{ij} F_{ij}(t,u_i,u_j), \quad t>0, \quad u_i \in D_i, \quad i=1,2,\cdots,n.$$

(2) $F_{ij}(t,u_i,u_j) \leq G_i(t,u_i) - G_j(t,u_j), 1 \leq i,j \leq n$, *if there exist functions* $G_i(t,u_i), i=1,2,\cdots,n$.
(3) Assume $n \geq 2$. *Then* $c_i = \sum_{T\in T_i} W(T)$, $i=1,2,\cdots,n$, *where* T_i *is the set of all spanning trees* T *of* $(\mathcal{G},A)$ *that are rooted at the vertex* i, *and* $W(T)$ *is the weight of* T. *In particular, if* $(\mathcal{G},A)$ *is strongly connected, then* $c_i > 0$ *for* $1 \leq i \geq n$.

Then $V(t,u) = \sum_{i=1}^{n} c_i V_i(t,u_i)$ *is a Lyapunov function for* $\dot{u}_i = f_i(t_i,u_i) + \sum_{j=1}^{N} g_{ij}(t,u_i,u_j)$, $i=1,2,\cdots,N$, *that is,* $\dot{V}(t,u) \leq 0$ *for* $t>0$ *and* $u \in D$.

Theorem 3.2. *If* $R_0 > 1$, *then the endemic equilibrium* $\mathcal{P}^*$ *is unique and globally asymptotically stable in the interior of* Δ.

proof. Let

$$f_{ij}(S_i, I_{u_j}) = \frac{S_i I u_j}{1+\delta_i I_{u_j}}.$$

Then, for $S_i \neq S_i^*$,

$$\begin{aligned} &(S_i - S_i^*)[f_{ii}(S_i, I_{u_i}^*) - f_{ii}(S_i^*, I_{u_i}^*)] \\ &= (S_i - S_i^*)[\frac{S_i I_{u_i}^*}{1+\delta_i I_{u_i}^*} - \frac{S_i^* I_{u_i}^*}{1+\delta_i I_{u_i}^*}] \\ &= (S_i - S_i^*)^2 \frac{I_{u_i}^*}{1+\delta_i I_{u_i}^*} > 0. \end{aligned} \tag{4}$$

And

$$\begin{aligned}
&[f_{ii}(S_i^*, I_{u_i}^*)f_{ij}(S_i, I_{u_j}) - f_{ij}(S_i^*, I_{u_j}^*)f_{ii}(S_i, I_{u_i}^*)] \\
&\times[\frac{f_{ii}(S_i^*, I_{u_i}^*)f_{ij}(S_i, I_{u_j})}{I_{u_j}} - \frac{f_{ij}(S_i^*, I_{u_j}^*)f_{ii}(S_i, I_{u_i}^*)}{I_{u_j}^*}] \\
&= [\frac{S_i^* I_{u_i}^* S_i I_{u_j}}{(1+\delta_i I_{u_i}^*)(1+\delta_i I_{u_j})} - \frac{S_i^* I_{u_j}^* S_i I_{u_i}^*}{(1+\delta_i I_{u_j}^*)(1+\delta_i I_{u_i}^*}] \\
&\times[\frac{S_i^* I_{u_i}^* S_i I_{u_j}}{(1+\delta_i I_{u_i}^*)(1+\delta_i I_{u_j})I_{u_j}} - \frac{S_i^* I_{u_j}^* S_i I_{u_i}}{(1+\delta_i I_{u_j}^*)(1+\delta_i I_{u_i})I_{u_j}^*}] \\
&= \frac{-\delta S_i^{*2} I_{u_i}^{*2} S_i^2 (I_{u_j} - I_{u_j}^*)^2}{(1+\delta_i I_{u_i}^*)^2(1+\delta_i I_{u_j})^2(1+\delta_i I_{u_j}^*)^2} \le 0.
\end{aligned} \tag{5}$$

Then let

$$\begin{aligned}
V_k &= \int_{S_k^*}^{S_k} \frac{f_{kk}(\zeta, I_{u_k}^*) - f_{kk}(S_k^*, I_{u_k}^*)}{f_{kk}(\zeta, I_{u_k}^*)} d\zeta + (I_{u_k} - I_{u_k}^* \ln I_{u_k}) \\
&+ (I_{k_k} - I_{k_k}^* \ln I_{k_k})(\frac{\beta_k + d_{1k}}{\beta_k}) \\
&= \int_{S_k^*}^{S_k} (1 - \frac{S_k^*}{\zeta}) d\zeta + (I_{u_k} - I_{u_k}^* \ln I_{u_k}) \\
&+ (I_{k_k} - I_{k_k}^* \ln I_{k_k})(\frac{\beta_k + d_{1k}}{\beta_k}).
\end{aligned}$$

Then by using the equilibrium equations (3), one obtains

$$\begin{aligned}
\dot{V}_k &= (1 - \frac{S_k^*}{S_k})[b_k - d_{1k} S_k - \sum_{j=1}^{N} \alpha_{kj} \frac{S_k I_{u_j}}{(1+\delta_k I_{u_j})} + \epsilon_i R_i] \\
&+ (1 - \frac{I_{u_k}^*}{I_{u_k}})[\sum_{j=1}^{N} \alpha_{kj} \frac{S_k I_{u_j}}{(1+\delta_k I_{u_j})} - (\beta_k + d_{1k}) I_{u_k}] \\
&+ (1 - \frac{I_{kk}^*}{I_{kk}})(\frac{\beta_k + d_{1k}}{\beta_k})[\beta_k I_{u_k} - (\gamma_k + d_{2k}) I_{k_k}]
\end{aligned}$$

$$
\begin{aligned}
&= (1 - \frac{S_k^*}{S_k})[\sum_{j=1}^{N} \alpha_{kj} \frac{S_k^* I_{u_j}^*}{(1+\delta_k I_{u_j}^*)} + d_{1k} S_k^* - \epsilon_k R_k^* \\
&+\epsilon_k R_k - d_{1k} S_k - \sum_{j=1}^{N} \alpha_{kj} \frac{S_k I_{u_j}}{(1+\delta_k I_{u_j})}] \\
&+(1 - \frac{I_{u_k}^*}{I_{u_k}})[\sum_{j=1}^{N} \alpha_{kj} \frac{S_k I_{u_j}}{(1+\delta_k I_{u_j})} - \sum_{j=1}^{N} \alpha_{kj} \frac{S_k^* I_{u_j}^*}{(1+\delta_k I_{u_j}^*)} \frac{I_{u_k}}{I_{u_k}^*}] \\
&+(1 - \frac{I_{k_k}^*}{I_{k_k}})(\frac{\sum_{j=1}^{N} \alpha_{kj} \frac{S_k^* I_{u_j}^*}{(1+\delta_k I_{u_j}^*)}}{I_{u_k}^* \beta_k})[\beta_k I_{u_k} - \beta_k I_{u_k}^* \frac{I_{k_k}}{I_{k_k}^*}] \\
&= -\frac{d_{1k}}{S_k}(S_k - S_k^*)^2 + \epsilon_k (R_k - R_k^*)(S_k - S_k^*) \\
&+ \sum_{j=1}^{N} \alpha_{kj} \frac{S_k^* I_{u_j}^*}{(1+\delta_k I_{u_j}^*)}[3 - \frac{S_k^*}{S_k} - \frac{I_{k_k}}{I_{k_k}^*} + \frac{S_k^*}{S_k} \frac{S_k I_{u_j}}{(1+\delta_k I_{u_j})} \frac{(1+\delta_k I_{u_j}^*)}{S_k^* I_{u_j}^*} \\
&- \frac{I_{u_k}^*}{I_{u_k}} \frac{S_k I_{u_j}}{(1+\delta_k I_{u_j})} \frac{(1+\delta_k I_{u_j}^*)}{S_k^* I_{u_j}^*} - \frac{I_{u_k} I_{k_k}^*}{I_{u_k}^* I_{k_k}}].
\end{aligned}
$$

Let $a_{kj} = \alpha_{kj} \frac{S_k^* I_{u_j}^*}{(1+\delta_k I_{u_j}^*)}$, and

$$
\begin{aligned}
F_{kj}(S_k, I_{u_k}, I_{u_j}, I_{k_k}) = 3 - \frac{S_k^*}{S_k} - \frac{I_{k_k}}{I_{k_k}^*} + \frac{S_k^*}{S_k} \frac{S_k I_{u_j}}{(1+\delta_k I_{u_j})} \frac{(1+\delta_k I_{u_j}^*)}{S_k^* I_{u_j}^*} \\
- \frac{I_{u_k}^*}{I_{u_k}} \frac{S_k I_{u_j}}{(1+\delta_k I_{u_j})} \frac{(1+\delta_k I_{u_j}^*)}{S_k^* I_{u_j}^*} - \frac{I_{u_k} I_{k_k}^*}{I_{u_k}^* I_{k_k}}
\end{aligned}
$$

Since recovered people can be susceptible to the disease again, so the susceptible people increase and the recovered ones decrease, and we have: $(R_k - R_k^*)(S_k - S_k^*) \leq 0$ and then equation (4) indicates

$$
\dot{V}_k \leq \sum_{j=1}^{N} a_{kj} F_{kj}(S_k, I_{u_k}, I_{u_j}, I_{k_k}).
$$

Let $\phi(c) = 1 - c + \ln c$, therefore $\phi(c) \leq 0$ for any $c > 0$ and the equality holds only when $c = 1$.

Furthermore, the equation (5) gives the result

$$
\begin{aligned}
&F_{kj} = \phi(\frac{S_k^*}{S_k}) - \ln(\frac{S_k^*}{S_k})\\
&+\phi(\frac{I_{u_k}^*}{I_{u_k}}\frac{S_k I_{u_j}}{(1+\delta_k I_{u_j})}\frac{(1+\delta_k I_{u_j}^*)}{S_k^* I_{u_j}^*}) - \ln(\frac{I_{u_k}^*}{I_{u_k}}\frac{S_k I_{u_j}}{(1+\delta_k I_{u_j})}\frac{(1+\delta_k I_{u_j}^*)}{S_k^* I_{u_j}^*})\\
&+\phi(\frac{I_{u_k} I_{k_k}^*}{I_{u_k}^* I_{k_k}}) - \ln(\frac{I_{u_k} I_{k_k}^*}{I_{u_k}^* I_{k_k}}) - \frac{I_{k_k}}{I_{k_k}^*} + 1 + \frac{I_{u_j}}{I_{u_j}^*}\\
&-\frac{I_{u_j}}{I_{u_j}^*}\frac{\frac{S_k I_{u_k}^*}{(1+\delta_k I_{u_k}^*)}}{\frac{S_k I_{u_j}}{(1+\delta_k I_{u_j})}}\frac{\frac{S_k^* I_{u_j}^*}{(1+\delta_k I_{u_j}^*)}}{\frac{S_k^* I_{u_k}^*}{(1+\delta_k I_{u_k}^*)}}\\
&+(\frac{\frac{S_k^* I_{u_k}^*}{(1+\delta_k I_{u_k}^*)}}{\frac{S_k^* I_{u_j}^*}{(1+\delta_k I_{u_j}^*)}}\frac{\frac{S_k I_{u_j}}{(1+\delta_k I_{u_j})}}{\frac{S_k I_{u_k}^*}{(1+\delta_k I_{u_k}^*)}} - 1)(1 - \frac{I_{u_j}}{I_{u_j}^*}\frac{\frac{S_k I_{u_k}^*}{(1+\delta_k I_{u_k}^*)}}{\frac{S_k I_{u_j}}{(1+\delta_k I_{u_j})}}\frac{\frac{S_k^* I_{u_j}^*}{(1+\delta_k I_{u_j}^*)}}{\frac{S_k^* I_{u_k}^*}{(1+\delta_k I_{u_k}^*)}})\\
&\le -\ln(\frac{S_k^*}{S_k}) - \ln(\frac{I_{u_k}^*}{I_{u_k}}\frac{S_k I_{u_j}}{(1+\delta_k I_{u_j})}\frac{(1+\delta_k I_{u_j}^*)}{S_k^* I_{u_j}^*}) - \ln(\frac{I_{u_k} I_{k_k}^*}{I_{u_k}^* I_{k_k}})\\
&-\frac{I_{k_k}}{I_{k_k}^*} + \frac{I_{u_j}}{I_{u_j}^*} + \phi(\frac{I_{u_j}}{I_{u_j}^*}\frac{\frac{S_k I_{u_k}^*}{(1+\delta_k I_{u_k}^*)}}{\frac{S_k I_{u_j}}{(1+\delta_k I_{u_j})}}\frac{\frac{S_k^* I_{u_j}^*}{(1+\delta_k I_{u_j}^*)}}{\frac{S_k^* I_{u_k}^*}{(1+\delta_k I_{u_k}^*)}})\\
&-\ln(\frac{I_{u_j}}{I_{u_j}^*}\frac{\frac{S_k I_{u_k}^*}{(1+\delta_k I_{u_k}^*)}}{\frac{S_k I_{u_j}}{(1+\delta_k I_{u_j})}}\frac{\frac{S_k^* I_{u_j}^*}{(1+\delta_k I_{u_j}^*)}}{\frac{S_k^* I_{u_k}^*}{(1+\delta_k I_{u_k}^*)}})\\
&+(\frac{\frac{S_k^* I_{u_k}^*}{(1+\delta_k I_{u_k}^*)}}{\frac{S_k^* I_{u_j}^*}{(1+\delta_k I_{u_j}^*)}}\frac{\frac{S_k I_{u_j}}{(1+\delta_k I_{u_j})}}{\frac{S_k I_{u_k}^*}{(1+\delta_k I_{u_k}^*)}} - 1)(1 - \frac{I_{u_j}}{I_{u_j}^*}\frac{\frac{S_k I_{u_k}^*}{(1+\delta_k I_{u_k}^*)}}{\frac{S_k I_{u_j}}{(1+\delta_k I_{u_j})}}\frac{\frac{S_k^* I_{u_j}^*}{(1+\delta_k I_{u_j}^*)}}{\frac{S_k^* I_{u_k}^*}{(1+\delta_k I_{u_k}^*)}})\\
&\le -\ln(\frac{I_{u_k}^* I_{u_j}}{I_{u_j} I_{u_j}^*}) - \ln(\frac{I_{u_k} I_{k_k}^*}{I_{u_k}^* I_{k_k}}) - \frac{I_{k_k}}{I_{k_k}^*} + \frac{I_{u_j}}{I_{u_j}^*}\\
&= -\ln(\frac{I_{u_j} I_{k_k}^*}{I_{u_j}^* I_{k_k}}) - \frac{I_{k_k}}{I_{k_k}^*} + \frac{I_{u_j}}{I_{u_j}^*}
\end{aligned}
$$

$$
\begin{aligned}
&= (-\frac{I_{k_k}}{I^*_{k_k}} + \ln\frac{I_{k_k}}{I^*_{k_k}}) - (-\frac{I_{u_j}}{I^*_{u_j}} + \ln\frac{I_{u_j}}{I^*_{u_j}}) \\
&= G_k(I_{k_k}) - G_j(I_{u_j}).
\end{aligned}
$$

Taking $G_k(I_{k_k}) = (-\frac{I_{k_k}}{I^*_{k_k}} + \ln\frac{I_{k_k}}{I^*_{k_k}})$, therefore V_k, F_{kj}, G_k, a_{kj} satisfies the assumptions of Theorem 3.1.
So, the function $V = \sum_{k=1}^{n} c_k V_k$ as defined in Theorem 3.1 is a Lyapunov function for system (2), namely, $\dot{V} \leq 0$ for all $(S_1, I_{u_1}, I_{k_1}, R_1, ..., S_n, I_{u_n}, I_{k_n}, R_n) \in \Delta$. One can only show that the largest invariant subset where $\dot{V} = 0$ is the singleton $\mathcal{P}^*$. By LaSalle's invariance principle, $\mathcal{P}^*$ is globally asymptotically stable in Δ.

4 Optimal Control

In this section, we calculate an optimal control model for the multigroup Covid-19 epidemic model.

If $R_0 > 1$, then the disease-free equilibrium $\mathcal{P}_0$ is unstable. In this case, we want to design a suitable control that is stable P_0 and its goal is to minimize the total number of infectious people symptomatic and asymptomatic.
There was no vaccine and treatment anywhere in the world until the end of 2020. During this time, scientists were researching for the correctness three control strategies to reduce the spread of Covid-19. First, washing hands regularly with soap, using face masks and stay away from infected people. Second, in case of contact with an infected person, go to quarantine. Third, inform infected people to go to hospital or isolation.

In the optimal control we tend to minimize the total number of asymptomatic and symptomatic infectious people. To do this, three types of control in the controlled system (6) are introduced. The first control w_i is to using face masks and washing hands with soap and stay away from infected people. The second control θ_i is to encourage them to join the quarantine if they come into contact with an infected person.

The third control v_i is to notify infected people to stay at home or in the hospital. In this case, the model of system (2) with control is as follows:

$$\begin{aligned}
\frac{dS_i}{dt} &= b_i - (1-w_i)\sum_{j=1}^{N}\frac{\alpha_{ij}}{1+\delta_i I_{u_j}}S_i I_{u_j} - d_{1i}S_i + \epsilon_i R_i,\\
\frac{dI_{u_i}}{dt} &= (1-w_i)\sum_{j=1}^{N}\frac{\alpha_{ij}}{1+\delta_i I_{u_j}}S_i I_{u_j} - (\beta_i + d_{1i})I_{u_i} - \theta_i I_{u_i},\\
\frac{dI_{k_i}}{dt} &= \beta_i I_{u_i} - (\gamma_i + d_{2i})I_{k_i} - v_i I_{k_i} + (1-t_i)\theta_i I_{u_i},\\
\frac{dR_i}{dt} &= \gamma_i I_{k_i} - d_{1i}R_i + t_i\theta_i I_{u_i} + v_i I_{k_i} - \epsilon_i R_i.
\end{aligned} \tag{6}$$

where $S_i(0) \geq 0, I_{u_i}(0) \geq 0, I_{k_i}(0) \geq 0$ and $R_i(0) \geq 0$ for all $i = 1, 2, \cdots, n$ are given initial values of variables. The objective function is as follows:

$$J(w_i, \theta_i, v_i) = \int_0^T (I_{u_i} + I_{k_i} + \frac{A_i^1}{2}w_i^2 + \frac{A_i^2}{2}\theta_i^2 + \frac{A_i^3}{2}v_i^2)dt, \tag{7}$$

where T is the given final time, $A_i^1 \geq 0, A_i^2 \geq 0, A_i^3 \geq 0$, are the cost coefficients at time t associated with applied controls, and $i = 1, \cdots, n$. Therefore, we want to obtain the controls of w_i^*, θ_i^*, v_i^* such that

$$J(w_i^*, \theta_i^*, v_i^*) = \min J(w_i, \theta_i, v_i), \quad (w_i, \theta_i, v_i) \in C_{ad},$$

where

$$C_{ad} = \{(w_i, \theta_i, v_i) : \quad 0 \leq w_i \leq 1, \quad 0 \leq \theta_i \leq 1, \quad 0 \leq v_i \leq 1; \quad t \in [0, T]\}.$$

Theorem 4.1. *There exists a triple $(w_i^*, \theta_i^*, v_i^*)$ of optimal controls for (6) with fixed initial conditions, such that*

$$J(w_i^*, \theta_i^*, v_i^*) = \min J(w_i, \theta_i, v_i),$$

if the following conditions hold:
(1) The set of controls and the corresponding state functions are non-empty.
(2) The control set C_{ad} is compact.
(3) The state system (6) is linear in terms of control variables with

coefficients depending on time and states.
(4) The integrand $L(S_i, I_{u_i}, I_{k_i}, R_i) = I_{u_i} + I_{k_i} + \frac{A_i^1}{2}w_i^2 + \frac{A_i^2}{2}\theta_i^2 + \frac{A_i^3}{2}v_i^2$ *of the objective functional is convex on* C_{ad} *and there exist constants* c_1 *and* c_2 *such that*

$$L(S_i, I_{u_i}, I_{k_i}, R_i) \geq -3c_1 + \frac{c_2}{2}(|w_i|^2 + |\theta_i|^2 + |v_i|^2)$$

proof. (1) C_{ad} is a non-empty set of real-valued measurable functions on $[0, T]$. The corresponding state variables exist and are bounded by Proposition 2.1.
(2) Consider

$$C_{ad} = \left\{u \in \mathbb{R}^3; ||u|| \leq 1\right\}.$$

Let $u_1, u_2 \in C_{ad}$ such that $||u_1|| \leq 1$ and $||u_2|| \leq 1$. Then for any $\rho \in [0, 1]$,

$$||\rho u_1 + (1-\rho)u_2|| \leq \rho||u_1|| + (1-\rho)||u_2|| \leq 1.$$

Therefore, C_{ad} is compact.
(3) The dynamical system (6) is linear in terms of control variables w_i, θ_i and v_i while coefficients are dependent on state variables. Therefore, (3) is satisfied.
(4) Let $\rho \in [0, 1]$, $u_1 = (w_{i1}, \theta_{i1}, v_{i1}) \in C_{ad}$, $u_2 = (w_{i2}, \theta_{i2}, v_{i2}) \in C_{ad}$, $g(x, u) = I_{u_i} + I_{k_i} + \frac{A_i^1}{2}w_i^2 + \frac{A_i^2}{2}\theta_i^2 + \frac{A_i^3}{2}v_i^2$ we have

$$\begin{aligned}
&g(x, (1-\rho)u_1 + \rho u_2) - [(1-\rho)g(x, u_1) + \rho g(x, u_2)] \\
&= I_{u_i} + I_{k_i} + \frac{A_i^1}{2}[(1-\rho)^2 w_{i1}^2 + 2\rho(1-\rho)w_{i1}w_{i2} + \rho^2 w_{i2}^2] \\
&+ \frac{A_i^2}{2}[(1-\rho)^2\theta_{i1}^2 + 2\rho(1-\rho)\theta_{i1}\theta_{i2} + \rho^2\theta_{i2}^2] \\
&+ \frac{A_i^3}{2}[(1-\rho)^2 v_{i1}^2 + 2\rho(1-\rho)v_{i1}v_{i2} + \rho^2 v_{i2}^2] \\
&-(1-\rho)(I_{u_i} + I_{k_i}) - [(1-\rho)(\frac{A_i^1}{2}w_{i1}^2 + \frac{A_i^2}{2}\theta_{i1}^2 + \frac{A_i^3}{2}v_{i1}^2)] - \rho(I_{u_i} + I_{k_i}) \\
&-\rho[\frac{A_i^1}{2}w_{i2}^2 + \frac{A_i^2}{2}\theta_{i2}^2 + \frac{A_i^3}{2}v_{i2}^2]
\end{aligned}$$

$$= (\rho^2 - \rho)[\frac{A_i^1}{2}(w_{i1} - w_{i2})^2 + \frac{A_i^2}{2}(\theta_{i1} - \theta_{i2})^2 + \frac{A_i^3}{2}(v_{i1} - v_{i2})^2]$$
$$= \frac{(\rho^2 - \rho)}{2}[(u_1 - u_2)^2] \leq 0.$$

Hence $g(x, (1-\rho)u_1 + \rho u_2) \leq (1-\rho)g(x, u_1) + \rho g(x, u_2)$, which proves the convexity of g(x,u) in C_{ad}.
So, let $c_1 = \sup_{t\in[0;T]}(I_{u_i}, I_{k_i})$ and $c_2 = \inf(A_i^1, A_i^2, A_i^3)$. Then $L(S_i, I_{u_i}, I_{k_i}, R_i) \geq -3c_1 + \frac{c_2}{2}(|w_i|^2 + |\theta_i|^2 + |v_i|^2)$. Therefore, from results stated in [33](Theorem 2.1 of Chapter 3), we conclude that there exists a set of optimal controls for system (6).

Now, the Pontryagin's maximum principle [16] is used to drive necessary conditions for existence of the optimal controls. This principle is based on minimizing Hamiltonian $H_i(t)$ which is defined by

$$H_i = I_{u_i} + I_{k_i} + \frac{A_i^1}{2}w_i^2 + \frac{A_i^2}{2}\theta_i^2 + \frac{A_i^3}{2}v_i^2 + \sum_{k=1}^{4}\psi_i^k f_k(S_i, I_{u_i}, I_{k_i}, R_i), \quad (8)$$

where ψ_i^k is the k^{th} adjoint variable at time t and f_k is the right-hand side function of system (6) corresponding to the k^{th} state at time t.

Theorem 4.2. *The optimal controls of control system (6) have the following form:*

$$w_i^* = \max(0, \min(1, \frac{-(\psi_i^1 - \psi_i^2)}{A_i^1}\sum_{j=1}^{N}\frac{\alpha_{ij}S_i I_{u_j}}{1 + \delta_i I_{u_j}})),$$
$$\theta_i^* = \max(0, \min(1, \frac{\psi_i^2 - t_i\psi_i^4 - (1-t_i)\psi_i^3}{A_i^2}I_{u_i})),$$
$$v_i^* = \max(0, \min(1, \frac{(\psi_i^3 - \psi_i^4)}{A_i^3}I_{k_i})).$$

where, S_i, I_{u_i}, I_{k_i} and R_i as related states.

proof. We know that,

$$
\begin{aligned}
f_1(S_i, I_{u_i}, I_{k_i}, R_i) &= b_i - (1-w_i)\sum_{j=1}^{N}\frac{\alpha_{ij}}{1+\delta_i I_{u_j}} S_i I_{u_j} - d_{1i}S_i + \epsilon_i R_i,\\
f_2(S_i, I_{u_i}, I_{k_i}, R_i) &= (1-w_i)\sum_{j=1}^{N}\frac{\alpha_{ij}}{1+\delta_i I_{u_j}} S_i I_{u_j} - (\beta_i + d_{1i})I_{u_i} - \theta_i I_{u_i},\\
f_3(S_i, I_{u_i}, I_{k_i}, R_i) &= \beta_i I_{u_i} - (\gamma_i + d_{2i})I_{k_i} - v_i I_{k_i} + (1-t_i)\theta_i I_{u_i},\\
f_4(S_i, I_{u_i}, I_{k_i}, R_i) &= \gamma_i I_{k_i} - d_{1i}R_i + t_i\theta_i I_{u_i} + v_i I_{k_i} - \epsilon_i R_i,
\end{aligned}
$$

Pontryagin's maximum principle determines the adjoint equations and transversality conditions:

$$
\begin{aligned}
\dot{\psi_i^1} &= -\frac{\partial H_i}{\partial S_i} = \psi_i^1 d_{1i} + (\psi_i^1 - \psi_i^2)(1-w_i)\sum_{j=1}^{N}\frac{\alpha_{ij} I_{u_j}}{1+\delta_i I_{u_j}}\\
\dot{\psi_i^2} &= -\frac{\partial H_i}{\partial I_{u_i}} = (\beta_i + d_{1i} + \theta_i)\psi_i^2 - \beta_i\psi_i^3\\
&\qquad + (\psi_i^1 - \psi_i^2)(1-w_i)S_i\sum_{j=1}^{N}\frac{\alpha_{ij}}{(1+\delta I_{u_j})^2}\\
&\qquad - (1-t_i)\theta_i\psi_i^3 - t_i\theta_i\psi_i^4 - 1\\
\dot{\psi_i^3} &= -\frac{\partial H_i}{\partial I_{k_i}} = (\gamma_i + d_{2i} + v_i)\psi_i^3 - \gamma_i\psi_i^4 - v_i\psi_i^4 - 1\\
\dot{\psi_i^4} &= -\frac{\partial H_i}{\partial R_i} = d_{1i}\psi_i^4 - (\psi_i^1 - \psi_i^4)\epsilon_i,
\end{aligned}
$$

The tranversality conditions at final time implies that $\psi_i^1(T) = 0$, $\psi_i^2(T) = 0$, $\psi_i^3(T) = 0, \psi_i^4(T) = 0$. Now, solving the optimality conditions

$$
\frac{\partial H_i}{\partial w_i} = 0, \quad \frac{\partial H_i}{\partial \theta_i} = 0, \quad \frac{\partial H_i}{\partial v_i} = 0.
$$

helps us to find the optimal controls w_i^*, θ_i^* and v_i^* for $t \in [0, T]$. Therefore, we obtain

$$
\begin{aligned}
&A_i^1 w_i + (-\psi_i^1 + \psi^2) \sum_{j=1}^{N} \frac{\alpha_{ij} S_i I_{u_j}}{1 + \delta I_{u_j}} = 0, \\
&A_i^2 \theta_i - \psi_i^2 I_{u_i} + t_i \psi_i^4 I_{u_i} + (1 - t_i) \psi_i^3 I_{u_i} = 0, \\
&A_i^3 v_i - \psi_i^3 I_{k_i} + \psi_i^4 I_{k_i} = 0.
\end{aligned}
$$

Form system of equations above, we obtain

$$
\begin{aligned}
w_i &= \frac{-(\psi_i^1 - \psi_i^2)}{A_i^1} \sum_{j=1}^{N} \frac{\alpha_{ij} S_i I_{u_j}}{1 + \delta_i I_{u_j}}, \\
\theta_i &= \frac{\psi_i^2 - t_i \psi_i^4 - (1 - t_i) \psi_i^3}{A_i^2} I_{u_i}, \\
v_i &= \frac{(\psi_i^3 - \psi_i^4)}{A_i^3} I_{k_i}.
\end{aligned}
$$

By using C_{ad}, the bounds of the controls, the optimal controls are obtained as:

$$
\begin{aligned}
w_i^* &= \max(0, \min(1, \frac{-(\psi_i^1 - \psi_i^2)}{A_i^1} \sum_{j=1}^{N} \frac{\alpha_{ij} S_i I_{u_j}}{1 + \delta_i I_{u_j}})), \\
\theta_i^* &= \max(0, \min(1, \frac{\psi_i^2 - t_i \psi_i^4 - (1 - t_i) \psi_i^3}{A_i^2} I_{u_i})), \\
v_i^* &= \max(0, \min(1, \frac{(\psi_i^3 - \psi_i^4)}{A_i^3} I_{k_i})).
\end{aligned}
$$

5 Numerical Results

In this section, we will simulate the model presented for Covid-19 in Iran in two situations without control and with control. We consider three groups as representatives which are respectively the provinces of Isfahan, Khorasan-Razavi, and Fars in Iran (See Figure 3). Due to lack of direct data regarding the transfer coefficient from S_i to I_{u_j} among the three provinces, we determined the pollution coefficients as the travel ratio in Nowruz 1400 (March 21, 2021) and this data is from the country's road management center (https://141.ir).

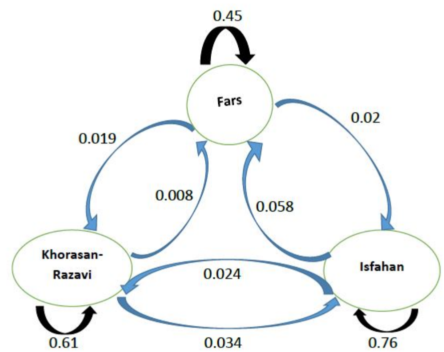


Figure 3: The cross infection among three Iranian

The matrix $A = (\alpha_{ij})_{N\times N}$ is

$$A = \begin{bmatrix} 0.76 & 0.024 & 0.058 \\ 0.034 & 0.61 & 0.008 \\ 0.02 & 0.019 & 0.45 \end{bmatrix}.$$

Information obtained from the Iranian National Bureau of Statistics (www.amar.org.ir) indicates that the annual death rate is equal to 0.00476 and the annual birth rate is equal to 0.0144.

According to the data in (https://behdasht.gov.ir), for COVID-19 the death rate of population is 0.00076. Hence, d_{2i} is the death rate due to COVID-19 plus d_{1i} of i^{th} group, is $d_{2i} = 0.00552$, $i = 1, 2, 3$. In addition, because there is no real data about δ_i, β_i, γ_i, ϵ_i so we assume $\delta_i = 2$, $\beta_i = 0.019$, $\gamma_i = 0.03$, $\epsilon_i = 0.001$, $i = 1, 2, 3$.

So, the matrix FV^{-1} is as follows, based on the above values of parameters:

$$FV^{-1} = \begin{bmatrix} 95.792 & 3.025 & 7.310 \\ 4.285 & 76.886 & 1.008 \\ 2.521 & 2.395 & 56.719 \end{bmatrix}.$$

Since $R_0 = \rho(FV^{-1}) = 96.9996 > 1$, one can know from Theorem 2.4 that P_0 is unstable.

We set March 10, 2021 as time zero. The number of people who remain susceptible in each province is similar to that of its resident population. The number of people who remain asymptomatic in an infected population of each province is equal to 0.1 of the population of the center of that province.

The data obtained from Isfahan University of Medical Sciences in March 10, 2021, show that the number of symptomatic infectious population in Isfahan is $I_{k_1}(0) = 286$, and the recovery population of Isfahan is $R_1(0) = 21$. (mui.ac.ir).

Furthermore, data from Razavi Khorasan University of Medical Sciences (https://www.mums.ac.ir) reveal that the number of symptomatic infectious populations in Razavi Khorasan is $I_{k_2}(0) = 170$, and the recovery population in Razavi Khorasan is $R_2(0) = 40$.

The data from Fars University of Medical Sciences (sums.ac.ir) indicate that the number of symptomatic infectious population in Fars is $I_{k_3}(0) = 151$, and the recovery population of Fars is $R_3(0) = 18$.

The parameters and initial conditions that describe the model are listed in Table 2, and Table 3.

Table 2: Value the parameters

Parameter	Value	Parameter	Value
$S_1(0)$	6120850	$R_1(0)$	21
$S_2(0)$	6434501	$R_2(0)$	40
$S_3(0)$	5054700	$R_3(0)$	18
$I_{u_1}(0)$	286126	b_i	0.0144
$I_{u_2}(0)$	361900	δ_i	2
$I_{u_3}(0)$	169081	β_i	0.019
$I_{k_1}(0)$	286	γ_i	0.03
$I_{k_2}(0)$	170	d_{1i}	0.00476
$I_{k_3}(0)$	151	d_{2i}	0.00552
		ϵ_i	0.001

Table 3: The values of cost coefficients associated with controls

A_1^1	A_2^1	A_3^1	A_1^2	A_2^2	A_3^2	A_1^3	A_2^3	A_3^3	t_1	t_2	t_3
7	3	3	1	1	1	1	1	1	0.001	0.001	0.001

The simulation of the COVID-19 outbreak in the Isfahan, Fars and Khorasan-Razavi provinces has been performed.

Furthermore, from Figure 4 we observe that the number of susceptible in the three provinces decreased rapidly. The reason is that susceptible people have entered other groups. Therefore, the infection rate has been very high, and I_u, I_k shows in the three provinces that is increasing rapidly.

Figure 5 depicts that S, and R in the three provinces will increase rapidly. The reason is that susceptible people do not enter other groups. Therefore, the infection rate has been very down and the number of those who have recovered has also increased.

Figure 6 shows the number of susceptible people, infected without symptoms, infected with symptoms, and those who have recovered in 60 days and without control, demonstrating the speed of spread of the disease. If we can identify infected people without symptoms, it will have a great impact on controlling this disease.

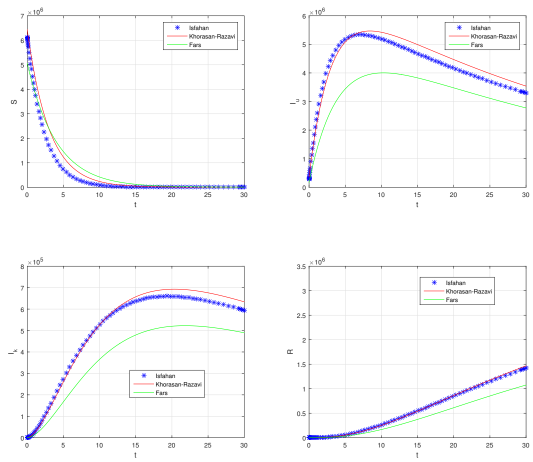


Figure 4: Plots for S, I_u, I_k, and R in the three provinces under no control using data from Table 2.

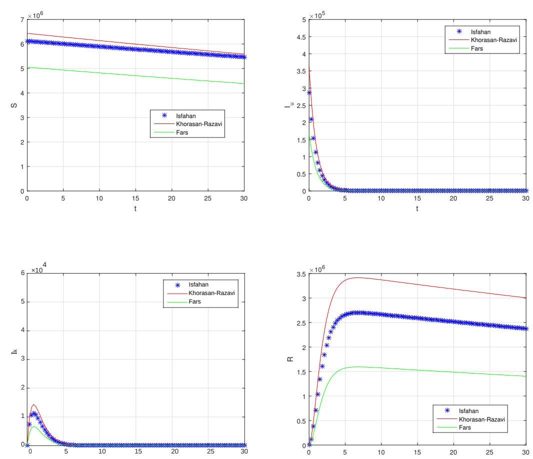


Figure 5: Plots for S, I_u, I_k, and R in the three provinces under control using data from Table 2.

Figure 7 shows the effectiveness of the controls in the number of susceptible people, infected without symptoms, infected with symptoms and recovered in 60 days, which aims to reduce the number of infected people.

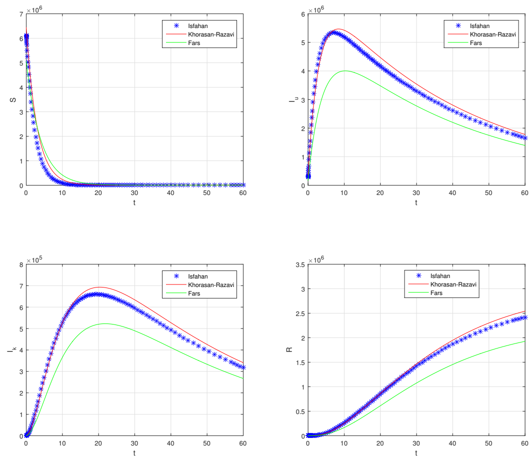


Figure 6: Time series of S, I_u, I_k, and R in the three provinces under no control (in 60 days).

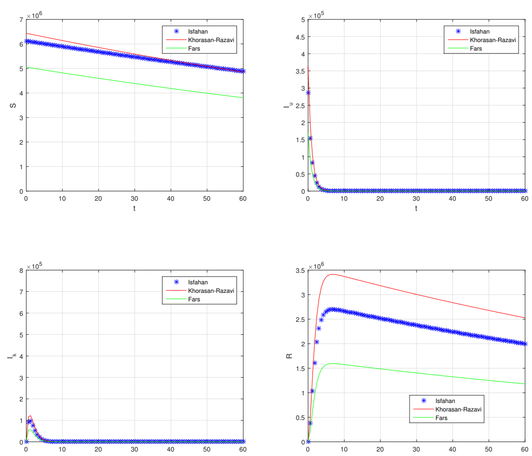


Figure 7: Time series of S, I_u, I_k, and R in the three provinces under control (in 60 days).

The obtained results show that with the application of controls, the number of susceptible and recovery people has increased, while the number of asymptomatic infectious population and the number of symptomatic infectious population has decreased drastically.

6 Conclusions

In this article, a multi-group model of covid-19 disease according to the outbreak process and measures taken in three provinces of Iran is presented. Also, the stability of disease-free and endemic equilibrium has been studied. To prevent the spread of the disease, we designed some suitable control strategies whose purpose was to reduce the number of symptomatic and asymptomatic infected people. Finally, to show the effectiveness of these results, we used almost real data from the provinces of the Isfahan, Khorasan-Razavi and Fars in Iran. The graphs obtained from the results of the simulations clearly show that the number of infected people with symptoms and without symptoms has greatly decreased.

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