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Original Research Paper

Connected Topological n-Ary Polygroups and Fundamental Relations

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Abstract. In this paper, we investigate the topological aspects of n-ary polygroups, a generalization of classical group-like structures within the framework of algebraic hyperstructures. We introduce the definition of n-ary polygroups. Then develop the theory of n-ary topological polygroups by equipping these structures with compatible topologies, and examine their continuity properties. Special attention is given to the subclass of connected n-ary topological polygroups, where we explore the influence of topological connectedness on the algebraic behavior. In particular, we prove some results about the quotient topological n-ary polygroups. Finally, we study the connection between the fundamental relation—a key concept in the theory of hyperstructures—and the topology of n-ary polygroups, providing new insights into their structural decomposition. Our results contribute to a deeper understanding of the interplay between topology and generalized algebraic operations. By using the notion of fundamental relation we make a connection between topological n-ary polygroups and topological n-ary groups.

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1 Introduction and Basic Definitions

The concept of a group generalizes to hypergroups by Marty [14]. One of the most important subclass of hypergroups is known as polygroups For details about polygroups we refer to [2, 4, 15, 18]. In [11], Heidari et al. introduced the notion of topological polygroups, also see [12].

Also, the concept of a group generalizes to hypergroups by Dörnte [8]. Davvaz and Vougiouklis [5] bring forward the class of n-ary hypergroups, which is a generalization of hypergroups and n-ary groups, for more information also see [6, 7]. Special subclass of n-ary hypergroups is n-ary polygroups. In Ghadiri and Waphare [10] introduced this subclass. For more details see [1, 19, 16].

Definition 1.1. [10] An n-ary polygroup (in short nAP) is a multi-valued system $M = \langle P, f, e,^{-1} \rangle$, where $e \in P$, $^{-1}$ is a unitary operation on P, f is an n-ary hyperoperation on P, i.e.,

$$f : \underbrace{P \times P \times ... \times P}_{\text{n- times}} \longrightarrow \mathcal{P}^*(P)$$

and for all $1 \leq i, j \leq n$, $x_1, ..., x_{2n-1}, x \in P$, we have

(1) $f(x_1^{i-1}, f(_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}), f(x_j^{n+j-1}, x_{n+j}^{2n-1})$,

(2) e is a unique element such that $f(e, ..., e, x, e, ..., e) = x, e^{-1} = e$,

(3) $x \in f(x_1^n) = f(x_1, ..., x_n)$ implies $x_i \in (x_{i-1}^{-1}, ..., x_1^{-1}, x, x_n^{-1}, ...x_{i+1}^{-1})$.

A non-empty subset S of an nAP P is called an n-ary subpolygroup (in short nASP) if, together with the hyperoperation defined on the nAP, it forms an nAP itself.

The motivation for this study stems from the desire to generalize classical algebraic structures, such as groups and polygroups, to the n-ary and topological setting. While binary hyperstructures have been widely studied, the n-ary case—especially when combined with topology—remains under explored despite its potential applications in abstract algebra, theoretical physics, and systems with multi-valued logic or interactions. Understanding topological nAP their fundamental relations allows us to capture more complex algebraic behaviors and topological features that cannot be modelled by binary operations alone.

This broader framework not only enriches the theory of hyperstructures but also creates new connections between algebra and topology.

2 On Topological nAP and Examples

Definition 2.1. An nAP P is zero-dimensional if it has a base Ω consisting of nAP s which are both open and closed in P.

Definition 2.2. A topological nAP P is paracompact if every open covering of P can be refined by locally finite open covering.

Definition 2.3. An nAP P is locally σ-compact if for every point x of P, there exists an open neighborhood V such that the closure of V is σ-compact.

Definition 2.4. A family γ of nAP P is star-finite if every element of γ intersects only finitely many elements of γ.

Definition 2.5. A nAP P is said to be strongly paracompact if every open covering of P can be refined by a star-finite open cover in.

Every regular Lindelof nAP P is strongly paracompact and since σ-compact nAP are Lindelof. All regular σ-compact nAP are strongly paracompact.

Definition 2.6. A topological nAP P is said to be disconnected if there exists U_1,...,U_n nASP of P such that $U_1 \cup ... U_n = P$ and $U_1 \cap ... \cap U_n = \phi$. In this case, we say that $(U_1, ..., U_n)$ is a disconnection on P.

A topological nAP P is connected if the only closed and open nASP are ϕ and P.

Definition 2.7. A topological nAP P is said to be connected if it is not disconnected.

For a topological nAP P we denote by $C(P)$ the connected component of e_P and we call it briefly CC of P.

Proposition 2.8. *Every totally disconnected (in short TD) locally compact Hausdorff nAP P is zero-dimensional.*

Proof. Since P is regular, we can assume that P is compact. Fix a point $a \in P$, and let q be the family of all open and closed nASP of P which contain a. Put $Q = \bigcap q$ clearly, Q closed in P and $a \in Q$. Notice also that the family q is closed under finite intersections. □

Theorem 2.9. *Every locally σ-compact topological nAP P is strongly paracompact.*

Proof. Take a symmetric open neighborhood V of the identity e in P such that $F = \overline{V}$ is σ-compact, and put $H = \cup F^n$. Clearly, H is nASP of P, and the interior of H contains V. Therefore, H is an open and closed nASP of P. It is also clear that each F^n is compact, therefore, the space H is σ-compact and hence lindelof. The space P is the free topological of the n-ary subspaces homomorphic to H. □

Theorem 2.10. *Every connected locally σ-compact topological nAP P is σ- compact.*

Proof. We repeat the argument in the proof of theorem 2.9. The open closed nASP H of P constructed there has to coincide with P, since H is a non-empty and the n-space$P$ is connected. Since H is σ-compact. It follows that P is σ-compact. □

Example 2.11. Every compact nAP P is totally bounded. Indeed, We know that an nAP P is totally bounded if every cover of P has a finite subcover. Since every cover is an open cover, we conclude that every compact nAP P is totally bounded.

Example 2.12. Let P be a set and S be a non-empty subset of P, for every $a_1, a_2, ..., a_n \in P$ we define $f(a_1, ..., a_n) = S$. Then, P is a left big in the nAP $(P, f, e,^{-1})$ if and only if $P = S$.

Example 2.13. Any compact nAP is locally compact, because if an nAP P is compact, then it is a compact neighborhood of every point.

Example 2.14. Any discrete nAP P is locally compact, because the singleton can serve as compact neighborhood.

Example 2.15. If P is locally compact, then P is compactly generated. To see this, let A be an nASP of P such that all compact nASP C of

P, $A, \cap C$ is open in P. Let $x \in A$, and choose any neighborhood U of x that lies in a compact nASP C of P, i.e., U is open in C, $A \cap C$ is open in C, so $A \cap C \cap U$ is open in $C \cap U$. This yields that $A \cap U$ is open in U. SO $A \cap U$ is open in P.

Example 2.16. Every open nASP of a locally compact Hausdorff nAP is locally compact. To see this, let P be a locally compact Hausdorff nAP and Y an open nASP of P. Since P is locally compact Hausdorff n-ary polygroup, it is a regular nAP and Y is regular nAP. Choose a random $x \in Y$ and $U \subseteq Y$ such that $\overline{U} \subseteq Y$ with $x \in U$, we can do this because Y is regular since $\overline{U}$ is the closure of an open nAP in a locally compact nAP we know that $\overline{U}$ is a compact in P. Therefore for every open cover of $\overline{U}$, there exists a finite subcover of $\overline{U}$. Because Y is open in P, the intersection of(any open cover of $\overline{U}$ in P)and Y would still be an open cover of $\overline{U}$ in P therefore there exists a finite subcover for this open cover contained in Y. therefore $\overline{U}$ is compact in Y which implies that Y is locally compact.

3 Connected Topological nAP s

Lemma 3.1. *Let $P = \langle P, f, e, ^{-1}, \tau \rangle$ be a topological nAP.*

(1) *If $C_1, ..., C_n$ are connected nAP s in P then, also $f(C_1, ..., C_n)$ is connected.*

(2) *If C is connected nAP in P then, the nAP C^{-1} is connected.*

Proof. (1) The nASP $C_1 \times ... \times C_n$ of $P \times ... \times P$ is connected. Now, the map $f : P \times ... \times P \longrightarrow \mathcal{P}^*(P)$ defined $(x_1, ..., x_n) \longmapsto f(x_1, ..., x_n)$ is continuous and $f(C_1 \times ... \times C_n)$ is connected.

(2) C^{-1} is a continuous image of C under the continuous map $x \mapsto x^{-1}$. Since C is connected, it follows that C^{-1} is connected. □

Theorem 3.2. *Let P be a topological nAP and let C be the component of identity e then, C is a closed normal nASP of P.*

Proof. Since inversion is a homomorphism of P, C^{-1} is a connected nAP containing e, hence, $C^{-1} \subseteq C$. If $a_1, ..., a_{n-1}$ are points of C then, also $a_1^{-1}, ..., a_{n-1}^{-1} \in C$. Therefore, $f(a_1, ..., a_{n-1}, C)$ is connected nAP containing e, and so $f(a_1, ..., a_{n-1}, C) \subset C$. Thus, $f(C, C, ..., C) \subset C$, so that C is an nASP of P. If $a \in P$ then, $f(a, C, a^{-1}, e_*) \subset C$ is a connected nAP containing e, so that $f(a, C, a^{-1}, e_*) \subset C$. This implies that C is a normal nASP of P. Therefore, C is closed. □

Theorem 3.3. *Let P be a topological nAP and let C be the component of the identity in P. Then, for all $a_1, ..., a_{n-1} \in P$, $f(a_1, ..., a_{n-1}, C) = f(C, a_1, ..., a_{n-1})$ is a component of $a_1, ..., a_{n-1}$.*

Proof. The mapping $x \longmapsto f(a_1, ..., a_{n-1}, x)$ is a homomorphism of P, and C is a normal nASP of P. □

Definition 3.4. Suppose that K is an nASP of P. We define the relation $\equiv_k$ on $P^{(m-1)}$ by

$$(x_2^m) \equiv_k (y_2^m) \iff f(k, x_2^m) = f(k, y_2^m),$$

for (x_2^m),$(y_2^m) \in P^{(m-1)}$. The class of $(x_2, ..., x_m) \in P^{(m-1)}$ is denoted by

$$\begin{aligned} K[x_2^m] &= \{(y_2^m) \mid f(K, y_2^m) \\ &= f(K, x_2^m), y_2, ..., y_m \in P\} \end{aligned}$$

and we set $\dfrac{P^{(m-1)}}{K} = \{K[x_2^m], x_2, ..., x_m \in P\}$. Also, we define the relation $=^K$ on P as follows:

$$x =^K y \Leftrightarrow \exists a_2, ..., a_m \in P \ wit \ x, y \in f(K, a_2^m),$$

for every $x, y \in P$.

Theorem 3.5. [17] *Let N be a normal nASP of a topological nAP P and every open subset of P be a complete part. Then, $\langle \dfrac{P^{(m-1)}}{N}, \odot, N, ^{-I} \rangle$ is a topological nAP , where $\odot(N[a_1, e_*], ..., N[a_m, e_*]) = \{N[t, e_*] \mid t \in f(a_1^m)\}$.*

Theorem 3.6. *Let P be a topological nAP and let C be the component of the identity in P. Then, $\dfrac{P^{(n-1)}}{C}$ is a TD Hausdorff nAP.*

Proof. Let $\{C[x, e_*] : x \in P\}$ be an nASP of $\dfrac{P^{(n-1)}}{C}$, which is properly containing $\{C\}$. We will show that $\{C[x, e_*] : x \in P\}$ is disconnected in $\dfrac{P^{(n-1)}}{C}$. Let π be the natural mapping of $P^{(n-1)}$ onto $\dfrac{P^{(n-1)}}{C}$, and let A nASP of $P^{(n-1)}$. It is easy to verify that $\pi(A \cap C[x, e_*]) = \pi(A) \cap \{C[x, e_*] : x \in P\}$. The nAP $C[x, e_*]$ properly contains C, and so is disconnected:
$C[P, e_*] = (U_1 \cap C[P, e_*]) \cup ... \cup (U_n \cap C[P, e_*])$, where
$(U_1 \cap C[P, e_*]) \cap ... \cap (U_n \cap C[P, e_*]) = \phi$, neither set is void, and $U_1, ..., U_n$ are open in P. Thus,
$\{C[x, e_*] : x \in P\} = (\pi(U_1) \cap \{C[x, e_*] : x \in P\}) \cup ... \cup (\pi(U_n) \cap \{C[x, e_*] : x \in P\})$, where $\pi(U_1), ..., \pi(U_n)$ are open in $\dfrac{P^{(n-1)}}{C}$, and π is an open mapping. For $x \in P$, we have $C[x, e_*] = (U_1 \cap C[x, e_*]) \cup ... \cup (U_n \cap C[x, e_*])$. Thus, since $C[x, e_*]$ is connected, either $C[x, e_*] \subseteq U_1 \cap C[x, e_*]$ or ... or $C[x, e_*] \subseteq U_n \cap C[x, e_*]$. Consequently, $U_1 \cap C[x, e_*]$,...,$U_n \cap C[x, e_*]$ are union of n-ary cosets of C and so they have disjoint images under π. Thus,

$$(\pi(U_1) \cap \{C[x, e_*] | x \in P\}) \cap ... \cap (\pi(U_n) \cap \{C[x, e_*] | x \in P\}) = \phi,$$

so that $\{C[x, e_*] : x \in P\}$ is disconnected. □

Theorem 3.7. *Let P be a topological nAP, C the component of e, and U be a neighborhood of e. Then, $C \subset \bigcup_{n=1}^{\infty} U_n$. In particular, if P is connected, then $P = \bigcup_{n=1}^{\infty} U_n$.*

Proof. If V is a symmetric neighborhood of e such that $V \subset U$, then $\bigcup_{n=1}^{\infty} V^n$ is open and close, since P is connected, it follows that $C \subset \bigcup_{n=1}^{\infty} V^n \subset \bigcup_{n=1}^{\infty} U^n$. □

Theorem 3.8. *Let P be a locally compact nAP and C be the component of the identity element in P. Then, C is the intersection of all open nASP of P.*

Proof. Obviously, C is contained in the intersection of all open nASPs of $P^{(n-1)}$. Suppose that x is an arbitrary element of P with $x \notin C$. Take nAP $\dfrac{P^{(n-1)}}{N}$. It is TD and locally compacted. So, there is a compact nASP $\{C[x, e_*] : u \in U\}$ of $\dfrac{P^{(n-1)}}{C}$ that does not contain the element $C[x, e_*]$ of $\dfrac{P^{(n-1)}}{C}$. We may take U to be a neighborhood of e in $P^{(n-1)}$. Then, $C[U, e_*]$ is open nASP of $P^{(n-1)}$ not containing x. □

Corollary 3.9. *Let P be a locally compact. The below conditions are equivalent:*

(1) *P is connected;*

(2) *P has no proper open nASP;*

(3) *For any neighborhood U of e, we have $\bigcup_{n=1}^{\infty} U^n = P$.*

Proof. According to Theorem 3.7 and Theorem 3.8, the result is straightforward. □

Theorem 3.10. *Let P be a locally compact nAP and let C be the component of e. If $\dfrac{P^{(n-1)}}{C}$ is compactly generated, then $P^{(n-1)}$ is compactly generated.*

Proof. The proof follows in a direct manner from the preceding results and requires no additional technical tools. □

Theorem 3.11. *Let P be a topological nAP and let N be an nASP of P. If N and $\dfrac{P^{(n-1)}}{N}$ are connected then, $P^{(n-1)}$ is connected.*

Proof. Assume that $P = U_1 \cup ... \cup U_n$, where $U_1, ..., U_n$are n-ary disjoint non-void n-ary open sets. Since N is connected, each coset of N is either an n-ary subset of U_1 or an n-ary subset U_n. Thus, the relation $\dfrac{P^{(n-1)}}{N} = \{N[x, e_*] : N[x, e_*] \subset U_1\} \cup ... \cup \{N[x, e_*] : N[x, e_*] \subset U_n\} = \{N[x, e_*], x \in U_1\} \cup ... \cup \{N[x, e_*], x \in U_n\}$ expresses $\dfrac{(n-1)}{N}$ as the union of disjoint non-void n-ary open sets. This contradicts the hypothesis that $\frac{P^{(n-1)}}{N}$ is connected. □

Theorem 3.12. *Let P be a topological nAP and N be a closed normal $nASP$ of P. The following statements are equivalent:*

(1) *If both N and $\frac{P^{(n-1)}}{N}$ are connected then, also $P^{(n-1)}$ is connected.*

(2) *If both N and $\frac{P^{(n-1)}}{N}$ are TD, then, also P is TD.*

Proof. (1) Let $A \neq \phi$ be an n-ary closed and open set of P. Since each coset $N[a, e_*]$ is connected, it follows that either $N[a, e_*] \subseteq A$ or $N[a, e_*] \cap A = \phi$. Hence, $\pi^{-1}(\pi(A)) = A$. This implies that $\pi(A)$ is a non-empty n-ary close and open set of the connected nAP $\frac{P^{(n-1)}}{N}$. Thus, $\pi(A) = \frac{P^{(n-1)}}{N}$.consequently $A = P^{(n-1)}$.

(2) Assume that C is connected n-ary set in P. Then, $\pi(C)$ is a connected set of $\frac{P^{(n-1)}}{N}$, so $\pi(C)$ is a singleton. Consequently, C is contained in some coset $N[x, e_*]$. Since $N[x, e_*]$ is TD as well, we deduce that C is singleton. This shows $P^{(n-1)}$ is TD. □

Lemma 3.13. *If P is a topological nAP, then the nAP $\frac{P^{(n-1)}}{C(P^{(n-1)})}$ is TD.*

Proof. Suppose that π from $P^{(n-1)}$ to $\frac{P^{(n-1)}}{N}$ is the canonical map, and N is the inverse image of $C(\frac{P^{(n-1)}}{C(P^{(n-1)})})$ under π. Next, we have

$$\frac{N}{C(N)} \cong C(\frac{P^{(n-1)}}{C(P^{(n-1)})}).$$

Hence, N is connect. Since it contains $C(P^{(n-1)})$, it follows that $N = C(P^{(n-1)})$. Therefore, $\frac{P^{(n-1)}}{C(P^{(n-1)})}$ is TD. □ For a topological nAP P denote by $a(P)$ the n-ary set of points $x \in P$ connected to e_p by an arc, i.e., a continuous map $g : [0, 1] \longrightarrow P$ such that $g(0) = e_p$ and $g(1) = x$.

Theorem 3.14. *Let P be a topological nAP the arc component $a(P)$ of P is an nASP of P.*

Proof. We have $f(a(P),...,a(P)) \subseteq a(P)$. Similarly, by using the continuity of the inverse $x \to x^{-1}$, we obtain that $a(P) \subseteq a(P)^{-1}$. This prove that $a(P)$ is an nASP of P. □

Definition 3.15. Let P be a topological nAP denote by $Q(p)$ the quasi component of the natural element e_p of P and call it quasi-component of P.

Corollary 3.16. *Let P be a locally compact nAP. Then, $Q(P) = C(p)$.*

Proof. $C(P)$ is an intersection of open nASPs. Hence, $C(p)$ contains $Q(P)$ which is true coincide with the intersection of all n-ary sets of P containing e_p. The inclusion $C(P) \subseteq Q(P)$ is always true. □

Proposition 3.17. *Let P be a topological nAP. The quasi-component $Q(P)$ is a closed normal nASP of P.*

Proof. Let $x_1, ..., x_n \in Q(P)$. To prove that $f(x_1, ..., x_n) \subseteq Q(P)$. We need to verify that $f(x_1, ..., x_n) \subseteq O$ for every n-ary close set O containing e_p. Let O be an n-ary set, then $x_1, ..., x_n \in O$. Obviously, $f(O, x_n^{-1}, ..., x_2^{-1})$ is n-ary close set containing e, hence $x_1 \in f(O, x_n^{-1}, ..., x_2^{-1})$. This implies that $f(x_1, ..., x_n) \subseteq O$. Hence, $Q(P)$ is stable under multiplication.
Each close set O containing e, the set O^{-1} has the same property. Thus, $Q(P)$ is stable we write the operation $a \to a^{-1}$. This implies that $Q(P)$ is an nASP. Moreover, for every $a \in P$ and for every close set O containing e also its image $f(a, o, a^{-1}, e_*$ under the conjugations a close set containing e_p. So $Q(P)$ is stable also under conjugation. Therefore, $Q(P)$ is n-ary normal and closed. □

Definition 3.18. When A is an n-ary subset of an nAP, $\langle A \rangle$ denotes the intersection of all nASPs containing A. We call $\langle A \rangle$ the nASP generated by A.

Proposition 3.19. *Let P be a topological nAP and A n-ary subset of P. If A is connected and $e \in A$ then, $\langle A \rangle$ is connected.*

Proof. Suppose that A is connected and $e \in A$. We see that for all $n = 0, 1, ..., C_n = (A \cup A^{-1})^n$ is connected (note that $A \cup A^{-1}$ is connected since $e \in A \cup A^{-1}$). But $e \in \bigcap_{n=0}^{\infty} C_n$, so it follows that $\langle A \rangle = \cup_{n=0}^{\infty} C_n$ is connected as well. □

Proposition 3.20. *If P is a connected nAP and U n-ary non-empty open subset of P then, P is the nAP generated by U. In otherworld, $P = \langle U \rangle$*

Proof. Since $\langle U \rangle$ is an nASP of P containing an n-ary non-empty open subset, we conclude that it is an nASP, $\langle U \rangle$ cannot be proper and it follows that $\langle U \rangle = P$. □

4 Fundamental Relations and Related Results

Definition 4.1. For all $n > 1$, we define the relation β_n on a semihypergroup $(H, \circ)$ as follows:

$$x\beta_n y \text{ if there exists } a_1, \ldots, a_n \text{ in } H \text{ such that} \{x, y\} \subseteq \prod_{i=1}^{n} a_i$$

and we set $\beta = \bigcup_{n \geq 1} \beta_n$, where $\beta_1 = \{(x, x) \mid x \in H\}$ is the diagonal relation on H.

This relation was introduced by Koskas [13] and studied by many authors, for instance [3, 4, 9, 20], and many others.

Definition 4.2. Let A be a subset of topological space X and $\sim$ be an equivalence relation on X. Then, the saturation of A with respect to $\sim$ is the set $\widehat{A} = \{x \in X | \exists a \in A, x \sim a\}$. If $\widehat{A} = A$ then, A is called saturated.

Let $\langle P, f, e, ^{-1}, \tau \rangle$ be a topological nAP and β^* be the fundamental relation on P. Then, $(\frac{P}{\beta^*}, \overline{\tau})$ is a topological space,where $\overline{\tau}$ is the quotient topology induce by the natural mapping $\pi : P \longrightarrow \frac{P}{\beta^*}$. That is $A \subseteq \frac{P}{\beta^*}$ is open in $\frac{P}{\beta^*}$ if and only if $\pi^{-1(A)}$ is open in P.

Lemma 4.3. *Let $\langle P, f, e,^{-1}, \tau\rangle$ be a topological nAP and β^* be the fundamental relation on P. Then, every saturation subset of P is a complete part.*

Proof. Suppose that that A is a saturated subset of P such that $A \cap \prod a_i \neq \phi$ for a nonzero natural number n and for some $x_1, ..., x_n \in P$. Hence, there exists $a \in A \cap \prod_{i=1}^{n} a_i$. Then, for every $x \in \prod_{i=1}^{n} x_i$, we have $x\beta^* a$. Thus, $x \in \widehat{A} = A$. There, A is a complete part. □

Lemma 4.4. *Let $\langle P, f, e,^{-1}, \tau\rangle$ be a topological nAP such that every n-ary open subset of P is a complete part. Then, the natural mapping $\pi : P \longrightarrow \frac{P}{\beta^*}$ is an open mapping.*

Proof. The proof follows in a direct manner from the preceding results and requires no additional technical tools. □

Theorem 4.5. *Let $\langle P, f, e,^{-1}, \tau\rangle$ be a topological nAP such that every open subset of P is a complete part. Then, $\langle \frac{P}{\beta^*}, \frac{f}{\beta^*}, \beta^*(e), -1, \overline{\tau}\rangle$ is topological n-group.*

Proof. We know $(\frac{P}{\beta^*}, \frac{f}{\beta^*}, \beta^*(e), \overline{\tau})$ is n-group. We show that the mapping
$(\beta^*(a_1), ..., \beta^*(a_n)) \longrightarrow \frac{f}{\beta^*}(\beta^*(a_1), ..., \beta^*(a_n))$ and $\beta^*(x) \mapsto (\beta^*(x))^{-1} = \beta^*(x^{-1})$ are continuous.

Suppose that A is open in $\frac{P}{\beta^*}$ such that $\frac{f}{\beta^*}(a_1), ..., \beta^*(a_n)) = \beta^*(a) \in A$ for every $a \in f(\beta^*(x_1), ..., \beta^*(x_n))$. So $f(a_1, ..., a_n) \subseteq \frac{f^{-1}}{\beta^*}(A)$. Since $\frac{f^{-1}}{\beta^*}(A)$ is open in P, it follows that there are open subsets $V_1, ..., V_n$ of P such that $a_1 \in V_1, ..., a_n \in V_n$ and $f(V_1, ..., V_n) \subseteq \pi^{-1}(A)$. Thus, $\frac{f}{\beta^*}(\beta^*(V_1), ..., \beta^*(V_n)) \subseteq A$ and $\beta^*(V_1), ..., \beta^*(V_n)$ are open in $\frac{P}{\beta^*}$. Hence, $\frac{f}{\beta^*}$ is continuous. Now, suppose that $\beta^*(x)^{-1} = \beta^*(x^{-1}) \in \frac{P}{\beta^*}$. Then, $x^{-1} \in \pi^{-1}(A)$. Thus, there exists an open subset U of P such that

$x^{-1} \in U^{-1} \subseteq \pi^{-1}(A)$. So $\pi(x^{-1}) \in \pi(U^{-1}) \subseteq A$. And $\pi^{-1}(U)$ is open in $\frac{P}{\beta^*}$ □

Corollary 4.6. *Let P be a topological nAP such that every saturated subset of P is open. Then, $(\frac{P}{\beta^*}, \frac{f}{\beta^*}, \beta^*(e), -1, \overline{\tau})$ is a topological n-group.*

Definition 4.7. Let G be an n-group and $\{A_g\}$ be a collection of disjoint non-empty sets. Let $P = \bigcup_{g \in P} A_g$, and for $x_1, ..., x_n \in P$ define $f(x_1, ..., x_n) = A_{g_{x_1} \cdots g_{x_n}}$, where $x_1 \in A_{g_{x_1}}$ ... and $x_n \in A_{g_{x_n}}$. Then, $\langle P, f, e,^{-1} \rangle$ is an nAP, and it is called (G, P) nAP.

Lemma 4.8. *Let $\langle G, \circ, \tau \rangle$ be a topological n-ary group and $\{A_g\}$ be a collection of disjoint non-empty sets, and let $P = \cup A_g$. Then, $\langle P, f, e,^{-1}, \tau_P \rangle$ is a topological nAP, where P is a (G, P), nAP and $\tau_P = \{\cup A_u, u \in U | U \in \tau\} \cup \{\phi\}$.*

Proof. One can show τ_P is a topology on P. Suppose that $A_U = \bigcup_{u \in U} A_u$ is an open subset of P such that $f(x_1, ..., x_n) \subseteq A_U$ for $x_1, ..., x_n \in P$. Since U is open in P and $g_{x_1}, ..., g_{x_n} \in U$, it follows that there exists open subsets $V_1, ..., V_n$ of P containing $g_{x_1}, ..., g_{x_n}$, respectively, such that $V_1 ... V_n \subseteq U$. So $A_{V_1}, ..., A_{V_n}$ are open in P containing $x_1, ..., x_n$, respectively, and $f(A_{V_1}, ..., A_{V_n}) \subseteq A_U$. Thus, f is continuous. Similarly, -1 is continuous. □

Theorem 4.9. *Let $(G, \circ, \tau)$ be a topological n-group and $\{A_g\}$ be collection of disjoint non-empty sets, and let $P = \bigcup_{g \in G} A_g$. Then, the fundamental nAP P and G are topological isomorphic.*

Proof. Let $P = \bigcup_{g \in G} A_g$. For every $x \in P$, there is $g_x \in G$ such that $x \in A_{g_x}$. Now, we define $\psi : \frac{P}{\beta^*} \longrightarrow G$ by $\psi(\beta^*(x)) = g_x$. We observe that ψ is a group isomorphism. Clearly, $\psi(\varphi(A_U)) = U$ for all open subset U of P. Hence, we conclude that ψ is open as well as continuous. □

Proposition 4.10. *Let $(G,\circ)$ be a topological n-group, and let $\frac{P}{N}$ be an n-ary normal subgroup of G. Then, $\frac{G^{n-1}}{N}=\{P[x,e_*]|x\in G\}$ is topological n-space with respect to the quotient topology induce by natural mapping $G^{(n-1)}\longrightarrow\frac{G^{(n-1)}}{N}$. Furthermore, every open subset of $\frac{G^{(n-1)}}{N}$ is the form $\{N[u,e_*]:u\in U\}$ for some open subset U of G.*

Proof. Let (P,f) be an n-ary group, and let N be a non-normal nASP of P. If we denote $\frac{P^{(n-1)}}{N}=\{N[x,e_*]|x\in P\}$. Then, $(\frac{P^{(n-1)}}{N},F,N[e],-1)$ is an nAP, where for all $N[x_1,e*],...,N[x_n,e*]$ of $\frac{P^{(n-1)}}{N}$, we have

$$F(N[x_1,e_*],...,N[x_n,e_*])=\{N[z,e_*]|z\in f(x_1,...,x_n)\}.$$

Let β^* be the fundamental relation of the nAP $\frac{P^{(n-1)}}{N},F,N,-1)$. Then, $\frac{\frac{P^{(n-1)}}{N}}{\beta^*}$ is a topological n-space with respect to the quotient topology induce by natural mapping $\pi:\frac{P^{(n-1)}}{N}\longrightarrow\frac{\frac{P^{(n-1)}}{N}}{\beta^*}$, where $\pi(N[x,e_*])=\beta^*(N[x,e_*])$. by Theorem 4.9, $A\subseteq\frac{\frac{P^{(n-1)}}{N}}{\beta^*}$ is open if and only if $\pi^{-1}(A)=\{N[u,e_*]|u\in U\}$ for some open subset U of $P^{(n-1)}$ □

Theorem 4.11. *Let (P,f) be an n-ary group, and let H be an n-ary non-normal subgroup of P. Consider β^* on the nAP $\langle\frac{P^{(n-1)}}{N},F,N[e_*],^{-1}\rangle$. Then, there exists an n-ary normal sungroup N of P such that the n-ary groups $\frac{\frac{P^{(n-1)}}{H}}{\beta^*}$ and $\frac{P^{(n-1)}}{N}$ are isomorphic.*

Proof. Suppose that N is the n-ary subgroup of P generated by the set $\{f(P^{-1},h,P,e_*)|p\in P,h\in H\}$. Then, N is an n-ary normal subgroup of P. Now, we define $\Phi : \dfrac{\frac{P^{(n-1)}}{H}}{\beta^*} \longrightarrow \dfrac{P^{(n-1)}}{N}$ by $\beta^*([x, e_*]) = N([x, e_*])$. Then, Φ is well define. Indeed, if $\beta^*(H[x, e_*]) = \beta^*(H[y, e_*])$, then there exists $n \in \mathbb{N}$ and $a_1, ..., a_n \in P$ such that $\{H[x, e_*] = H[y, e_*]\} \subseteq F(H[a_1, e_*], ..., H[a_n, e_*])$. So $N[x, e_*] = N[y, e_*]$. Thus, we get

$$\Phi(\beta^*(H[x, e_*])) = \Phi(\beta^*(H[x, e_*])).$$

Also, Φ is homomorphism. Indeed, for every $x_1, ..., x_n \in H$, we have $\Phi(\dfrac{F}{\beta^*}(\beta^*(H[x_1, e_*]), ..., \beta^*(H[x_n, e_*]) = F^{'}(N[x_1, e_*], ..., N[x_1, e_*]) = F^{'}(\beta^*(H[x_1, e_*], ..., \beta^*(H[x_1, e_*])$. Obviously, Φ is onto. It remains to show that Φ is injective. If $\beta^*(H[x, e_*]) \in Ker\Phi\Phi$, then $x \in N$. Hence, there exists $n \in \mathbb{N}$ and $P_1, ..., P_{n+1} \in P$ and $h_1, ..., h_n \in H$ such that $x = \pi_{i=1}^{n} f(P_i^{-1}, h_i, g_i, e)$. Thus, we have

$$\{H[x, e_*], H[e_*]\} \subseteq F(H[g_1^{-1}, e_*], H[g_1, e_*], ..., H[g_n^{-1}, e_*], H[g_n, e_*]).$$

Hence, $H[x, e_*]\beta^* H[e_*]$. So, $Ker\Phi$ is trivial. Therefore, Φ is an isomorphism. □

Theorem 4.12. *Let (P, f) be a topological n-group, and let H be an n-ary non normal subgroup of P. Let β^* be the fundamental relation of the nAP $\langle \dfrac{P(n-1)}{H}, F, H[e_*],^{-1} \rangle$. Then, the natural mapping π : $\dfrac{P^{(n-1)}}{H} \longrightarrow \dfrac{\frac{P^{(n-1)}}{H}}{\beta^*}$ is open.*

Proof. Suppose that A is open in $\dfrac{P^{(n-1)}}{H}$. Then, there is an open subset V in $P^{(n-1)}$ such that $A = \{H[v, e_*] | v \in V\}$. First, we verify that $\pi^{-1}(\pi(A)) = \{H[n, v, e_*] | n \in N, v \in V\}$. Let $H[x, e_*] \in \pi^{-1}(\pi(A))$. So there is $v \in V$ such that $\pi(H[x, e_*]) = \pi(H[v, e_*])$. It implies that $\{H[x, e_*], H[v, e_*]\} \subseteq F(H[a_1, e_*]), ..., H[a_k, e_*])$ for $k \in \mathbb{N}$ and for some $a_1, ..., a_k \in P$. Therefore $H[x, e_*] \in [H[n, v, e_*]] | n \in N, v \in V\}$.

For the converse, let $v \in V$ and $n \in N$. Then, there are $k \in \mathbb{N}$ and $a_1, ..., a_k \in P$ and $h_1, ..., h_k \in H$ and we have

$$\{H[n, v, e_*], H[v, e_*]\} \subseteq F(H[a_1^{-1}, e_*], ..., H[a_k^{-1}, e_*], H[a_k^{-1}, e_*]).$$

Thus, $H[n,v,e_*]\beta^*H[v,e_*]$. So $H[n,v,e_*] \in \pi^{-1}((\{H[v,e_*]|v \in V\}))$. Hence, we conclude that $\pi(A)$ is open in $\dfrac{\frac{P^{(n-1)}}{H}}{\beta^*}$. This yields that $\pi^{-1}(\pi(A)) = \{H[n,v,e_*]|n \in N, v \in V\}$ is open in $\dfrac{P^{(n-1)}}{H}$. More precisely, $f(V,N,e_*)$ is open in $P^{(n-1)}$. □

5 Conclusion

In this work, we have extended the study of polygroups to the setting of n-ary hyperoperations equipped with topological structures. After laying out the foundational definitions, we introduced the concept of topological nAP and analyzed their continuity properties. Our focus on connected topological nAP highlighted how topological connectedness imposes structural constraints and enriches the algebraic framework. Moreover, we examined the role of the fundamental relation in connecting the algebraic and topological aspects of these hyperstructures. The compatibility between this equivalence relation and the underlying topology sheds light on the decomposition of nAPs into simpler components. These results not only generalize known properties from binary hyperstructures to the n-ary case but also open up avenues for further research in topological hyperalgebraic systems and their applications.

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