Chebyshev Finite Difference Method for Fractional Boundary Value Problems

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Abstract. This paper presents a numerical method for fractional differential equations using Chebyshev finite difference method. The fractional derivatives are described in the Caputo sense. Numerical results show that this method is of high accuracy and is more convenient and efficient for solving boundary value problems involving fractional ordinary differential equations.

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1. Introduction

The idea of a derivative which interpolates between the familiar integer order derivatives was introduced many years ago and has gained increasing importance only in recent years due to the development of mathematical models of a certain situations in engineering, materials science, control theory, polymer modelling etc. For example see [20, 22, 25, 26].
Most fractional order differential equations describing real life situations, in general do not have exact analytical solutions. Several numerical and approximate analytical methods for ordinary differential equation

have been extended to solve fractional order differential equations. According to [26], these methods include Adomian decomposition method [6], homotopy perturbation method [23], finite difference method [33], fractional linear multi-step method [19], extrapolation method [11] and predictor-corrector method [7, 10]. Whilst much work has been published on fractional differential equation to date, most of the work is focused on initial value problems. Boundary value problems for fractional differential equations has received attention in [1, 2, 3, 26, 29]. Also other studies that has been done in the field of fractional differential equations can be found in [16, 4, 15].

In numerical analysis, Taylor polynomials are often used to expand functions. Chebyshev polynomials can be used to expand functions. An advantage of using Chebyshev polynomials to expand functions is the good representation of smooth function by finite Chebyshev expansion if the function $y(t)$ is infinitely differentiable [12]. Chebyshev finite difference method (ChFD) has been successfully used in the numerical solution of boundary value problems, boundary layer equations, nonlinear system of second-order boundary value problems and Fredholm integro-differential equations [12, 13, 27, 8].

Since the Chebyshev finite difference method for solving ordinary differential equations is an appropriate method, we have extended it to the fractional differential equations. So, the objective of this work is to use ChFD method to solve boundary value problems involving fractional differential equations.

2. Preliminaries

To develop our proposed technique, we require the following definitions of fractional integral, fractional derivatives and Chebyshev polynomials.

Definition 2.1. [25] *Let $\alpha \in R^+$. The operator J_a^α, defined on the usual Lebesgue space $L_1[a, b]$ by*

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} f(t) dt,$$

$$J_a^0 f(x) = f(x),$$

for $a \leqslant x \leqslant b$, is defined to be the Riemann-Liouville fractional integral operator of order α . Some properties of the operator J_a^α are as follows: For $f \in L_1\left[a, b\right], \alpha, \beta \geqslant 0$ and $\gamma > -1$,

(1) $J_a^\alpha f(x)$ exists for almost every $x \in [a, b]$,

(2) $J_a^\alpha J_a^\beta f(x) = J_a^{\alpha+\beta} f(x)$,

(3) $J_a^\alpha J_a^\beta f(x) = J_a^\beta J_a^\alpha f(x)$,

(4) $J_a^\alpha (x - a)^\gamma = \dfrac{\Gamma(\gamma + 1)}{\Gamma(\alpha + \gamma + 1)} (x - a)^{\alpha+\gamma}$.

Definition 2.2. [25] *The fractional derivative of $f(t)$ in the Caputo sense can be defined by*

$$_aD_*^\alpha f(t) = \frac{1}{\Gamma(m - \alpha)} \int_a^t (t - s)^{m-\alpha-1} f^{(m)}(s)ds.$$

Here $m - 1 < \alpha \leqslant m, m \in N, t > a$.
If $m - 1 < \alpha \leqslant m, t > a$ *then the following properties hold:*

(1) $_aD_*^\alpha k = 0$,

(2) $_aD_*^\alpha (J_a^\alpha f(t)) = f(t)$,

(3) $J_a^\alpha (_aD_*^\alpha) = f(t) - \sum\limits_{k=0}^{m-1} f^{(k)}(a) \dfrac{(t - a)^k}{k!}$.

Definition 2.3. [21] *The Chebyshev polynomials of the first kind of degree n are defined on the interval* $[-1, 1]$ *by*

$$T_n(t) = \cos(n \arccos(t)), \qquad n = 0, 1, ...,$$

Obviously, $T_0(t) = 1$, $T_1(t) = t$ and they satisfy the following recurrence relations:

$$T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t), \quad n = 1, 2, \dots.$$

[5] has given the following approximation of the function $y(t)$

$$y(t) = \sum_{n=0}^{N} {}'' r_n T_n(t), \tag{1}$$

where

$$r_n = \frac{2}{N} \sum_{j=0}^{N} {}'' y(t_j) T_n(t_j),$$

and $t_j = \cos(\frac{j\pi}{N})$, $j = 0, 1, ..., N$ are Gauss-Lobatto points. As in [12] the summation symbol with double primes denotes a sum with both the first and last terms halved.

The derivatives of the Chebyshev function are formed as follows:

$$T_n'(t) = \sum_{\substack{k=0 \\ (n+k)odd}}^{n-1} \frac{2n}{c_k} T_k(t), \tag{2}$$

$$T_n''(t) = \sum_{\substack{k=0 \\ (n+k)even}}^{n-2} \frac{n}{c_k}(n^2 - k^2) T_k(t), \tag{3}$$

where $c_0 = 2$ and $c_i = 1$ for $i \geqslant 1$.

From equation (2) and (3) it can be obtained

$$y'(t) = \frac{4}{N} \sum_{n=0}^{N} {}'' \sum_{j=0}^{N} {}'' \sum_{\substack{k=0 \\ (n+k)odd}}^{n-1} \frac{n}{c_k} y(t_j) T_n(t_j) T_k(t), \tag{4}$$

$$y''(t) = \frac{2}{N} \sum_{n=0}^{N} {}'' \sum_{j=0}^{N} {}'' \sum_{\substack{k=0 \\ (n+k)even}}^{n-2} \frac{n}{c_k}(n^2 - k^2) y(t_j) T_n(t_j) T_k(t), \tag{5}$$

The derivatives of the function $y(t)$ at the points $t_k, k = 0, 1, ...N$ are given by [8, 12, 14]

$$y^{(n)}(t_k) = \sum_{j=0}^{N} d_{k,j}^{(n)} y(t_j), \quad n = 1, 2 \tag{6}$$

where

$$d_{k,j}^{(1)} = \frac{4\mu_j}{N} \sum_{n=0}^{N} \sum_{\substack{l=0 \\ (n+l)odd}}^{n-1} \frac{n\mu_n}{c_l} T_n(t_j) T_l(t_k), \quad k,j = 0,1,...,N, \qquad (7)$$

$$d_{k,j}^{(2)} = \frac{2\mu_j}{N} \sum_{n=0}^{N} \sum_{\substack{l=0 \\ (n+l)even}}^{n-2} \frac{n\mu_n(n^2 - l^2)}{c_l} T_n(t_j) T_l(t_k), \quad k,j = 0,1,...,N,$$

$$(8)$$

with $\mu_0 = \mu_N = \frac{1}{2}$ and $\mu_j = 1$ for $j = 1, 2, ..., N - 1$.

As can be seen from equation (6), the derivatives of $y(t)$ at any point from the Gauss-Lobatto nodes are expanded as a linear combination of the values of the function at these points.

3.   Description of the Method

In this section by using Chebyshev finite difference method, the boundary value problem of fractional order of the form

$$_{-1}D_*^\alpha y(t) + a_2(t)y''(t) + a_1(t)y'(t) + a_0(t)y(t) = f(t), \quad -1 \leqslant t \leqslant 1 \ (9)$$

with the boundary condition

$$y(-1) = \beta_1, \quad y(1) = \beta_2 \qquad (10)$$

where $1 < \alpha \leqslant 2$, $a_0(t), a_1(t), a_2(t)$ and $f(t)$ are given function in $L^2[-1,1]$ and β_1, β_2 are real given numbers is solved.

In order to find the solution $y(t)$ in equation (9), one needs to first calculate equation (9) in Gauss-Lobatto nodes t_k for $k = 1, 2, ..., N - 1$ and use equations (6), (7), (8) as well as definition 2.2 to obtain

$$\frac{1}{\Gamma(2-\alpha)} \int_{-1}^{t_k} (t_k - s)^{-\alpha} \sum_{p=0}^{N}{}'' \sum_{q=0}^{N}{}'' \sum_{\substack{r=0 \\ (p+r)even}}^{p-2} \frac{2p}{Nc_r}(p^2 - r^2)y(t_q)T_p(t_q)T_r(s)ds$$

$$+ \sum_{j=0}^{N} y(t_j) a_2(t_k) d_{k,j}^{(2)} + \sum_{j=0}^{N} y(t_j) a_1(t_k) d_{k,j}^{(1)} + a_0(t_k) y(t_k) = f(t_k). \quad (11)$$

For $k = 0$ and $k = N$ the boundary condition (10) is used to obtain

$$y(t_0) = y(1) = \beta_2, \quad y(t_N) = y(-1) = \beta_1. \quad (12)$$

Therefore equations (11) and (12) generate a set of $N + 1$ algebraic equations, which can be solved for unknown $y(t_0), y(t_1), ..., y(t_{N-1})$ and $y(t_N)$ and thus $y(t)$ in equation (1) can be calculated.

Theorem 3.1. ([18]) *The error* $|E_T(N)| = |D_*^\alpha y(x) - D_*^\alpha y_N(x)|$ *in approximation* $D_*^\alpha y(x)$ *by* $D_*^\alpha y_N(x)$ *is bounded by*

$$|E_T(N)| \leqslant \left| \sum_{i=N+1}^{\infty} r_i \left(\sum_{k=[\alpha]}^{i} \sum_{j=0}^{k-[\alpha]} \theta_{i,j,k} \right) \right|,$$

where $\theta_{i,j,k} = \dfrac{(-1)^{i-k} \ 2i \ (i+k-1)! \ \Gamma\left(k-\alpha+\frac{1}{2}\right)}{h_j \ \Gamma\left(k+\frac{1}{2}\right) \ (i-k)! \ \Gamma(k-\alpha-j+1) \ \Gamma(k+j-\alpha+1)}$, $h_0 = 2, \ h_j = 1, \ j = 1, 2, \ldots.$

Remark 3.2. *For system of fractional differential equation we can apply Chebyshev finite difference method similarly. Therefore, we solve two examples in the next section.*

Remark 3.3. *If the interval under consideration in the boundary condition is* $[a, b]$, *we transform it to the interval* $[-1, 1]$ *by the change of variable,*

$$t = \frac{b-a}{2} x + \frac{b+a}{2}.$$

4. Numerical Examples

In this section, we apply the ChFD method to solve different types of linear fractional differential equations to show the accuracy of the proposed method. All results were computed using Maple 13.

4.1   Ordinary fractional differential equations

Example 4.1.1. Consider the boundary value problem of fractional order

$$_{-1}D_*^{1.4}y(t) + ty(t) = \frac{(t+1)^{\frac{3}{5}}}{4\Gamma(0.6)}(25t - 15) + t^4, \quad -1 \leqslant t \leqslant 1,$$

$$y(-1) = -1, \quad y(1) = 1.$$

The exact solution of this problem is $y(t) = t^3$. We apply the proposed method for this problem and achieve the exact solution.

Example 4.1.2. [26] Consider the boundary value problem

$$_0D_*^\alpha y(t) - \frac{1-t}{(1+t)^2}y(t) = \frac{1}{(1+t)^2}, \quad 1 < \alpha \leqslant 2$$

$$y(0) = 1, \quad y(1) = \frac{1}{2}.$$

We apply the ChFD method and the approximate solution of this problem for different values of α are shown in figure 1.

When $\alpha = 2$, the exact solution is $y(t) = \frac{1}{1+t}$ [32]. For this case we compare our results with methods based on Taylor's expansion [32] and Haar wavelet method [26]. The evaluated absolute errors between the exact solution and numerical solution are shown in Table 1.
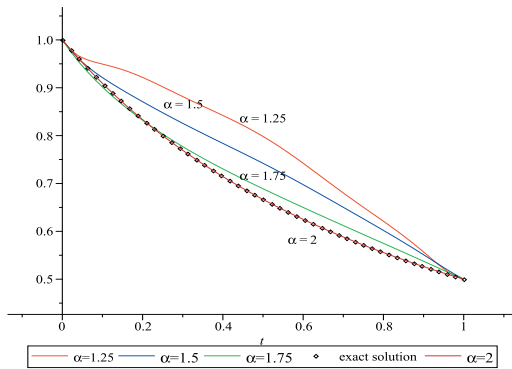

Figure 1. Approximate solution with N=8 and different values of α for example 4.1.2.

Table 1: Comparison of the ChFD method for N=20 with the haar wavelet [26] and methods in ref. [32].

	Methods in ref. [32]		Haar wavlet [26]	ChFD method
t	$|\tilde{y}(t) - y(t)|$	$|\hat{y}(t) - y(t)|$	$|y_{Haar} - y(t)|$	$|y_{ChFD}(t) - y(t)|$
0.1	1.590477×10^{-4}	2.281673×10^{-4}	1.855711×10^{-7}	5.196667×10^{-17}
0.2	1.047506×10^{-4}	2.469834×10^{-4}	1.307581×10^{-7}	8.692174×10^{-17}
0.3	2.784706×10^{-5}	1.858469×10^{-4}	5.238972×10^{-8}	2.484844×10^{-16}
0.4	4.550068×10^{-5}	1.157510×10^{-4}	2.279295×10^{-8}	1.933657×10^{-16}
0.5	1.167467×10^{-4}	6.412753×10^{-5}	6.420743×10^{-9}	1.365740×10^{-17}
0.6	1.812416×10^{-4}	3.403860×10^{-5}	2.6709901×10^{-9}	1.654892×10^{-16}
0.7	2.228086×10^{-4}	2.153643×10^{-5}	5.052727×10^{-9}	1.822994×10^{-16}
0.8	2.213083×10^{-4}	1.719699×10^{-5}	1.285991×10^{-8}	4.105945×10^{-17}
0.9	1.543950×10^{-4}	1.205234×10^{-5}	2.474367×10^{-8}	3.606577×10^{-17}

4.2 Multi-order fractional differential equations

Example 4.2.1. Consider the boundary value problem of fractional order

$$_0D_*^{1.5}y(t) + y'(t) + \frac{y(t)}{e^t} = e^t + erf(\sqrt{t})e^t + 1, \quad 0 \leqslant t \leqslant 1,$$

$$y(0) = 1, \quad y(1) = e,$$

with the exact solution $y(t) = e^t$.

Now we define the maximum errors for $y_N(t) = \sum_{n=0}^{N} {}''r_n T_n(t)$ as

$$e_N = ||y_N(t) - y_{exact}(t)||_\infty = max\{|y_N(t) - y_{exact}(t)|, \ 0 \leqslant t \leqslant 1\}$$

where y_N is the computed result with N and y_{exact} is the exact solution. In Table 2 we give the errors e_N for different values of N.

Table 2: The maximum errors of e_N for different values of N for example 4.2.1

N	4	6	8	10	12
e_N	10^{-4}	10^{-7}	10^{-10}	5×10^{-14}	2.5×10^{-16}

Example 4.2.2. [26] Consider the boundary value problem

$$_0D_*^\alpha y(t) = {}_0D_*^\beta y(t) - e^{t-1} - 1, \quad 1 < \alpha \leqslant 2, \quad 0 < \beta \leqslant 1,$$

$$y(0) = 0, \quad y(1) = 0.$$

For the general case the exact solution of the problem is not known. However, for $\alpha = 2$ and $\beta = 1$ the problem has the exact solution $y(t) = t(1 - e^{t-1})$. This problem was solved numerically in [30] using combined homotopy perturbation method and Green function method. Also in [26] it was solved by Haar wavelet method. We compare ChFD method and the two mentioned methods in Table 3. Computer plots for $\beta = 1$ and different values of α given in Figure 2 show that as α approaches to 2, the corresponding solutions of fractional order differential equation approach to the solutions of integer order differential equation.
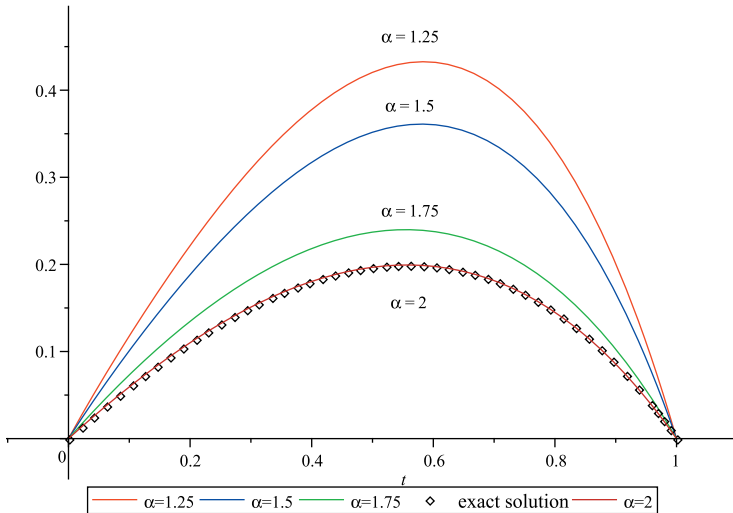

Figure 2. Approximate solution with N=8, $\beta = 1$ and different values of α for example 4.2.2.

Table 3: Comparison of the ChFD method for N=20 with the haar wavelet [26] and HPM [30].

t	HPM [30]	Haar wavlet [26]	exact solution	$|y_{ChFD} - y(t)|$
0.1	0.05934820	0.05934300	0.05934303	1.0213×10^{-26}
0.2	0.11014318	0.11013418	0.11013421	1.0682×10^{-26}
0.3	0.15103441	0.15102438	0.15102441	1.1182×10^{-26}
0.4	0.18048329	0.18047531	0.18047535	1.1682×10^{-26}
0.5	0.19673826	0.19673463	0.19673467	1.2201×10^{-26}
0.6	0.19780653	0.19780792	0.19780797	1.2733×10^{-26}
0.7	0.18142196	0.18142718	0.18142725	1.3297×10^{-26}
0.8	0.14500893	0.14501532	0.14501540	1.3946×10^{-26}
0.9	0.08564186	0.08564623	0.08564632	1.4753×10^{-26}

Example 4.2.3. The Bagley-Torvik equation is of the form

$$ay''(t) + b \, _0D_*^\alpha y(t) + cy(t) = g(t), \quad t \in [0,1],$$

$$y(0) = 0, \quad y(1) = y_0,$$

where $a, b, c \in \mathbb{R}$ and $a \neq 0$. The Bagley-Torvik equation which involves fractional derivative of order $\frac{1}{2}$ or $\frac{3}{2}$ arises in the modelling of the motion of a rigid plate in a Newtonian fluid and a gas in a fluid. According to [26] several authors, such as Podlubny [25], Saha Ray and Bera [28], Diethelm and Ford [9], Diethelm et al. [10] and Wang and Wang [31], have proposed different techniques for the solution of Bagley-Torvik equation. In the present work, we solve this equation with two-point boundary conditions by using the Chebyshev finite difference method.

We choose $\alpha = \frac{3}{2}$, $a = b = c = 1$, $y_0 = 1$ and $g(t) = 4\sqrt{\frac{t}{\pi}} + t^2 + 2$. It can be easily verified that the exact solution is $y(t) = t^2$. For solving this special case we applied the ChFD method with $N = 2$ and obtained the exact solution.

4.3 System of fractional differential equations

Example 4.3.1. Consider the system of fractional differential equations

$$\begin{cases} _0D_*^{1.5}y_1(t) + y_2'(t) + t^2 y_1'(t) = f_1(t), & 0 \leqslant t \leqslant 1 \\ _0D_*^{1.5}y_2(t) + (t-1)y_1''(t) = f_2(t), & 0 \leqslant t \leqslant 1, \end{cases}$$

with the boundary conditions

$$y_1(0) = 0, y_1(1) = 1, \quad y_2(0) = y_2(1) = 0,$$

where $f_1(t) = \frac{4}{\Gamma(0.5)}\sqrt{t} + 2t^3 + 3t^2 - 1$ and $f_2(t) = \frac{8}{\Gamma(0.5)}t^{\frac{3}{2}} + 2(t-1)$. The exact solutions are $y_1(t) = t^2$ and $y_2(t) = t^3 - t$. For solving the above equations we put

$$y_i(t) = \sum_{n=0}^{N} {}''r_n^i T_n(t), \quad r_n^i = \frac{2}{N}\sum_{j=0}^{N} {}'' y_i(t_j)T_n(t_j), \ i = 1, 2,$$

by using ChFD method and boundary conditions we obtain $2N+2$ algebraic equations. By solving this system we obtain unknown coefficients $y_1(t_k)$ and $y_2(t_k)$ for $k = 0, 1, ..., N$. Now if we put $N = 3$ and using the ChFD method for solving the above system of fractional differential equation we obtain the exact solutions $y_1(t) = t^2$ and $y_2(t) = t^3 - t$.

Example 4.3.2 [17] Consider the system of fractional differential equation

$$\begin{cases} {}_0D_*^\alpha y_1(t) + (2t-1)y_1'(t) + \cos(\pi t)y_2'(t) = f_1(t), & 0 \leqslant t \leqslant 1 \\ {}_0D_*^\beta y_2(t) + ty_1(t) = f_2(t), & 0 \leqslant t \leqslant 1, \end{cases}$$

with the boundary conditions

$$y_1(0) = 0, y_1(1) = 0, \quad y_2(0) = y_2(1) = 0,$$

where $1 < \alpha, \beta \leqslant 2$ and $f_1(t), f_2(t)$ are determined by substituting exact solutions in the above equations. For $\alpha = \beta = 2$ the system have exact solutions $y_1(t) = sin(\pi t)$ and $y_2(t) = t^2 - t$. This problem was solved in [17] using variational iteration method. We compare the absolute error for $y_1(t)$ in Table 4. Note that by using VIM and ChFD method we can obtain the exact solution for $y_2(t)$. Now we suppose $\alpha = 1.5, \beta = 1.2$ and the exact solutions for this case is the same as above. We apply ChFD method and obtain the exact solution for $y_2(t)$ and for $y_1(t)$ the absolute errors are reported in Table 5.

Table 4: Comparison of the CHFD method for N=16
with the VIM [17] .

t	VIM [17]	ChFD method
0.1	0.0003	8.1492×10^{-17}
0.2	0.0025	6.5735×10^{-18}
0.3	0.0078	5.0842×10^{-18}
0.4	0.0166	1.2769×10^{-18}
0.5	0.0277	3.5625×10^{-18}
0.6	0.0387	1.2770×10^{-18}
0.7	0.0459	5.0842×10^{-18}
0.8	0.0449	6.5832×10^{-18}
0.9	0.0309	8.1498×10^{-17}

Table 5: Absolute error for $\alpha = 1.5, \beta = 1.2$ with N=16.

t	$|y_{ChFD} - y_1(t)|$
0.1	4.3004×10^{-16}
0.2	7.9298×10^{-16}
0.3	1.1305×10^{-15}
0.4	1.2186×10^{-15}
0.5	1.2259×10^{-15}
0.6	1.2196×10^{-15}
0.7	8.9839×10^{-16}
0.8	6.6944×10^{-16}
0.9	2.1484×10^{-16}

5.   Conclusion

In this paper we have developed a numerical scheme for solving fractional boundary value problems. Our proposed scheme was based on the Chebyshev finite difference method. The results obtained were compared with exact solutions and the results agree well with exact solutions even for small values of N. In some examples, the method achieved the exact solutions. The obtained results indicate that this approach can solve boundary value problems involving fractional ordinary differential equation effectively.

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