Journal of Mathematical Extension
Vol. 9, No. 3, (2015), 87-96
ISSN: 1735-8299
URL: http://www.ijmex.com

The Average Shadowing Property in Continuous Iterated Function Systems

M. Fatehi Nia
Yazd University

Abstract. In this paper, we introduce a new type of iterated function systems, named; CIFS. Actually in a CIFS we have some flows instead of some functions in iterated function systems. Then, we generalize the notions of average shadowing property, chain transitivity, and attractor sets on a CIFS. It is shown that every uniformly contracting CIFS has the average shadowing property. We also prove that if a CIFS, $\mathcal{F}$ on a compact metric space X has the average shadowing property, then $\mathcal{F}$ is chain transitive, but the converse is not always true. As a result, this proves that if $\mathcal{F}$ is an uniformly contracting CIFS on compact metric space X, then X is the only nonempty attractor of $\mathcal{F}$.

AMS Subject Classification: 37C50; 37C15
Keywords and Phrases: Attractor set, average shadowing, chain recurrent, iterated function systems, uniformly contracting

1. Introduction

The shadowing property of a dynamical system arises from the study related to Anosov diffeomorphisms [4]. "This property is very important and especially desirable in the case of chaotic dynamical systems in which the truncation errors and the approximations of the numerical integration scheme grow drastically fast under forward iterations implying that the numerically obtained orbit diverges from the true orbit after just very few iterations [10]." In [3], Blank introduced the concept of average shadowing property, which is a good tool to characterize Anosov diffeomorphisms. This notion was further studied, with emphasis on connections with other notions known from discrete dynamical systems, by Niu [11], Sakai [13], Gu [7], Park and Zhang [12]. In [8], the

Received: September 2014; Accepted: Jonuary 2015

authors define the average shadowing property for continuous flows and prove the following theorem:

Theorem 1.1. *Let X be a compact metric space and $\varphi : R \times X \to X$ be a continuous flow. If φ has the positive (or negative)-average-shadowing property, then φ is chain transitive.*

On the other hand, basic concepts in topological dynamics like attractors, minimality, transitivity and shadowing can be extended to *iterated function systems* (briefly: IFS) [1, 2, 5, 6]. Specially, Glavan and Gutu defined the shadowing property for a parameterized iterated function system and prove that every uniformly contracting IFS has the shadowing property [5].
In this paper, firstly, we define continuous iterated function systems (briefly: CIFS) on a complete metric space. Then we extend the average shadowing property and chain transitivity notions on a CIFS. Theorem 3.1 shows that every uniformly contracting CIFS has the average shadowing property. Then we give an example that has the average shadowing property. Theorem 3.3 is the main result of this paper. Actually, this theorem and the method of its proof is a generalization of Theorem 1.1. Then, we define the attractor set for a CIFS and as a corollary of Theorem 3.3, we show that average shadowing property implies that the whole of the state space is the only nonempty attractor set. In Example 3.7, we give a CIFS which is chain transitive but does not have the average shadowing property.

2. Definitions

In this section, we introduce some definitions related to the average shadowing property and chain transitivity similar to those given in [5, 8, 9].
Let (X, d) be a complete metric space with a metric d and Λ be a nonempty finite set (as an indexing set). Write $R = (-\infty, +\infty)$. Let $\varphi_\lambda : R \times X \to X$ be a continuous flow, for each $\lambda \in \Lambda$, that is, $\varphi_\lambda : R \times X \to X$ is a continuous map and satisfies the following conditions:
(1) $\varphi_\lambda(0, x) = x$ for any $x \in X$.
(2) $\varphi_\lambda(s, \varphi_\lambda(t, x)) = \varphi_\lambda(s + t, x)$ for any $x \in X$ and $t, s \in R$.
We call $\mathcal{F} = \{X; \varphi_\lambda | \lambda \in \Lambda\}$ as a *continuous iterated function systems (CIFS)*. A sequence $\{x_i\}_{i\geqslant 0}$ of points in X is said to be an orbit of the CIFS $\mathcal{F}$ if for any $i \geqslant 0$, there exists $\lambda_i \in \Lambda$ and $t_i > 0$ such that $\varphi_{\lambda_i}(t_i, x_i) = x_{i+1}$. Given $\delta > 0$ and $T > 0$. A bi-sequence $(\{t_i\}_{a\leqslant i\leqslant b}, \{x_i\}_{a\leqslant i\leqslant b})(-\infty \leqslant a < b \leqslant \infty)$ is said to be a $(\delta, T)-$ pseudo-orbit of the CIFS $\mathcal{F}$ if for any $a \leqslant i \leqslant b$, $t_i \geqslant T$ and there exists $\lambda_i \in \Lambda$ such that $d(\varphi_{\lambda_i}(t_i, x_i), x_{i+1}) < \delta$.
Let $\Lambda^{\mathbb{Z}_+}$ denote the set of all infinite sequences $\{\lambda_i\}_{i\geqslant 0}$ of elements in Λ.

For $\delta > 0$, a bi-sequence $(\{t_i\}_{0\leqslant i<\infty}, \{x_i\}_{0\leqslant i<\infty})$ is said to be a (δ, T)-average pseudo-orbit of $\mathcal{F}$ if $t_i \geqslant T$ and for any $0 \leqslant i < \infty$ there exists a natural number $N = N(\delta) > 0$ and $\sigma = \{\lambda_0, \lambda_1, \lambda_2, ...\}$ in $\Lambda^{\mathbb{Z}_+}$, such that for all $n \geqslant N$,

$$\frac{1}{n}\sum_{i=0}^{n-1} d(x_{i+1}, \varphi_{\lambda_i}(t_i, x_i)) < \delta.$$

A $(\delta, T)-$ average pseudo-orbit $(\{t_i\}_{0\leqslant i<\infty}, \{x_i\}_{0\leqslant i<\infty})$ of the CIFS $\mathcal{F}$ is said to be $\epsilon-$ shadowed in average by the orbit $\{y_i\}_{0\leqslant i<\infty}$, if there is an orientation preserving homeomorphism $\alpha : R \to R$ with $\alpha(0) = 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1}\int_0^{t_i} d(\varphi_{\lambda_i}(\alpha(t), y_i), \varphi_{\lambda_i}(t, x_i))dt < \epsilon,$$

where $\{\lambda_0, \lambda_1, \lambda_2, ...\} \in \Lambda^{\mathbb{Z}_+}$ and $y_{i+1} = \varphi_{\lambda_i}(\alpha(t_i), y_i)$ for all $i \geqslant 0$. A CIFS $\mathcal{F}$ is said to have the average shadowing property if for any $\epsilon > 0$ there is $\delta > 0$ such that every $(\delta, 1)-$ average pseudo-orbit of $\mathcal{F}$ can be $\epsilon-$shadowed in average by some orbit of $\mathcal{F}$.

Definition 2.1. *A CIFS $\mathcal{F} = \{X; \varphi_\lambda | \lambda \in \Lambda\}$ is uniformly contracting, if there is a positive number $\alpha < 1$ such that*

$$\frac{d(\varphi_\lambda(t, x), \varphi_\lambda(t, y))}{d(x, y)} \leqslant \alpha^t,$$

for all $t \geqslant 0$, $\lambda \in \Lambda$ and $x, y \in X$.

Definition 2.2. *Let $x, y \in X$, a finite sequence $\{x_i\}_{0\leqslant i\leqslant k}$ of the CIFS $\mathcal{F}$ is said to be a (δ, T)-chain from x to y if $x_0 = x$, $x_k = y$ and for any $0 \leqslant i < k$ we can find a finite sequence $t_0^i, t_1^i, ..., t_{l_i}^i$ of real numbers and a finite sequence $\mu_0^i, \mu_1^i, \mu_2^i, ..., \mu_{l_i}^i$ of Λ members such that $\Sigma_{j=0}^{l_i} t_j \geqslant T$ and*
$d(x_{i+1}, \varphi_{\mu_{l_i}^i}(t_{l_i}^i, \varphi_{\mu_{l_i-1}^i}(t_{l_i-1}^i, ...(\varphi_{\mu_0^i}(t_0^i, x_i))...) < \delta$

We say that x can be chained to y under $\mathcal{F}$, denoted by $x \to_{\mathcal{F}} y$, if for any $\delta > 0$ and $T > 0$ there is a $(\delta, T)-$chain from x to y. We say that x is chain equivalence to y, denoted by $x \sim_{\mathcal{F}} y$, if $x \to_{\mathcal{F}} y$ and $y \to_{\mathcal{F}} x$. The set $CR(\mathcal{F}) = \{x \in X : x \sim_{\mathcal{F}} x\}$ is said to be chain recurrent set of $\mathcal{F}$ and each point in $CR(\mathcal{F})$ is chain recurrent point of $\mathcal{F}$. A CIFS $\mathcal{F}$ is said to be chain transitive if for any $x, y \in X$, $x \to_{\mathcal{F}} y$.

3. Results

In this section we discuss some properties of the average shadowing property on CIFS. First, we investigate the average shadowing property on uniformly

contracting CIFS. Then we consider the the relation between the average shadowing and chain transitivity properties.
The following theorem is one of the main results of this paper.

Theorem 3.1. *If a CIFS $\mathcal{F} = \{X; \varphi_\lambda | \lambda \in \Lambda\}$ is uniformly contracting, then it has the average shadowing property.*

Proof. Given $\epsilon > 0$. Let α be the uniformly contracting ratio number of CIFS $\mathcal{F}$ and $\delta > 0$ be a number such that $\frac{-1}{\ln \alpha} \frac{1}{1-\alpha} \delta \leqslant \epsilon$. Suppose that the bi-sequence $(\{t_i\}_{0 \leqslant i < \infty}, \{x_i\}_{0 \leqslant i < \infty})$ is a $(\delta, 1)$-average pseudo-orbit of $\mathcal{F}$. So there exists a natural number $N = N(\delta) > 0$ and $\sigma = \{\lambda_0, \lambda_1, \lambda_2, ...\}$ in $\Lambda^{\mathbb{Z}_+}$, such that for all $n \geqslant N$,

$$\frac{1}{n} \sum_{i=0}^{n-1} d(x_{i+1}, \varphi_{\lambda_i}(t_i, x_i)) < \delta.$$

Put $\beta_i = d(x_{i+1}, \varphi_{\lambda_i}(t_i, x_i))$, for all $i \geqslant 0$. Consider an orbit $\{y_i\}_{i \geqslant 0}$ such that $y_0 = x_0$ and $y_{i+1} = \varphi_{\lambda_i}(t_i, y_i)$, for all $i \geqslant 0$. Obviously
$d(\varphi_{\lambda_0}(t, x_0), \varphi_{\lambda_0}(t, y_0)) = 0$ for every $0 \leqslant t \leqslant t_0$ and

$$\begin{aligned} d(\varphi_{\lambda_1}(t, x_1), \varphi_{\lambda_1}(t, y_1)) &\leqslant \alpha^t d(x_1, y_1) \\ &\leqslant \alpha^t (d(x_1, \varphi_{\lambda_0}(t_0, x_0)) + d(\varphi_{\lambda_0}(t_0, x_0), \varphi_{\lambda_0}(t_0, y_0))) \\ &\leqslant \alpha^t \beta_0. \end{aligned}$$

Similarly

$$\begin{aligned} d(\varphi_{\lambda_2}(t, x_2), \varphi_{\lambda_2}(t, y_2)) &\leqslant \alpha^t d(x_2, y_2) \\ &\leqslant \alpha^t (d(x_2, \varphi_{\lambda_1}(t_1, x_1)) + d(\varphi_{\lambda_1}(t_1, x_1), \varphi_{\lambda_1}(t_1, y_1))) \\ &\leqslant \alpha^t (\beta_1 + \alpha^{t_1} d(x_1, y_1)) \\ &\leqslant \alpha^t (\beta_1 + \alpha^{t_1} \beta_0), \end{aligned}$$

and

$$\begin{aligned} d(\varphi_{\lambda_3}(t, x_3), \varphi_{\lambda_3}(t, y_3)) &\leqslant \alpha^t d(x_3, y_3) \\ &\leqslant \alpha^t (d(x_3, \varphi_{\lambda_2}(t_2, x_2)) + d(\varphi_{\lambda_2}(t_2, x_2), \varphi_{\lambda_2}(t_2, y_2))) \\ &\leqslant \alpha^t (\beta_2 + \alpha^{t_2} d(x_2, y_2)) \\ &\leqslant \alpha^t (\beta_2 + \alpha^{t_2} \beta_1 + \alpha^{t_2 + t_1} \beta_0). \end{aligned}$$

So, by induction we have

$$d(\varphi_{\lambda_{i+1}}(t, x_{i+1}), \varphi_{\lambda_{i+1}}(t, y_{i+1})) \leqslant \alpha^t (\beta_i + \alpha^{t_i} \beta_{i-1} + \alpha^{t_i + t_{i-1}} \beta_{i-2} + ... + \alpha^{r_i} \beta_0),$$

where $r_i = t_i + t_{i-1} + ... + t_1$.
Consider the identity map on R as an orientation preserving homeomorphism.

This implies that

$$\begin{aligned}
&\int_0^{t_{i+1}} d(\varphi_{\lambda_{i+1}}(t, y_{i+1}), \varphi_{\lambda_{i+1}}(t, x_{i+1}))dt \\
&\qquad \leqslant \tfrac{\alpha^{t_{i+1}}-1}{\ln\alpha}(\beta_i + \alpha^{t_i}\beta_{i-1} + \alpha^{t_i+t_{i-1}}\beta_{i-2} + ... + \alpha^{r_i}\beta_0) \\
&\qquad \leqslant \tfrac{-1}{\ln\alpha}(\beta_i + \alpha^{t_i}\beta_{i-1} + \alpha^{t_i+t_{i-1}}\beta_{i-2} + ... + \alpha^{r_i}\beta_0).
\end{aligned}$$

For all $i \geqslant 0$. Then

$$\begin{aligned}
&\tfrac{1}{n}\sum_{i=0}^{n-1}\int_0^{t_i} d(\varphi_{\lambda_i}(t, y_i), \varphi_{\lambda_i}(t, x_i))dt \\
&\qquad \leqslant \tfrac{1}{n}\sum_{i=0}^{n-1}\tfrac{-1}{\ln\alpha}(\beta_i + \alpha^{t_i}\beta_{i-1} + \alpha^{t_i+t_{i-1}}\beta_{i-2} + ... + \alpha^{r_i}\beta_0) \\
&\quad \leqslant \tfrac{1}{n}\tfrac{-1}{\ln\alpha}(\beta_0(\alpha^{r_0}+\alpha^{r_1}+...+\alpha^{r_{n-1}})+\beta_1(\alpha^{r_0}+\alpha^{r_1}+...+\alpha^{r_{n-2}})+...+\beta_{n-1}) \\
&\qquad \leqslant \tfrac{1}{n}\tfrac{-1}{\ln\alpha}\sum_{i=0}^{n-1}\tfrac{1}{1-\alpha}\beta_i \\
&\qquad < \tfrac{-1}{\ln\alpha}\tfrac{1}{1-\alpha}\delta \leqslant \epsilon. \quad \square
\end{aligned}$$

Example 3.2. Let $\varphi_1, \varphi_2 : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ be two flows given by:

$$\varphi_1(t, \mathbf{x}) = e^{-2t}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\mathbf{x}$$

and

$$\varphi_2(t, \mathbf{x}) = \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{-3t} \end{bmatrix}\mathbf{x}.$$

This is clear that $\mathcal{F} = \{\mathbb{R}^2; \varphi_1, \varphi_2\}$ is uniformly contracting and by Theorem 3.1 has the average shadowing property.

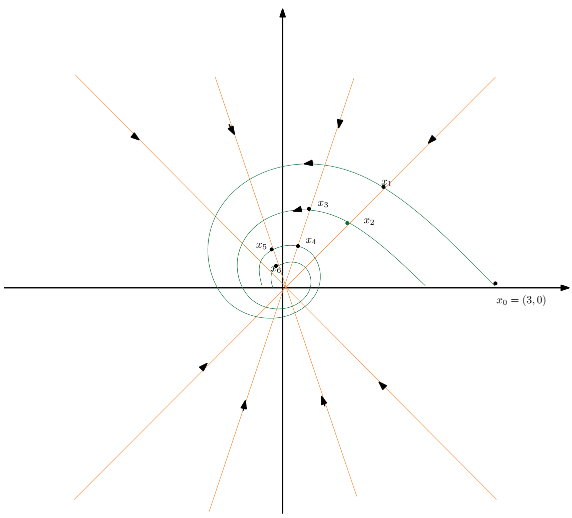

Figure 1. Graphical analysis of an orbit of $\mathcal{F}$ in Example 3.2. $x_1 = \varphi_1(t_0, x_0),\ x_2 = \varphi_2(t_1, x_1),\ x_3 = \varphi_1(t_2, x_2), ...$

The following theorem is the main result of this paper.

Theorem 3.3. *Let X be a compact metric space and $\mathcal{F} = \{X; \varphi_\lambda | \lambda \in \Lambda\}$ be a CIFS. If $\mathcal{F}$ has the average shadowing property then $\mathcal{F}$ is chain transitive.*

Proof. Let x, y be two distinct points of X. Given $\epsilon > 0$ and $T > 0$. Since X is a compact metric space. Let $\delta = \delta(\frac{\epsilon}{6}) > 0$ be a number as in the definition of the average shadowing property of $\mathcal{F}$, that is, every $(\delta, 1)-$ average pseudo-orbit $(\{t_i\}_{0\leqslant i<\infty}, \{x_i\}_{0\leqslant i<\infty})$ of $\mathcal{F}$ can be $\frac{\epsilon}{6}$-shadowed in average by an orbit of $\mathcal{F}$.
Let D be the diameter of X, that is, $D = \sup\{d(x, y) : x, y \in X\}$. Fix a sufficient larger $N_0 > T + 1$ such that $\frac{D}{N_0} < \delta$ and $\kappa \in \Lambda$. Define a periodic sequence $\{x_i\}_{0\leqslant i<\infty}$ such that
$x_i = \varphi_{\lambda_\kappa}([i \mod 6N_0] - 1, x)$ if $[i \mod 6N_0] \in \{1, 2, ..., 3N_0\}$;
$x_i = \varphi_{\lambda_\kappa}([i \mod 6N_0] - 6N_0, y)$ if $[i \mod 6N_0] \in \{3N_0 + 1, ..., 6N_0\}$.
That is, the terms of the sequence from $i = 1$ to $i = 6N_0$ are
$x_1 = x, \quad \varphi_{\lambda_\kappa}(1, x), ..., \varphi_{\lambda_\kappa}(3N_0 - 1, x) = x_{3N_0}$,
$x_{3N_0+1} = \varphi_{\lambda_\kappa}(-(3N_0 - 1, y), ..., \quad \varphi_{\lambda_\kappa}(-1, y), y = x_{6N_0}$.
So for any $n \geqslant N_0$

$$\frac{1}{n}\sum_{i=0}^{n-1} d(\varphi_{\lambda_i}(1, x_i), x_{i+1}) < \frac{\frac{n}{N_0} \times D}{n} \leqslant \frac{D}{N_0} < \delta.$$

Then, the sequence $(\{s_i\}_{0\leqslant i<\infty}, \{x_i\}_{0\leqslant i<\infty})$ is a $(\delta, 1)-$average pseudo-orbit of $\mathcal{F}$ through x, where $s_i = 1$ for all $i \geqslant 0$. Therefore it can be $\frac{\epsilon}{6}$-shadowed in average by the orbit of $\mathcal{F}$, that is, there is a point z in X, a sequence $\sigma = \{\lambda_0, \lambda_1, \lambda_2, ...\} \in \Lambda^{\mathbb{Z}_+}$ and an orientation preserving homeomorphism $\alpha : R \to R$ with $\alpha(0) = 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \int_0^1 d(\varphi_{\lambda_i}(\alpha(t), z_i), \varphi_{\lambda_i}(t, x_i))dt < \frac{\epsilon}{6}.$$

Where $z_0 = z$ and $z_{i+1} = \varphi_{\lambda_i}(\alpha(1), z_i)$ for all $i \geqslant 0$. By proof of Theorem 2.1. in [8], we have the following statements:
(1) There are infinitely many positive integers i and $t_i \in [0, 1]$ such that
$x_i \in \{\varphi_{\lambda_\kappa}(N_0, x), \varphi_{\lambda_\kappa}(N_0 + 1, x), ..., \varphi_{\lambda_\kappa}(2N_0 - 1, x)\}$ and
$d(\varphi_{\lambda_i}(\alpha(t_i), z_i), \varphi_{\lambda_i}(\alpha(t_i, x_i)) < \epsilon$.
(2) There are infinitely many positive integers j and $t_j \in [0, 1]$ such that
$x_j \in \{\varphi_{\lambda_\kappa}(-(2N_0 - 1), y), \varphi_{\lambda_\kappa}(-(2N_0 + 1), y), ..., \varphi_{\lambda_\kappa}(-N_0, y)\}$ and
$d(\varphi_{\lambda_j}(\alpha(t_j), z_j), \varphi_{\lambda_i}(\alpha(t_j, x_j)) < \epsilon$.
Now, we chose $0 < i_0 < j_0$ such that
(*i*) $x_{i_0} \in \{\varphi_{\lambda_\kappa}(N_0, x), \varphi_{\lambda_\kappa}(N_0 + 1, x), ..., \varphi_{\lambda_\kappa}(2N_0 - 1, x)\}$ and

$d(\varphi_{\lambda_{i_0}}(\alpha(t_{i_0}), z_{i_0}), \varphi_{\lambda_{i_0}}(t_{i_0}, x_{i_0}) < \epsilon$.
(ii) $x_{j_0} \in \{\varphi_{\lambda_\kappa}(-(2N_0-1), y), \varphi_{\lambda_\kappa}(-(2N_0+1), y), ..., \varphi_{\lambda_\kappa}(-N_0, y)\}$ and
$d(\varphi_{\lambda_{j_0}}(\alpha(t_{j_0}), z_{j_0}), \varphi_{\lambda_{j_0}}(t_{j_0}, x_{j_0})) < \epsilon$.
It may be assumed
$x_{i_0} = \varphi_{\lambda_\kappa}(i_1, x)$ for some $i_1 \in \{N_0, N_1, ..., 2N_0-1\}$,
and
$x_{j_0} = \varphi_{\lambda_\kappa}(-j_1, y)$ for some $j_1 \in \{N_0, N_1, ..., 2N_0-1\}$.
Let
$w_0 = x$,
$w_1 = x_{i_0}$,
$w_2 = \varphi_{\lambda_{i_0}}(t_{i_0} - T, x_{i_0})$,
$w_3 = \varphi_{\lambda_{i_0}}(\alpha(t_{i_0}), z_{i_0})$,
$w_4 = z_{i_0+N_0}$,
$w_5 = \varphi_{\lambda_{j_0}}(\alpha(t_{j_0}), x_{j_0})$,
$w_6 = y$,
Now we show that $x = w_0, w_1, w_2, w_3, w_4, w_5, w_6 = y$ is a $(\epsilon, T)-$chain.
$d(w_1, \varphi_{\lambda_\kappa}(i_1, w_0)) = d(x_{i_0}, x_{i_0}) = 0$,
$d(w_2, \varphi_{\lambda_{i_0}}(t_{i_0} - T, w_1)) = d(w_2, w_2) = 0$,
$d(w_3, \varphi_{\lambda_{i_0}}(T, w_2)) = d(\varphi_{\lambda_{i_0}}(\alpha(t_{i_0}), z_{i_0}), \varphi_{\lambda_{i_0}}(t_{i_0}, x_{i_0})) < \epsilon$,
$d(w_4, \varphi_{\lambda_{i_0+N_0-1}}[\alpha(1), \varphi_{\lambda_{i_0+N_0-2}}(\alpha(1), ...(\varphi_{\lambda_{i_0}}(\alpha(1) - \alpha(t_{i_0}), w_3))...)]) = d(w_4, w_4)$
$= 0 < \epsilon$.
The proof of the other statements are similar, so there exist a $(\epsilon, T)-$chain from x to y. $\square$

Let $\mathcal{F} = \{X; \varphi_\lambda | \lambda \in \Lambda\}$ and U is a subset of X, we put $\mathcal{F}(U, t) = \cup_{\lambda \in \Lambda} \varphi_\lambda(U, t)$.
For $T > 0$, we say that $y \in \mathcal{F}(U.[T, \infty))$ if there is an orbit $(\{t_i\}_{0 \leqslant i \leqslant n}, \{x_i\}_{0 \leqslant i \leqslant n})$ such that $\Sigma_{i-0}^{n-1} t_i > T$, $x_0 \in U$ and $x_n = y$.
A subset $A \subseteq X$ is said to be an attractor of $\mathcal{F}$ if there is a neighborhood U of A in X such that $\bigcap_{s \geqslant 0}\{\mathcal{F}(U, t) | t \geqslant s\} = A$.
The proof of the following lemma is straightforward from the definitions, so omitted.

Lemma 3.4. *Let X be a compact metric space and $\mathcal{F} = \{X; \varphi_\lambda | \lambda \in \Lambda\}$ be a CIFS.*
(i) There exist an attractor for $\mathcal{F}$;
(ii) Every attractor set is a closed set.

Corollary 3.5. *Let X be a compact metric space and $\mathcal{F}$ be a CIFS on X. If $\mathcal{F}$ has the average shadowing property, then*
(i) X is the only one chain component of $\mathcal{F}$.
(ii) X is the only nonempty attractor of $\mathcal{F}$.
(iii) X is chain recurrent, that is, $CR(\mathcal{F}) = X$.

Proof. We only prove (ii), the others are clear.
Suppose that there exist an attractor $A \neq \emptyset$ with basin attraction U, we show that $A = X$. Let $a \in A$ and $b \in U - A$. By Lemma 3.4, A is a closed set, then we can find $\delta > 0$ such that $B_\delta(b) \cap A = \emptyset$. This implies that $B_{\frac{\delta}{2}}(b) \cap \mathcal{F}(U.[T, \infty)) = \emptyset$, for some $T > 0$. So we can not have a $(\frac{\delta}{3}, T)$-chain from a to b that is a contradict from Theorem 3.3. This implies that $U - A = \emptyset$. Then A is a nonempty closed and open set. Hence $A = X$. $\square$

By Theorem 3.1 and Corollary 3.5 we have the following corollary.

Corollary 3.6. *Let X be a compact metric space and $\mathcal{F}$ be a CIFS on X. If $\mathcal{F}$ is uniformly contracting, then*
(i) X is the only one chain component of $\mathcal{F}$.
(ii) X is the only nonempty attractor of $\mathcal{F}$.
(iii) X is chain recurrent, that is, $CR(\mathcal{F}) = X$.

The following example shows that the converse of Theorem 3.3, is not true. The method is based on the description of the orbits.

Example 3.7. Let ψ and ϕ are two flows on $\mathbb{S}^2$ described in Figures 2 (a) and (b), respectively. In Figure 2 (a) $O_\psi(S) = \{S\}$, $O_\psi(N) = \{N\}$, γ and ν are two periodic orbits parallel to Equator. If $p \in U_1$ then $\{\psi(t,p)\}_{t \geqslant 0}$ asymptotically converges to γ and $\lim_{t \to -\infty}\{\psi(t,p)\} = S$. If $p \in U_2$ then $\{\psi(t,p)\}_{t \geqslant 0}$ asymptotically converges to ν and $\{\psi(t,p)\}_{t \leqslant 0}$ asymptotically converges to γ (when $t \to -\infty$). If $p \in U_3$ then $\{\psi(t,p)\}_{t \geqslant 0}$ asymptotically converges to N and $\{\psi(t,p)\}_{t \leqslant 0}$ asymptotically converges to ν(when $t \to -\infty$). In Figure 2 (b) $O_\phi(S) = \{S\}$, $O_\phi(N) = \{N\}$ and the all other orbits are periodic orbits parallel to Equator.

Now, consider the CIFS, $\mathcal{F} = \{\mathbb{S}^2; \phi, \psi\}$. Let δ be an arbitrary positive number.

By construction of ϕ and ψ we can find a $(\delta, 1)-$average pseudo orbit $(\{t_i\}_{0 \leqslant i < \infty}, \{x_i\}_{0 \leqslant i < \infty})$ of $\mathcal{F}$ such that $A_j = \{i \in N : x_i \in U_j\} \subset N$ has nonzero density, for each $j \in \{1, 2, 3\}$. But every orbit of $\mathcal{F}$ is a subset of closure of one U_j where $j \in \{1, 2, 3\}$. So $\mathcal{F}$ does not have the shadowing property.

Suppose that x and y are two arbitrary point in $\mathbb{S}^2$. Since the orbits of ϕ are periodic and the orbits of ϕ and ψ are transversal, then for every $\delta > 0$ and $T > 0$ we can find a $(\delta, T)-$chain from x to y. Then $\mathcal{F}$ is chain transitive.

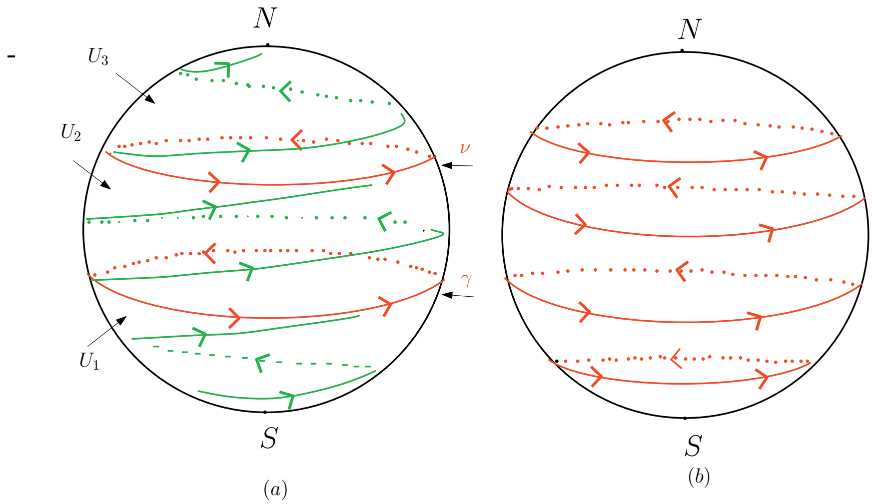

Figure 2.

shadowing property.

Suppose that x and y are two arbitrary point in $\mathbb{S}^2$. Since the orbits of ϕ are periodic and the orbits of ϕ and ψ are transversal, then for every $\delta > 0$ and $T > 0$ we can find a $(\delta, T)-$chain from x to y. Then $\mathcal{F}$ is chain transitive.

4. Conclusion

In this paper, we define the continuous iterated function systems (CIFS) and extend some important notions, like; attractor set, average shadowing property, and chain transitivity on CIFS. Moreover we prove that the average shadowing property implies the chain transitivity, but Example 3.7 shows that the converse statement is not true.

This is very important to find a CIFS on a compact metric space, especially on a compact manifold, which has the average shadowing property. It should be noted, there isn't any well known uniformly contracting CIFS on a compact metric space, it is therefore difficult to construct a CIFS on a compact metric space which has the shadowing property. But I guess that a CIFS on the torus $\mathbb{T}^2$ contains the rational and irrational flows has the average shadowing property. Up to this point and Example 3.7, the consideration of average shadowing property for a CIFS when X is $\mathbb{T}^2$ or $\mathbb{S}^2$ will be a new our research topic. Also, in the further research we are going to define the minimality on a CIFS and discuss about the following question:

Which CIFS with the average shadowing property is a minimal CIFS?

References

[1] M. F. Barnsley and A. Vince, The Conley attractor of an iterated function system, *Bull. Aust. Math. Soc.*, 88 (2013), 267-279.

[2] M. F. Barnsley and A. Vince, Fractal continuation of analytic (fractal) functions, ArXiv: 1209. 6100v1.

[3] M. L. Blank, Metric properties of $\epsilon-$trajectories of dynamical systems with stochastic behaviour, *Ergodic Theory Dynam. Systems*, 8 (1988), 365-378.

[4] R. Bowen, *Equilibrium States and the Ergodic Theory of Axiom A Diffeomorphisms*, Springer, New York, (1975), 68-87.

[5] V. Glavan and V. Gutu, Shadowing in parameterized IFS, *Fixed Point Theory*, 7 (2006), 263-274.

[6] V. Glavan and V. Gutu, Attractors and fixed points of weakly contracting relations, *Fixed Point Theory*, 5 (2004), 265-284.

[7] R. Gu, The average-shadowing property and topological ergodicity, *J. of Comput. and Appl. Math.*, 206 (2007), 796-800.

[8] R. Gu, Y. Sheng, and Z. Xia, The average-shadowing property and transitivity for continuous flows, *Chaos, solitons and Fractals*, 23 (2005), 989-995.

[9] F. N. Hu, Some properties of the chain recurrent set of dynamical systems. *Chaos, Soliton and Fractals*, 14 (2002), 1309-1314.

[10] Z. Maczyn'ska and J. Tabor, Shadowing with multidimensional time in Banach spaces. *J. Math. Anal. Appl.*, 331 (2007), 866-872.

[11] Y. Niu, The average-shadowing property and strong ergodicity, *J. Math. Anal. Appl.*, 376 (2011), 528-534.

[12] J. J. Park and Y. Zhang, Average shadowing properties on compact metric spaces, *Commun. Korean Math. Soc.*, 21 (2006), 355-361.

[13] K. Sakai, Various shadowing properties for positively expansive maps, *Topology. Appl.*, 131 (2003), 15-31.

Mehdi Fatehi Nia
Department of Mathematics
Assistant Professor of Mathematics
Yazd University
Yazd, Iran
E-mail: fatehiniam@yazd.ac.ir