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# $m$-Polar Fuzzy $d$-Ideals on $d$-Algebras

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**Abstract.** In recent years, the idea of $m$-polar fuzzy sets has formed the basis of many studies. This idea has been rapidly carried over to many algebraic structures in the last few years. We present this article as the first step in studying this structure in $d$-algebras. It discusses a generalization of the concepts of fuzzy $d$-subalgebra, fuzzy $d$-ideal and bi-polar fuzzy $d$-ideal in $d$-algebras. We introduce the notions of $m$-polar fuzzy $d$-subalgebras, (commutative, closed) $m$-polar fuzzy $d$-ideals on $d$-algebras. Moreover, connections between them and various characterizations are obtained.

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## 1 Introduction

Algebraic structures play an important role in mathematics with wide range of applications in many disciplines such as theoretical physics,

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computer sciences, control engineering, information sciences. One of the frequently used in these application areas is fuzzy structures. Fuzzy sets attract the attention of many researchers as they are one of the most used tools of artificial intelligence when modeling the uncertainty in human thoughts. Fuzzy sets, introduced by Zadeh [23], deal with probabilistic uncertainty associated with the uncertainty of situations, preferences and perceptions. Since 1965, fuzzy set theory, which helps to numerically express uncertainty in solving real-life problems, has continued to advance rapidly. Over the years, fuzzy set theory has found research in many fields such as statistics, engineering, graph theory, computer network, artificial intelligence, pattern recognition, decision making and social sciences. A fuzzy set $\mu$ defined on a crisp set $A$ is characterized by a membership function $\mu(a)$ that assigns an element $a \in A$ to a value in the interval $[0,1]$. Note that $\mu(a)$ "represents the degree of membership of $a$". Any subset $B$ of a set $A$ can be identified with it is characteristic function $\kappa_B : A \to \{0,1\}$ defined by $\kappa_B(b) = \left\{ \begin{array}{ll} 1, & b \in B \\ 0, & b \notin B \end{array} \right\}$. The characteristic functions of subsets of a set are referred to as the crisp fuzzy sets in $A$. The union and intersection of the fuzzy sets $\{\mu_i\}_{i\in I}$ are defined by $(\bigcup\limits_{i\in I} \mu_i)(a) = \sup_{i\in I}\{\mu_i(a)\}$ and $(\bigcap\limits_{i\in I} \mu_i)(a) = \inf_{i\in I}\{\mu_i(a)\}$, respectively. BCK/BCI-algebras first appeared in 1966, in works by Imai and Iséki [13], [14]. These works are created from two distinct approaches: propositional calculi and set theory. The name of BCK/BCI-algebras arises from the combinatories B, C, K, I in combinatory logic. BCK-algebra is an algebraic structure that has many generalizations and is actively studied. One of the generalizations of BCK-algebras, $d$-algebra, was presented by Neggers and Kim [19]. After that, $d$-subalgebra, $d$-ideal, $d^{\#}$-ideal, $d^{*}$-ideal in $d$-algebras were defined by Neggers, Jun and Kim [20]. Since the class of $d$-algebras contains large classes of algebras over fields (which are not BCK-algebras), it has been necessary to investigate more general situation. The concept of fuzzy sets was carried over to BCK-algebras by Xi [22]. Recently, related constructions on the ideals of BCK-algebras have been rapidly progressing (see [16], [1], [18], [21], [8]). The notion of fuzzy subalgebras and $d$-ideals in $d$-algebras were introduced by Akram and Dar [2]. For many properties provided by $d$-algebras and different types of fuzzy ideals on $d$-algebras (see [9],

[17], [12] and their references). Zhang [24] presented bi-polar fuzzy set as a generalization of Zadeh's fuzzy set idea, which extends the classical sets whose elements have positive and negative degrees of membership. Chen et al. [10] extended the concept of bi-polar fuzzy sets to obtain the notion of $m$-polar fuzzy sets. The elements of $m$-polar fuzzy set consist of $m$ components. Multi-polar ($m$-polar) vagueness in data plays a prominent role in several areas of the sciences. In multi-agent decision making, that is, in cases where there are multi-attributes, $m$-polar fuzzy sets and interval-valued fuzzy sets have been proposed to be used in cognitive modeling. In recent years, the idea of $m$-polar fuzzy sets has attracted much attention. For example, $m$-polar fuzzy set theory has been studied on decision making [3], Lie subalgebras [4], groups [11], graph theory [5], [6], [15], BCK/BCI algebras [7], etc. In this paper, we introduce the notions of $m$-polar fuzzy $d$-subalgebras and $m$-polar fuzzy $d$-ideals and examine the relationships between them. Some characterization theorems of edge $d$-algebras are established.

## 2 Preliminaries

This section is devoted to presenting the concepts and results that will form a basis for the next section.

**Definition 2.1.** [19] A $d$-algebra is a non-empty set $\mathcal{A}$ with a constant 0 and a binary operation $*$ such that it satisfies the following axioms: For all $a, b \in \mathcal{A}$,

$(d1)$ $a * a = 0$
$(d2)$ $0 * a = 0$
$(d3)$ $a * b = 0$ and $b * a = 0$ implies $a = b$.

**Definition 2.2.** [9] Suppose $\mathcal{A}$ be a $d$-algebra and $\emptyset \neq I \subseteq \mathcal{A}$. If $a*b \in I$ for all $a, b \in I$, then $I$ is called a $d$-subalgebra of $\mathcal{A}$. Moreover, $I$ is called a $d$-ideal of $\mathcal{A}$ if it satisfies:

$(dI-1)$ $0 \in I$
$(dI-2)$ $a * b \in I$ and $b \in I$ imply $a \in I$.
$(dI-3)$ $a \in I$ and $b \in \mathcal{A}$ imply $a * b \in I$, i.e., $I * \mathcal{A} \subseteq I$.

**Definition 2.3.** [9] Let $\mathcal{A}$ be a $d$-algebra. A binary relation $\leq$ on $\mathcal{A}$ given by $a \leq b$ iff $a * b = 0$. According this relation, $(\mathcal{A}, \leq)$ is a partially ordered set.

**Proposition 2.4.** *[9] A $d$-algebra $\mathcal{A}$ has the following properties: For all $a, b, c \in \mathcal{A}$*

(1) $0 * (a * b) = (0 * a) * (0 * b)$
(2) $((a * c) * (b * c)) * (a * b) = 0$
(3) $a \leq b \Rightarrow a * c \leq b * c$ and $c * b \leq c * a$
(4) $((a * b) * (a * c)) * (c * b) = 0$
(5) $(a * b) * 0 = (a * 0) * (b * 0)$
(6) $a * (b * c) \geq (a * b) * c$
(7) $(a * c) * (b * c) = a * b$
(8) $(a * 0) * 0 = a$.

**Definition 2.5.** [9] Let $\mathcal{A}$ be a $d$-algebra and $a \in \mathcal{A}$. $\mathcal{A}$ is called an edge $d$-algebra if $a * \mathcal{A} = \{a, 0\}$ for any $a \in \mathcal{A}$, where $a * \mathcal{A} = \{a * b | \, b \in \mathcal{A}\}$.

**Lemma 2.6.** *[9] If $\mathcal{A}$ is an edge $d$-algebra, then $a * 0 = a$ for any $a \in \mathcal{A}$.*

**Lemma 2.7.** *[9] Let $\mathcal{A}$ be an edge $d$-algebra. Thus, $(a * b) * c = (a * c) * b$ holds for all $a, b, c \in \mathcal{A}$.*

**Lemma 2.8.** *[9] Let $\mathcal{A}$ be an edge $d$-algebra. Thus, $(a * (a * b)) * b = 0$ holds for all $a, b \in \mathcal{A}$.*

Similar to BCK-algebras, we can write the following definitions in $d$-algebras.

**Definition 2.9.** A non-empty subset $I$ of a $d$-algebra $\mathcal{A}$ is called a commutative $d$-ideal of $\mathcal{A}$ if it satisfies $0 \in I$ and $(a * b) * c \in I$, $c \in I \Rightarrow a * (b \wedge a) \in I$ for all $a, b, c \in \mathcal{A}$, where $a \wedge b = b * (b * a)$.

**Definition 2.10.** [10] An $m$-polar fuzzy set $P_m$ on a non-empty set $\mathcal{A}$ is a mapping $P_m : \mathcal{A} \to [0, 1]^m$. The membership value of every element $a \in \mathcal{A}$ is denoted by $P_m(a) = (p_1 \circ P_m(a), p_2 \circ P_m(a), ..., p_m \circ P_m(a))$, where $p_i \circ P_m : [0, 1]^m \to [0, 1]$ is defined the $i$-th projection mapping.

Note that $[0,1]^m$ ($m$-th power of $[0,1]$) is considered as a poset with the pointwise order $\leq$, where $m$ is an arbitrary ordinal number (we make an appoint that $m = \{n|\, n < m\}$ when $m > 0$), $\leq$ is defined by $a \leq b \Leftrightarrow p_i(a) \leq p_i(b)$ for each $i \in m$ ($a, b \in [0,1]^m$) and $p_i : [0,1]^m \to [0,1]$ is the $i$-th projection mapping. Clearly, $0_m = (0, 0, ...0)$ is the smallest value in $[0,1]^m$ and $1_m = (1, 1, ..., 1)$ is the largest value in $[0,1]^m$.

## 3 $m$-Polar Fuzzy $d$-Subalgebras and $d$-Ideals

Throughout this section, except for some special cases, $\mathcal{A}$ will denote a $d$-algebra.

**Definition 3.1.** An $m$-polar fuzzy set $P_m$ of $\mathcal{A}$ is called an $m$-polar fuzzy $d$-subalgebra if the following property is hold: For all $a, b \in \mathcal{A}$, $P_m(a * b) \geq \inf\{P_m(a), P_m(b)\}$. That is; $\forall a, b \in \mathcal{A}$, $p_i \circ P_m(a * b) \geq \inf\{p_i \circ P_m(a), p_i \circ P_m(b)\}$ for all $i = \overline{1, m}$.

**Example 3.2.** Consider the following $d$-algebra $\mathcal{A} = \{0, a, b, c\}$, which is not a BCK-algebra, since $((a * c) * (a * b)) * (b * c) = a \neq 0$.

**Table 1:** Cayley table for the operation $*$

| * | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | a | 0 | 0 | a |
| b | b | b | 0 | 0 |
| c | c | c | c | 0 |

Define a 4-polar fuzzy set $P_m : \mathcal{A} \to [0,1]^4$ by:

$$P_m(x) = \begin{cases} (0.7, 0.5, 0.4, 0.8), & x = 0 \\ (0.3, 0.4, 0.4, 0.7), & x = a \\ (0.2, 0.1, 0.3, 0.2), & x = b \\ (0.6, 0.5, 0.4, 0.7), & x = c \end{cases}.$$

It is routine to verify that $P_m$ is a 4-polar fuzzy $d$-subalgebra of $\mathcal{A}$.

**Example 3.3.** Let $\mathcal{A} = \{0, a, b, c, d, e\}$ be a $d$-algebra with the following table:

**Table 2:** Cayley table for the operation $*$

| $*$ | 0 | a | b | c | d | e |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| a | a | 0 | e | d | c | b |
| b | b | b | 0 | c | 0 | c |
| c | c | c | b | 0 | b | 0 |
| d | d | b | a | e | 0 | c |
| e | e | c | d | a | b | 0 |

Since $b * \mathcal{A} = \{0, b, c\} \neq \{0, b\}$, $\mathcal{A}$ is not an edge $d$-algebra. Define a 5-polar fuzzy set $P_m : \mathcal{A} \to [0,1]^5$ by:

$$P_m(x) = \left\{ \begin{array}{ll} (0.8, 0.9, 0.8, 0.5, 0.6), & x = 0 \\ (0.7, 0.5, 0.7, 0.4, 0.3), & x = b \\ (0.4, 0.2, 0.5, 0.1, 0.1), & x = a, c, d, e \end{array} \right\}$$

Then, $P_m$ is a 5-polar fuzzy $d$-subalgebra of $\mathcal{A}$.

**Definition 3.4.** For any $m$-polar fuzzy set $P_m$ on $\mathcal{A}$ and $\widehat{\delta} = (\delta_1, \delta_2, ..., \delta_m) \in [0,1]^m$, the set

$$P_m^{\widehat{\delta}} = \left\{ a \in \mathcal{A} |\, P_m(a) \geq \widehat{\delta} \right\}$$

is called the $\widehat{\delta}$-level cut set of $P_m$ and the set

$$P_m^{\widehat{\delta}_s} = \left\{ a \in \mathcal{A} |\, P_m(a) > \widehat{\delta} \right\}$$

is called the strong $\widehat{\delta}$-level cut set of $P_m$.

**Theorem 3.5.** *Let $P_m$ be an $m$-polar fuzzy set of $\mathcal{A}$. Then, $P_m$ is an $m$-polar fuzzy $d$-subalgebra of $\mathcal{A}$ if and only if $P_m^{\widehat{\delta}} \neq \emptyset$ is a $d$-subalgebra of $\mathcal{A}$ for all $\widehat{\delta} = (\delta_1, \delta_2, ..., \delta_m) \in [0,1]^m$.*

**Proof.** Suppose that $P_m$ is an $m$-polar fuzzy $d$-subalgebra of $\mathcal{A}$, $\widehat{\delta} \in [0,1]^m$ and $P_m^{\widehat{\delta}} \neq \emptyset$. Let $a, b \in P_m^{\widehat{\delta}}$. Hence, $P_m(a) \geq \widehat{\delta}$ and $P_m(b) \geq \widehat{\delta}$. Using Definition 3.1, we have $P_m(a * b) \geq \inf \{P_m(a), P_m(b)\} \geq \widehat{\delta}$ and so $a * b \in P_m^{\widehat{\delta}}$. It implies that $P_m^{\widehat{\delta}}$ is a $d$-subalgebra of $\mathcal{A}$. Now, let $P_m^{\widehat{\delta}}$

be a $d$-subalgebra of $\mathcal{A}$. Suppose that there exist $a, b \in \mathcal{A}$ such that $P_m(a * b) < \inf\{P_m(a), P_m(b)\}$. Thus, there exists $\widehat{\delta} = (\delta_1, \delta_2, ..., \delta_m) \in [0,1]^m$ such that $P_m(a * b) < \widehat{\delta} \leq \inf\{P_m(a), P_m(b)\}$. It implies that $a, b \in P_m^{\widehat{\delta}}$, but $a * b \notin P_m^{\widehat{\delta}}$. This is a contradiction. Therefore, $P_m(a * b) \geq \inf\{P_m(a), P_m(b)\}$ for all $a, b \in \mathcal{A}$ and $P_m$ is an $m$-polar fuzzy $d$-subalgebra of $\mathcal{A}$. □

**Corollary 3.6.** *If $P_m$ is an $m$-polar fuzzy $d$-subalgebra of $\mathcal{A}$, then $P_m^{\widehat{\delta_s}} \neq \emptyset$ is a $d$-subalgebra of $\mathcal{A}$ for all $\widehat{\delta} \in [0,1]^m$.*

**Lemma 3.7.** *Every $m$-polar fuzzy $d$-subalgebra $P_m$ of $\mathcal{A}$ satisfies the following:*
*$\forall a \in \mathcal{A}$, $P_m(0) \geq P_m(a)$.*

**Proof.** Straightforward. □

**Proposition 3.8.** *Let $\mathcal{A}$ be an edge $d$-algebra. If every $m$-polar fuzzy $d$-subalgebra $P_m$ of $\mathcal{A}$ satisfies the following inequality: $P_m(a * b) \geq P_m(b)$ for all $a, b \in \mathcal{A}$, then $P_m(a) = P_m(0)$.*

**Proof.** Let $a \in \mathcal{A}$. By hypothesis, we have $P_m(a) = P_m(a * 0) \geq P_m(0)$. Using Lemma 3.7, we get $P_m(a) = P_m(0)$. □

**Definition 3.9.** Let $P_m$ be an $m$-polar fuzzy set of $\mathcal{A}$. Then, $P_m$ is called an $m$-polar fuzzy $d$-ideal if $P_m(0) \geq P_m(a) \geq \inf\{P_m(a * b), P_m(b)\}$ for all $a, b \in \mathcal{A}$. That is, $p_i \circ P_m(0) \geq p_i \circ P_m(a) \geq \inf\{p_i \circ P_m(a * b), p_i \circ P_m(b)\}$ for all $a, b \in \mathcal{A}$, $i = \overline{1, m}$.

**Example 3.10.** Let $\mathcal{A} = \{0, a, b, c, d\}$ be a $d$-algebra (which is not a BCK-algebra) with the Cayley table as follows:

**Table 3:** Cayley table for the operation $*$

| $*$ | 0 | a | b | c | d |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| a | a | 0 | a | 0 | a |
| b | b | b | 0 | c | 0 |
| c | c | c | b | 0 | c |
| d | c | c | a | a | 0 |

Define a 4-polar fuzzy set $P_m : \mathcal{A} \to [0,1]^4$ by

$$P_m(x) = \begin{cases} (0.5, 0.8, 0.8, 0.6), & x = 0 \\ (0.4, 0.6, 0.5, 0.5), & x = a \\ (0.1, 0.2, 0.4, 0.3), & x = b, c, d \end{cases}$$

It is clear that $P_m$ is a 4-polar fuzzy $d$-ideal of $\mathcal{A}$.

**Proposition 3.11.** *Let $P_m$ be an $m$-polar fuzzy $d$-ideal of $\mathcal{A}$. Thus, $a \leq b \Rightarrow P_m(a) \geq P_m(b)$ for all $a, b \in \mathcal{A}$.*

**Proof.** Let $a, b \in \mathcal{A}$ such that $a \leq b$. Hence, we have $a * b = 0$ and so $P_m(a) \geq \inf\{P_m(a * b), P_m(b)\} = \inf\{P_m(0), P_m(b)\} = P_m(b)$. □

**Proposition 3.12.** *Let $\mathcal{A}$ be an edge $d$-algebra, $P_m$ be an $m$-polar fuzzy $d$-ideal of $\mathcal{A}$. Then, the following are equivalent:*

($i$) $P_m(a * b) \geq P_m((a * b) * b)$ for all $a, b \in \mathcal{A}$,
($ii$) $P_m((a * c) * (b * c)) \geq P_m((a * b) * c)$ for all $a, b, c \in \mathcal{A}$.

**Proof.** Suppose that ($i$) holds and $a, b \in \mathcal{A}$. By Lemma 2.7, we have $((a*(b*c))*c)*c = ((a*c)*(b*c))*c$. On the other hand, by Proposition 2.4 (7), $((a * c) * (b * c)) * c = (a * b) * c$. Therefore, $((a * (b * c)) * c) * c = ((a * c) * (b * c)) * c = (a * b) * c$. Using ($i$), we get

$$\begin{aligned} P_m((a * c) * (b * c)) &= P_m((a * (b * c)) * c) \\ &\geq P_m(((a * (b * c)) * c) * c) \\ &= P_m((a * b) * c). \end{aligned}$$

Now, suppose that ($ii$) is valid. Repalcing $b$ by $c$ in ($ii$), we get $P_m((a * c) * (c * c)) = P_m((a * c) * 0) = P_m(a * c)$ and by using ($ii$) and Lemma 2.6, $P_m(a * c) = P_m((a * c) * (c * c)) \geq P_m((a * c) * c)$, since $\mathcal{A}$ is an edge $d$-algebra. □

**Theorem 3.13.** *Let $P_m$ be an $m$-polar fuzzy set of an edge $d$-algebra $\mathcal{A}$. Then, $P_m$ is an $m$-polar fuzzy $d$-ideal of $\mathcal{A}$ if and only if it satisfies $P_m^{\widehat{\delta}} \neq \emptyset \Rightarrow P_m^{\widehat{\delta}}$ is a $d$-ideal of $\mathcal{A}$ for all $\widehat{\delta} \in [0,1]^m$.*

**Proof.** Suppose that $P_m$ is an $m$-polar fuzzy $d$-ideal of $\mathcal{A}$. Let $\widehat{\delta} = (\delta_1, \delta_2, ..., \delta_m) \in [0,1]^m$ be such that $P_m^{\widehat{\delta}} \neq \emptyset$. It is clear that $0 \in$

$P_m^{\widehat{\delta}}$. Let $a, b \in \mathcal{A}$ be such that $a * b \in P_m^{\widehat{\delta}}$ and $b \in P_m^{\widehat{\delta}}$. Thus, we have $P_m(a * b) \geq \widehat{\delta}$ and $P_m(b) \geq \widehat{\delta}$. It follows from Definition 3.9 that $P_m(a) \geq \inf\{P_m(a * b), P_m(b)\} \geq \widehat{\delta}$. Hence, $a \in P_m^{\widehat{\delta}}$. Let $a, b \in \mathcal{A}$ be such that $a \in P_m^{\widehat{\delta}}$. Then, $P_m(a) \geq \widehat{\delta}$ and so

$$\begin{aligned} P_m(a * b) &\geq \inf\{P_m((a * b) * a), P_m(a)\} \\ &= \inf\{P_m((a * a) * b), P_m(a)\} \\ &= \inf\{P_m(0), P_m(a)\} \\ &= P_m(a) \geq \widehat{\delta} \end{aligned}$$

which implies that $a * b \in P_m^{\widehat{\delta}}$.

Conversely, suppose that $P_m^{\widehat{\delta}}$ is a $d$-ideal of $\mathcal{A}$. If there exists $x \in \mathcal{A}$ such that $P_m(0) < P_m(x)$, then $P_m(0) < \widehat{\delta}_x \leq P_m(x)$ for some $\widehat{\delta}_x = (\delta_{x_1}, \delta_{x_2}, ..., \delta_{x_m}) \in [0,1]^m$. Thus, $0 \notin P_m^{\widehat{\delta}_x}$ and we have a contradiction. Therefore, $P_m(0) \geq P_m(a)$ for all $a \in \mathcal{A}$. Let $x, y \in \mathcal{A}$ be such that $P_m(x) < \inf\{P_m(x * y), P_m(y)\}$. Thus, there exists $\widehat{\gamma} = (\gamma_1, \gamma_2, ..., \gamma_m) \in [0,1]^m$ such that $P_m(x) < \widehat{\gamma} \leq \inf\{P_m(x * y), P_m(y)\}$. It implies that $x * y \in P_m^{\widehat{\gamma}}$ and $y \in P_m^{\widehat{\gamma}}$, but $x \notin P_m^{\widehat{\gamma}}$. It is a contradiction and so we have $P_m(a) \geq \inf\{P_m(a * b), P_m(b)\}$ for all $a, b \in \mathcal{A}$. Therefore, $P_m$ is an $m$ -polar fuzzy $d$-ideal of $\mathcal{A}$. □

**Corollary 3.14.** *If $P_m$ is an $m$-polar fuzzy $d$-ideal of an edge $d$-algebra $\mathcal{A}$, then $P_m^{\widehat{\delta}_s} \neq \emptyset$ is a $d$-ideal of $\mathcal{A}$ for all $\widehat{\delta} \in [0,1]^m$.*

**Definition 3.15.** For any element $\theta$ of $\mathcal{A}$, we define the set

$$\mathcal{A}_\theta = \{a \in \mathcal{A} |\, P_m(a) \geq P_m(\theta)\}.$$

Clearly, $\theta \in \mathcal{A}_\theta$ and so $\mathcal{A}_\theta$ is a non-empty subset of $\mathcal{A}$.

**Theorem 3.16.** *Suppose that $\theta$ is an element of an edge $d$-algebra $\mathcal{A}$ and $P_m$ is an $m$-polar fuzzy $d$-ideal of $\mathcal{A}$. Thus, the set $\mathcal{A}_\theta$ is a $d$-ideal of $\mathcal{A}$.*

**Proof.** By Definition 3.9, we have $0 \in \mathcal{A}_\theta$. Let $a, b \in \mathcal{A}$ be such that $a * b \in \mathcal{A}_\theta$ and $b \in \mathcal{A}_\theta$. Hence, $P_m(a * b) \geq P_m(\theta)$ and $P_m(b) \geq P_m(\theta)$. Since $P_m$ is an $m$-polar fuzzy $d$-ideal of $\mathcal{A}$, we get $P_m(a) \geq$

$\inf\{P_m(a*b), P_m(b)\} \geq P_m(\theta)$ and so $a \in \mathcal{A}_\theta$. Let $a, b \in \mathcal{A}$ be such that $a \in \mathcal{A}_\theta$. Thus, we have $P_m(a) \geq P_m(\theta)$. Hence,

$$\begin{aligned} P_m(a*b) &\geq \inf\{P_m((a*b)*a), P_m(a)\} \\ &= \inf\{P_m(0), P_m(a)\} \\ &= P_m(a) \geq P_m(\theta) \end{aligned}$$

and so $a*b \in \mathcal{A}_\theta$. Therefore, $\mathcal{A}_\theta$ is a $d$-ideal of $\mathcal{A}$. □

**Proposition 3.17.** *Let $P_m$ be an $m$-polar fuzzy $d$-ideal of $\mathcal{A}$. If $a*b \leq c$ for all $a, b, c \in \mathcal{A}$, then $P_m(a) \geq \inf\{P_m(b), P_m(c)\}$.*

**Proof.** Suppose that $a*b \leq c$ is valid in $\mathcal{A}$. Thus,

$$\begin{aligned} P_m(a*b) &\geq \inf\{P_m((a*b)*c), P_m(c)\} \\ &= \inf\{P_m(0), P_m(c)\} \\ &= P_m(c) \end{aligned}$$

which implies that $P_m(a) \geq \inf\{P_m(a*b), P_m(b)\} \geq \inf\{P_m(b), P_m(c)\}$ for all $a, b, c \in \mathcal{A}$. □

**Theorem 3.18.** *Every $m$-polar fuzzy $d$-ideal of an edge $d$-algebra $\mathcal{A}$ is an $m$-polar fuzzy $d$-subalgebra.*

**Proof.** Let $P_m$ be an $m$-polar fuzzy $d$-ideal of $\mathcal{A}$ and $a, b \in \mathcal{A}$. Using Lemma 2.7, we get

$$\begin{aligned} P_m(a*b) &\geq \inf\{P_m((a*b)*a), P_m(a)\} \\ &= \inf\{P_m((a*a)*b), P_m(a)\} \\ &= \inf\{P_m(0), P_m(a)\} \\ &\geq \inf\{P_m(a), P_m(b)\}. \end{aligned}$$

□

The following example shows that the converse of Theorem 3.18 is not true in general.

**Example 3.19.** Consider an edge $d$-algebra $\mathcal{A} = \{0, a, b, c\}$ which is given in Example 3.2 :

Define a 3-polar fuzzy set $P_m : \mathcal{A} \to [0,1]^3$ by:

$$P_m(x) = \begin{cases} (0.7, 0.8, 0.3), & x = 0, a \\ (0.5, 0.1, 0.1), & x = b \\ (0.6, 0.4, 0.3), & x = c \end{cases}$$

Thus, $P_m$ is a 3-polar fuzzy $d$-subalgebra of $\mathcal{A}$. But $P_m$ is not a 3-polar fuzzy $d$-ideal of $\mathcal{A}$, since $P_m(b) = (0.5, 0.1, 0.1) < (0.6, 0.4, 0.3) = \inf\{P_m(b * c), P_m(c)\}$.

Theorem 3.18 is not valid in a $d$-algebra, that is, if $\mathcal{A}$ is a $d$-algebra, then there exists an $m$-polar fuzzy $d$-ideal that is not an $m$-polar fuzzy $d$-subalgebra, as seen in the following example.

**Example 3.20.** Let $\mathcal{A} = \{0, a, b, c\}$ be a $d$-algebra with the following Cayley table:

**Table 4:** Cayley table for the operation $*$

| * | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | a | 0 | 0 | b |
| b | b | c | 0 | 0 |
| c | c | c | c | 0 |

Since $a * \mathcal{A} = \{0, a, b\} \neq \{0, a\}$, $\mathcal{A}$ is not an edge $d$-algebra. Define a 3-polar fuzzy set $P_m : \mathcal{A} \to [0,1]^3$ by:

$$P_m(x) = \left\{ \begin{array}{ll} (0.9, 0.8, 0.5), & x = 0 \\ (0.6, 0.7, 0.4), & x = a \\ (0.5, 0.4, 0.3), & x = b \\ (0.3, 0.2, 0.2), & x = c \end{array} \right\}$$

Then, $P_m$ is a 3-polar fuzzy $d$-ideal of $\mathcal{A}$. Since $P_m(b * a) = P_m(c) = (0.3, 0.2, 0.2) < (0.5, 0.4, 0.3) = \inf\{P_m(b), P_m(a)\}$, $P_m$ is not a 3-polar fuzzy $d$-subalgebra of $\mathcal{A}$.

**Definition 3.21.** Let $\mathcal{A}$ be a $d$-algebra. An $m$-polar fuzzy $d$-ideal $P_m$ of $\mathcal{A}$ is called closed if it is also an $m$-polar fuzzy $d$-subalgebra of $\mathcal{A}$.

**Example 3.22.** Let $\mathcal{A} = \{0, a, b, c, d, e\}$ be a $d$-algebra which in Example 3.3. Since $b * \mathcal{A} = \{0, b, c\} \neq \{0, b\}$, $\mathcal{A}$ is not an edge $d$-algebra. Define a 5-polar fuzzy set $P_m : \mathcal{A} \to [0, 1]^5$ by:

$$P_m(x) = \left\{ \begin{array}{cc} (0.9, 0.9, 0.8, 0.5, 0.6), & x = 0 \\ (0.4, 0.2, 0.1, 0.1, 0.1), & x = a, d, e \\ (0.6, 0.4, 0.5, 0.2, 0.2), & x = b, c \end{array} \right\}$$

Then, $P_m$ is a closed 5-polar fuzzy $d$-ideal of $\mathcal{A}$.

**Example 3.23.** Consider an edge $d$-algebra $\mathcal{A} = \{0, a, b, c\}$ in Example 3.2. Define a 4-polar fuzzy set $P_m : \mathcal{A} \to [0, 1]^4$ by:

$$P_m(x) = \left\{ \begin{array}{cc} (0.8, 0.7, 0.9, 0.6), & x = 0 \\ (0.4, 0.6, 0.8, 0.4), & x = a \\ (0.2, 0.5, 0.7, 0.2), & x = b, c \end{array} \right\}$$

Hence, $P_m$ is a closed 4-polar fuzzy $d$-ideal of $\mathcal{A}$.

**Theorem 3.24.** *Let $\mathcal{A}$ be a $d$-algebra and let $P_m$ be an $m$-polar fuzzy set of $\mathcal{A}$ given as follows:*

$$P_m(a) = \left\{ \begin{array}{cc} \widehat{k} = (k_1, k_2, ..., k_m), & a \in \mathcal{A}^+ \\ \widehat{l} = (l_1, l_2, ..., l_m), & otherwise \end{array} \right\}$$

*where $\widehat{k}, \widehat{l} \in [0, 1]^m$ with $\widehat{k} > \widehat{l}$ and $\mathcal{A}^+ = \{a \in \mathcal{A} | \, 0 \leq a\}$. Thus, $P_m$ is a closed $m$-polar fuzzy $d$-ideal of $\mathcal{A}$.*

**Proof.** Since $0 \in \mathcal{A}^+$, we have $P_m(0) = \widehat{k} = (k_1, k_2, ..., k_m) \geq P_m(a)$ for all $a \in \mathcal{A}$. Let $a, b \in \mathcal{A}$. If $a \in \mathcal{A}^+$, then

$$\begin{array}{rcl} P_m(a) & = & \widehat{k} = (k_1, k_2, ..., k_m) \\ & \geq & \inf \{P_m(a * b), P_m(b)\}. \end{array}$$

Suppose $a \notin \mathcal{A}^+$. Two situations arise:

$(I)$ If $a * b \in \mathcal{A}^+$, then $b \notin \mathcal{A}^+$;

$(II)$ If $b \in \mathcal{A}^+$, then $a * b \notin \mathcal{A}^+$.

In both case, we obtain

$$\begin{aligned} P_m(a) &= \widehat{l} = (l_1, l_2, ..., l_m) \\ &= \inf\{P_m(a*b), P_m(b)\}. \end{aligned}$$

For any $a, b \in \mathcal{A}$, if any one of $a$ and $b$ does not belong to $\mathcal{A}^+$, then $P_m(a*b) \geq \widehat{l} = (l_1, l_2, ..., l_m) = \inf\{P_m(a), P_m(b)\}$. If $a, b \in \mathcal{A}^+$, then $a * b \in \mathcal{A}^+$. Thus, we get $P_m(a * b) = \widehat{k} = (k_1, k_2, ..., k_m) = \inf\{P_m(a), P_m(b)\}$. Therefore, $P_m$ is a closed $m$-polar fuzzy $d$-ideal of $\mathcal{A}$. □

**Proposition 3.25.** *Every $m$-polar fuzzy $d$-ideal of an edge $d$-algebra $\mathcal{A}$ is a closed $m$-polar fuzzy $d$-ideal of $\mathcal{A}$.*

**Proof.** The proof is a consequence of Theorem 3.18. □

**Definition 3.26.** Let $\mathcal{A}$ be a $d$-algebra. An $m$-polar fuzzy set $P_m$ of $\mathcal{A}$ is called a commutative $m$-polar fuzzy $d$-ideal of $\mathcal{A}$ if the following conditions are valid:

$(i)$ $P_m(0) \geq P_m(a)$ for all $a \in \mathcal{A}$
$(ii)$ $P_m(a * (b \wedge a)) \geq \inf\{P_m((a * b) * c), P_m(c)\}$ for all $a, b, c \in \mathcal{A}$.
That is,
$(i)$ $p_i \circ P_m(0) \geq p_i \circ P_m(a)$ for all $a \in \mathcal{A}$
$(ii)$ $p_i \circ P_m(a * (b \wedge a)) \geq \inf\{p_i \circ P_m((a * b) * c), p_i \circ P_m(c)\}$ for all $a, b, c \in \mathcal{A}$, for each $i = \overline{1, m}$.

**Example 3.27.** Let $\mathcal{A} = \{0, a, b, c\}$ be an edge $d$-algebra in Example 3.2. Define an $m$-polar fuzzy set $P_m : \mathcal{A} \to [0, 1]^m$ by:

$$P_m(x) = \left\{ \begin{array}{ll} \widehat{k} = (k_1, k_2, ..., k_m), & x = 0, a, b \\ \widehat{l} = (l_1, l_2, ..., l_m), & x = c \end{array} \right\}$$

where $\widehat{k}, \widehat{l} \in [0, 1]^m$ and $\widehat{k} > \widehat{l}$. Then, $P_m$ is a commutative $m$-polar fuzzy $d$-ideal of $\mathcal{A}$.

**Theorem 3.28.** *Let $\mathcal{A}$ be an edge $d$-algebra. Then, $P_m$ is a commutative $m$-polar fuzzy $d$-ideal of $\mathcal{A}$ if and only if $P_m$ is an $m$-polar fuzzy $d$-ideal of $\mathcal{A}$.*

**Proof.** Assume that $P_m$ is a commutative $m$-polar fuzzy $d$-ideal of $\mathcal{A}$ and $a, b \in \mathcal{A}$. Hence, we have

$$\begin{aligned} P_m(a) &= P_m(a * (0 \wedge a)) \\ &\geq \inf\{P_m((a * 0) * b), P_m(b)\} \\ &= \inf\{P_m(a * b), P_m(b)\}, \end{aligned}$$

as desired. Conversely, suppose that $P_m$ is an $m$-polar fuzzy $d$-ideal of $\mathcal{A}$. Using Lemma 2.7 and Proposition 2.4, we have

$$\begin{aligned} ((a * (b \wedge a)) * ((a * b) * c)) * c &= ((a * (b \wedge a)) * c) * ((a * b) * c) \\ &\leq (a * (b \wedge a)) * (a * b) \\ &= (b \wedge a) * (b \wedge a) \\ &= 0 \end{aligned}$$

which implies that $(a * (b \wedge a)) * ((a * b) * c) \leq c$ for all $a, b, c \in \mathcal{A}$. Since $P_m$ is an $m$-polar fuzzy $d$-ideal of $\mathcal{A}$, $P_m(a * (b \wedge a)) \geq \inf\{P_m((a * b) * c), P_m(c)\}$ from Proposition 3.17. Therefore, $P_m$ is a commutative $m$-polar fuzzy $d$-ideal of $\mathcal{A}$. $\square$

**Theorem 3.29.** *In a commutative edge $d$-algebra $\mathcal{A}$, every $m$-polar fuzzy $d$-ideal is a commutative $m$-polar fuzzy $d$-ideal.*

**Proof.** The proof is similar to that of Theorem 3.28. $\square$

**Theorem 3.30.** *Assume that $P_m$ is an $m$-polar fuzzy set of an edge $d$-algebra $\mathcal{A}$. Then, $P_m$ is a commutative $m$-polar fuzzy $d$-ideal of $\mathcal{A}$ if and only if it satisfies $P_m^{\widehat{\delta}} \neq \emptyset \Rightarrow P_m^{\widehat{\delta}}$ is a commutative $d$-ideal of $\mathcal{A}$ for all $\widehat{\delta} \in [0,1]^m$.*

**Proof.** Suppose that $P_m$ is a commutative $m$-polar fuzzy $d$-ideal of $\mathcal{A}$. Then, $P_m$ is an $m$-polar fuzzy $d$-ideal of $\mathcal{A}$ and so each non-empty $\delta$-level cut set $P_m^{\widehat{\delta}}$ of $P_m$ is a $d$-ideal of $\mathcal{A}$. Let $a, b, c \in \mathcal{A}$ such that $(a*b)*c \in P_m^{\widehat{\delta}}$ and $c \in P_m^{\widehat{\delta}}$. Hence, we have $P_m((a * b) * c) \geq \widehat{\delta}$ and $P_m(c) \geq \widehat{\delta}$. Thus, we get $P_m(a * (b \wedge a)) \geq \inf\{P_m((a * b) * c), P_m(c)\} \geq \widehat{\delta}$ which implies that $a * (b \wedge a) \in P_m^{\widehat{\delta}}$. Hence, $P_m^{\widehat{\delta}}$ is a commutative $d$-ideal of $\mathcal{A}$.

Conversely, suppose that $\emptyset \neq P_m^{\widehat{\delta}}$ is a commutative $d$-ideal of $\mathcal{A}$ for all $\widehat{\delta} \in [0,1]^m$. Clearly, $P_m(0) \geq P_m(a)$ for all $a \in \mathcal{A}$. Let $P_m((a *$

$b) * c) = \widehat{k} = (k_1, k_2, ..., k_m)$ and $P_m(c) = \widehat{l} = (l_1, l_2, ..., l_m)$ for all $a, b, c \in \mathcal{A}$. Hence, $(a * b) * c \in P_m^{\widehat{k}}$ and $c \in P_m^{\widehat{l}}$. Without loss of generality, we may suppose that $\widehat{k} \leq \widehat{l}$. Hence, $P_m^{\widehat{l}} \subseteq P_m^{\widehat{k}}$ and so $c \in P_m^{\widehat{k}}$. Since $P_m^{\widehat{k}}$ is a commutative $d$-ideal of $\mathcal{A}$ by hypothesis, we have $a * (b \wedge a) \in P_m^{\widehat{k}}$ which implies that $P_m(a * (b \wedge a)) \geq \widehat{k} = \inf \left\{\widehat{k}, \widehat{l}\right\} = \inf \{P_m((a * b) * c), P_m(c)\}$. Therefore, $P_m$ is a commutative $m$-polar fuzzy $d$-ideal of $\mathcal{A}$. □

**Corollary 3.31.** *If $P_m$ is a commutative $m$-polar fuzzy $d$-ideal of an edge $d$-algebra $\mathcal{A}$, then $P_m^{\widehat{\delta}_s} \neq \emptyset$ is a commutative $d$-ideal of $\mathcal{A}$ for all $\widehat{\delta} \in [0,1]^m$.*

**Theorem 3.32.** *Any commutative $d$-ideal of an edge $d$-algebra $\mathcal{A}$ can be realized as level commutative $d$-ideals of some commutative $m$-polar fuzzy $d$-ideal of $\mathcal{A}$.*

**Proof.** Assume that $I$ is a commutative $d$-ideal of $\mathcal{A}$ and $P_m$ be an $m$-polar fuzzy set in $\mathcal{A}$ defined by:

$$P_m(a) = \left\{ \begin{array}{cc} \widehat{k} = (k_1, k_2, ..., k_m), & a \in I \\ \widehat{0} = (0, 0, ..., 0), & a \notin I \end{array} \right\}$$

where $\widehat{k} \in [0,1]^m$. Let $a, b, c \in \mathcal{A}$. We will divide into the following cases to verify that $P_m$ is a commutative $m$-polar fuzzy $d$-ideal of $\mathcal{A}$. If $(a * b) * c \in I$ and $c \in I$, then $a * (b \wedge a) \in I$. Hence,

$$\begin{aligned} P_m((a * b) * c) &= P_m(c) \\ &= P_m(a * (b \wedge a)) \\ &= \widehat{k} = (k_1, k_2, ..., k_m) \end{aligned}$$

and the required result is verified. If $(a * b) * c \notin I$ and $c \notin I$, then $P_m((a * b) * c) = P_m(c) = \widehat{0} = (0, 0, ..., 0)$. Hence, $P_m(a * (b \wedge a)) \geq \inf \{P_m((a * b) * c), P_m(c)\}$. If exactly one of $(a * b) * c$ and $c$ belongs to $I$, then exactly one of $P_m((a * b) * c)$ and $P_m(c)$ is equal to $\widehat{0} = (0, 0, ..., 0)$. Thus, we have $P_m(a * (b \wedge a)) \geq \inf \{P_m((a * b) * c), P_m(c)\}$ for all $a, b, c \in \mathcal{A}$. Clearly, $P_m(0) \geq P_m(a)$ for all $a \in \mathcal{A}$. Therefore, $P_m$ is a commutative $m$-polar fuzzy $d$-ideal of $\mathcal{A}$ and so $P_m^{\widehat{k}} = I$. □

# 4 Conclusion

Fuzzy sets introduce mathematical concepts when modeling human reasoning. As a generalization of fuzzy sets, bi-polar fuzzy sets with membership degree range $[-1, 1]$ have been studied. Afterwards, many concepts were transferred by defining $m$-polar fuzzy sets as the input of bi-polar fuzzy sets, which are more sensitive when we encounter more than one situation. Because of these reasons, we have introduced a new generalized fuzzy structures of $d$-algebras, namely, $m$-polar fuzzy $d$-subalgebras and (commutative, closed) $m$-polar fuzzy $d$-ideals. We examine the characterizations of $m$-polar fuzzy $d$-subalgebras and $m$-polar fuzzy $d$-ideals. The main motivation that encouraged us to work on this topic is that we think that these structures are portable over different ideal types of $d$-algebras. In the future, the following basic topics can be investigated as a continuation of this article: $m$-polar fuzzy $q$-ideals and their generalizations, interval valued $m$-polar fuzzy $d$-subalgebras and $d$-ideals, $m$-polar (alpha, beta)-fuzzy $d$-ideals. Moreover, $m$-polar cubic set theory can be applied to $d$-algebras.

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