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Two Constraint Qualifications for Non-Differentiable Semi-Infinite Programming Problems Using Fréchet and Mordukhovich Subdifferentials

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Abstract. In this paper, we consider the semi-infinite programming problems with non-differentiable emerging functions. Firstly, we give a counterexample showing that Theorem 3.1 of Ref. [10] is not true. Then, by modifying the assumptions of this theorem, we establish a new necessary Theorem for optimal solution of the problem.

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1. Introduction

Given the locally Lipschitz functions $f,\ g_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, where $i \in I$, and I is an arbitrary set, not necessarily finite (but nonempty). We consider the following semi-infinite programming problem:

$$\text{SIP}: \ \inf \big\{ f(x) \mid g_i(x) \leqslant 0 \quad i \in I \big\}.$$

Optimality conditions of SIP have been studied by many authors; see for example [1, 2, 3, 5, 6, 8, 9, 15] in linear, or convex, or smooth, or locally Lipshitz cases. Recently, Mishra *et al.* in [10] proved that, the Karush-Kuhn-Tucker (KKT, briefly) necessary condition holds for nonsmooth SIPs. However, Theorem 3.1 in [10], which states the KKT property of

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the optimal point, is not correct. The purpose of this paper is to correct the result of this paper.
We organize the paper as follows. In Section 2 basic notations and results of nonsmooth analysis are reviewed. In Section 3, several versions of KKT type necessary optimality conditions for nonsmooth SIP are derived under some suitable qualification conditions.

2. Notations and Preliminaries

In this section we describe the notations used throughout this paper and present some preliminary results on nonsmooth analysis. For more details, discussions, and applications see [12,13]. To simplify the definitions, we assume in this section that φ is a locally Lipschitz function from $\mathbb{R}^n$ into the extended real line $\mathbb{R}\cup\{\infty\}$, finite at $x_0\in\mathbb{R}^n$.

The set

$$\partial_F\varphi(x_0) := \Big\{\xi\in\mathbb{R}^n \mid \liminf_{x\to x_0}\frac{\varphi(x)-\varphi(x_0)-\langle\xi, x-x_0\rangle}{\|x-x_0\|}\geqslant 0\Big\},$$

is called the Fréchet lower subdifferential of φ at x_0. Its elements are referred to as Fréchet subgradients.

The set

$$\partial_L\varphi(x_0) := \limsup_{x\to x_0}\partial_F\varphi(x),$$

is called the Mordukhovich subdifferential. We say that φ is regular at x_0 if $\partial_F\varphi(x_0)=\partial_L\varphi(x_0)$. The finite convex functions and continuously differentiable functions are examples of regular functions.
We observe that for two locally Lipschitz functions φ_1 and φ_2 from $\mathbb{R}^n$ to $\mathbb{R}\cup\{+\infty\}$, finite at x_0, the following subadditive formula holds:

$$\partial_L\big(\varphi_1+\varphi_2\big)(x_0)\subseteq\partial_L\varphi(x_0)+\partial_L\varphi_2(x_0). \tag{1}$$

Notice that if φ is regular at x_0, then the equality fulfilled in (1). Also, if the locally Lipschitz function φ is finite in a neighborhood of x_0, then the subdifferential $\partial_F\varphi(x_0)$ and $\partial_L\varphi(x_0)$ are compact subsets of $\mathbb{R}^n$, with $\partial_F\varphi(x_0)$ convex.

The following theorem will be useful in what follows.

Theorem 2.1. *If x_0 is a local minimizer point of the locally Lipschitz function φ on $\mathbb{R}^n$, then one has $0 \in \partial_L \varphi(x_0)$.*

The Fréchet upper subdifferential, the upper limiting subdifferential, and the symmetric limiting subdifferential of φ at x_0 are respectively defined as:

$$\partial_F^+ \varphi(x_0) := -\partial_F\big(-\varphi\big)(x_0),$$
$$\partial_L^+ \varphi(x_0) := -\partial_L\big(-\varphi\big)(x_0),$$
$$\partial_L^0 \varphi(x_0) := \partial_L \varphi(x_0) \cup \partial_L^+ \varphi(x_0).$$

It is easy to check that when one of the sets $\partial_F^+ \varphi(x_0)$ and $\partial_F \varphi(x_0)$ is not a singleton, the other is empty. This distinguishes the latter constructions from the limiting ones $\partial_L^+ \varphi(x_0)$ and $\partial_L \varphi(x_0)$, which are nonempty simultaneously for ever locally Lipschitzian function. Note that $\partial_L \varphi$ and $\partial_L^+ \varphi$ may be considerably different even in the case of convex and concave functions. The simple example is given by $\varphi(x) := -|x|$ at $x_0 := 0 \in \mathbb{R}$, where $\partial_L \varphi(0) = \{-1, 1\}$ while $\partial_L^+ \varphi(0) = [-1, 1]$. If φ is concave, $\partial_L^+ \varphi(x_0)$ reduces to the classical upper subdifferential of convex analysis.
Recall also that the normal cone of a closed subset $A \subseteq \mathbb{R}^n$ at $x_0 \in A$ is defined by $N_A(x_0) := \partial_L \chi_A(x_0)$, where $\chi_A(x_0)$ denotes the indicator function of A at x_0, i.e., $\chi_A(x) := 0$ for $x \in A$, and $\chi_A(x) := +\infty$ otherwise.
The negative polar cone and strictly negative polar cone of A is respectively defined as

$$A^0 := \big\{y \in \mathbb{R}^n \mid \big\langle y, a \big\rangle \leqslant 0 \quad \text{for all } a \in A\big\},$$
$$A^s := \big\{y \in \mathbb{R}^n \mid \big\langle y, a \big\rangle < 0 \quad \text{for all } a \in A\big\}.$$

The symbol $conv(A)$ denotes the convex hull of A, respectively.
Having the generally infinite index set T, denote by $\mathbb{R}^{(T)}$ the collection of multipliers $\tau := (\tau_t \mid t \in T)$ with $\tau_t \in \mathbb{R}$ and $\tau_t \neq 0$ for finitely many $t \in T$. Let $\mathbb{R}_+^{(T)}$ is defined by

$$\mathbb{R}_+^{(T)} := \big\{\tau \in \mathbb{R}^{(T)} \mid \tau_t \geqslant 0 \ \text{ for all } t \in T\big\}.$$

3. Main Results

Mishra et al. in [10, Theorem 3.1] proved that: if $\widehat{x}$ is an optimal solution for SIP, then there exists some $\lambda = (\lambda_i),\ i \in I$ where $\lambda_i \geqslant 0$ and $\lambda_i \neq 0$ for finitely many $i \in I$ such that

$$0 \in \partial_L f(\widehat{x}) + \sum_{i\in I} \lambda_i \partial_L g_i(\widehat{x}) \quad \text{and} \quad \lambda_i g_i(\widehat{x}) = 0 \quad \forall\, i \in I. \qquad (2)$$

They used [14, Theorem 3.2] in their proof, but this result is not necessarily correct when $|I| = \infty$ (In the proof of [14, Theorem 3.2], if $|I| = \infty$, then it is possible $\delta = 0$). We observe that their result is wrong even when $|I| < \infty$, since they did not consider any constraint qualification. As illustrated by Example 3. below, their theorem does not hold even for the finite differentiable case. Also, their result is not true even for Fritz-Juhn type condition, i.e.,

$$0 \in \lambda_0 \partial_L f(\widehat{x}) + \sum_{i\in I} \lambda_i \partial_L g_i(\widehat{x}) \quad \text{and} \quad \lambda_i g_i(\widehat{x}) = 0 \quad \forall\, i \in I, \ \text{and} \quad \lambda_0 \geqslant 0.$$

Example 3.2 below, is a counterexample for this fact.

Example 3.1. This is a well-known example. Consider the following problem:

$$\inf \big\{ f(x) \mid g(x) \leqslant 0,\ x \in \mathbb{R} \big\},$$

where $f(x) := x$ and $g(x) := x^2$. It is easy to see that $\hat{x} := 0$ is optimal solution of problem, $\partial_L f(\hat{x}) = \{1\}$, $\partial_L g(\hat{x}) = \{0\}$. Thus, there is no scalar $\lambda > 0$ satisfying (2).

Example 3.2. Consider the following problem:

$$\begin{aligned} &\inf u(x_1, x_2) := x_1 \\ \text{s.t.} \quad & v_j(x_1, x_2) := \sup \Big\{ a_1 x_1 + a_2 x_2 \mid (a_1, a_2) \in \\ & conv \{(-\sqrt[j]{\alpha}, -\alpha) \mid 0 \leqslant \alpha \leqslant 1\} \Big\} \leqslant 0 \quad j \in J, \quad (x_1, x_2) \in \mathbb{R}^2, \end{aligned}$$

where $J := \mathbb{N} - \{1\}$.

Obviously, $\big\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geqslant 0,\ x_1 + x_2 \geqslant 0\big\}$ is the feasible set of this problem, and $x_0 := (0, 0)$ is its optimal solution. Since u is continuously differentiable and v_j' s are support function of convex sets, we obtain

$$\partial_L u(x_0) = \{\nabla u(x_0)\} = \{(1,0)\},$$
$$\partial_L v_j(x_0) = conv\left\{(-\sqrt[j]{\alpha}, -\alpha) \mid 0 \leqslant \alpha \leqslant 1\right\}.$$

A short calculation shows that

$$\Big\{ \textstyle\sum_{i\in I} \lambda_j \partial_L v_j(x_0) \mid \lambda_j \geqslant 0, and \lambda_j \neq 0$$

for finitely many $j \in J$, and $\lambda_j v_j(x_0) = 0\Big\}$

$$= \Big\{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geqslant x_1,\ x_1 < 0,\ x_2 < 0\Big\} \cup \big\{(0,0)\big\} := \Pi(x_0).$$

Thus, it is easy to see that for each $\lambda_0 \geqslant 0$ we have

$$(0,0) \notin \lambda_0 \partial_L u(x_0) + \Pi(x_0),$$

Let M denote the feasible solutions of the SIP,

$$M := \{x \in \mathbb{R}^n \mid g_i(x) \leqslant 0 \quad \forall\, i \in I\}.$$

For each $x_0 \in M$, the symbol $I(x_0)$ denotes the set of active constraints at x_0, i.e.,

$$I(x_0) := \{i \in I \mid g_i(x_0) = 0\}.$$

Here, we recall the following theorem from [3, Theorem 3.3].

Theorem 3.3. *Let I be a compact subset of $\mathbb{R}^n$, and the functions $(x,i) \to \xi_i(x)$ ant $x \to \xi(x)$ are continuously differentiable on $\mathbb{R}^n \times I$ and $\mathbb{R}^n$, respectively. suppose further that $\hat{x}$ is an optimal solution of the following problem*

$$\inf\big\{\xi(x) \mid \xi_i(x) \leqslant 0 \quad i \in I\big\}, \tag{3}$$

and there exists a $\widehat{u} \in \mathbb{R}^n$ such that

$$\big\langle \widehat{u}, \nabla \xi_i(\widehat{x})\big\rangle < 0 \qquad \forall\, i \in I(\widehat{x}). \tag{4}$$

Then, there exists a set $J \subseteq I(\widehat{x})$ with $|J| < \infty$, and scalars $\lambda_j > 0$ for $j \in J$, such that

$$\nabla \xi(\widehat{x}) + \sum_{j\in J} \lambda_j \nabla \xi_j(\widehat{x}) = 0.$$

In the purpose of extension the above theorem for nonsmooth SIP, we shall use to the following theorem. This theorem provides important variational descriptions of Fréchet subgradients of nonsmooth functions in terms of smooth supports.

Theorem 3.4. (Variational descriptions of fréchet subgradients) *$\xi \in \partial_F \varphi(x_0)$ if and only if there exist a function $\vartheta : U \to \mathbb{R}$ defined on a neighborhood of x_0 and continuously differentiable on U such that*

$$\vartheta(x_0) = \varphi(x_0) \text{ and } \nabla\vartheta(x_0) = \xi \text{ and } \vartheta(x) \leqslant \varphi(x) \text{ for all } x \in U.$$

The above theorem leads us to the next definition.

Definition 3.5. *For a given SIP,* `condition A` *is said to hold at $x_0 \in M$, if the following conditions are satisfied:*

i) *I is a compact subset of $\mathbb{R}^m$,*
ii) *For some choice $\{\rho_i\}_{i\in I}$ with $\rho_i \in \partial_F\big(-g_i\big)(x_0)$, there exists a neighborhood $\widehat{U}$ of x_0 together with the functions $\vartheta_i : \widehat{U} \to \mathbb{R}$ such that the mapping $(x, i) \to \vartheta_i(x)$ is continuously differentiable on $\widehat{U} \times I$, and*

$$\vartheta_i(x_0) = -g_i(x_0) \text{ and} \nabla\vartheta_i(x_0) = \rho_i \text{and} \vartheta_i(x) \leqslant -g_i(x) \forall\ (x, i) \in \widehat{U} \times I. \quad (5)$$

Remark 3.6. *Owning to the Theorem 3.4, condition A holds for finite problems (i.e., $|I| < \infty$).*

Now, we prove the nonsmooth counterpart of Theorem 3.3.

Theorem 3.7. (KKT type necessary condition for SIP using $\partial^+_{\mathbf{F}}$) *Let $\widehat{x}$ be an optimal solution of SIP, condition A satisfy at $\hat{x}$, and the following qualification condition holds*

$$\Big(\bigcup_{i\in I(\widehat{x})} \partial_F g_i(\widehat{x})\Big)^s \neq \emptyset. \quad (6)$$

If $\partial^+_F f(\hat{x})$ and $\partial^+_F g_i(\hat{x}),\ i \in I(\widehat{x})$ are nonempty, then following assertions hold:

($\mathfrak{a}$) *There are Fréchet upper subgradients* $\xi_i \in \partial_F^+ g_i(\hat{x}),\ i \in I$, *such that for each* $\xi \in \partial_F^+ f(\hat{x})$, *there exist a set* $J \subseteq I(\widehat{x})$ *with* $|J| < \infty$, *and scalars* $\lambda_j > 0$ *for* $j \in J$, *such that*

$$\xi + \sum_{j \in J} \lambda_j \xi_i = 0.$$

($\mathfrak{b}$) *there exist a set* $J \subseteq I(\widehat{x})$ *with* $|J| < \infty$, *and scalars* $\lambda_j > 0$ *for* $j \in J$, *such that*

$$0 \in \partial_F^+ f(\hat{x}) + \sum_{j \in J} \lambda_j \partial_F^+ g_j(\hat{x}).$$

Proof. To prove ($\mathfrak{a}$) under the general assumptions made, choose the elements $\rho_i \in \partial_F\big(-g_i\big)(\hat{x}),\ i \in I$ at which condition A holds. Then, there exist a neighborhood $\widehat{U}_1$ of $\hat{x}$ and functions $\vartheta_i : \widehat{U}_1 \to \mathbb{R}$ such that the mapping $(x, i) \to \vartheta_i(x)$ is continuously differentiable on $\widehat{U}_1 \times I$, and the condition(5) satisfies.
Take $\xi_i(x) := -\vartheta_i(x)$ for all $(x, i) \in \hat{U}_1 \times I$. The condition(5) implies that

$$\xi_i(\widehat{x}) = g_i(\widehat{x}) \text{ and } \nabla \xi_i(\widehat{x}) = -\rho_i \text{ and } \xi_i(x) \geqslant g_i(x) \quad \forall (x, i) \in \widehat{U} \times I.$$

Take $\xi_i := -\rho_i$ for each $i \in I$. With regard to the above relation and the definition of Fréchet upper subdifferential we conclude that

$$\nabla \xi_i(\hat{x}) = \xi_i \in -\partial_F\big(-g_i\big)(\hat{x}) = \partial_F^+ g_i(\hat{x}), \tag{7}$$
$$\hat{x} \in \big\{x \in \hat{U}_1 \mid \xi_i(x) \leqslant 0 \quad i \in I\big\} \subseteq M \cap \hat{U}_1. \tag{8}$$

Take an arbitrary element $\xi \in \partial_F^+ f(\hat{x})$ and apply the variational description from Theorem 3.4 for $-\xi \in \partial_F\big(-f\big)(\hat{x})$. In this way we find a function ϑ form a neighborhood $\widehat{U}_2$ of $\hat{x}$ to $\mathbb{R}$, which is continuously differentiable on $\hat{U}_2$ satisfying

$$\vartheta(\hat{x}) = -f(\hat{x}) \text{ and } \nabla \vartheta(\hat{x}) = -\xi \text{ and } \vartheta(x) \leqslant -f(x) \text{ for all } x \in \hat{U}_2.$$

If $\xi(x) := -\vartheta(x)$, by optimality of $\hat{x}$ we obtain that

$$\nabla\xi(\hat{x}) = \xi \in \partial_F^+ f(\hat{x}), \tag{9}$$
$$\xi(\hat{x}) = f(\hat{x}) \leqslant f(x) \leqslant \xi(x), \quad \text{for all } x \in M \cap \hat{U}_2. \tag{10}$$

Owning to (8) and (10), It is easy to check that $\hat{x}$ is a local optimal solution to the following problem of the type (3)

$$\inf\big\{\xi(x) \mid \xi_i(x) \leqslant 0 \quad i \in I\big\}.$$

Since the functions $(x,i) \to \xi_i(x)$ ant $x \to \xi(x)$ are continuously differentiable and the condition (6) implies (4), owning to Theorem 3.3 we obtain that there exist a $J \subseteq I(\widehat{x}),\ |J| < \infty$, and $\lambda_j > 0,\ j \in J$, such that

$$\nabla\xi(\widehat{x}) + \sum_{j\in J} \lambda_j \nabla\xi_j(\widehat{x}) = 0.$$

Hence, ($\mathfrak{a}$) is proved by (7) and (9). The assertion ($\mathfrak{b}$) is immediate from ($\mathfrak{a}$). $\square$

Remark 3.8. *The qualification condition as (4) is introduced by Cottle in the case $|I| < \infty$ (see [11]). Thus the condition (6) is referred by us to as the* ***`Cottle constraint qualification`****, denoted for briefly by CCQ_F (The index F shows that it is defined by Fréchet subdifferential). The following example shows that we can not replace ∂_F^+ by ∂_F in Theorem 3.7($\mathfrak{a}$), even when the set I is finite. It also shows that the assumption of $\partial_F^+ f(\hat{x}) \neq \emptyset$ and $\partial_F^+ g_i(\hat{x}) \neq \emptyset$ for $i \in I(\widehat{x})$ can not be dropped.*

Example 3.9. We consider the following problem:

$$\inf\big\{f(x) \mid g(x) \leqslant 0,\ x \in \mathbb{R}\big\},$$

where $f(x) := |x|$ and $g(x) := x$. Note that $\hat{x} := 0$ is the local minimizer of this problem, $1 = \partial_F\ g(0)$, CCQ$_F$ holds at $\hat{x}$, and $1 \in \partial_F f(0)$. However, one cannot find any $\lambda > 0$ such that $1 + \lambda 1 = 0$. Thus, the assertion ($\mathfrak{a}$) in Theorem 3.7 is wrong. Also, $\partial_F^+ f(\hat{x}) = \emptyset$ and $\partial_F^+ g(\hat{x}) = \{1\}$, and hence there is not any $\lambda > 0$ satisfying $0 \in \partial_F^+ f(\hat{x}) + \lambda \partial_F^+ g(\hat{x})$.
The following theorems are immediate from Theorem 3.7, and the virtue of $\partial_F^+ \varphi(\hat{x}) \subseteq \partial_L^+ \varphi(\hat{x})$ for $\varphi = f$ and $\varphi = g_i,\ i \in I(\hat{x})$.

Theorem 3.10. (KKT Type Necessary Condition for SIP Using $\partial_{\mathbf{L}}^+$) *Let $\widehat{x}$ be an optimal solution of SIP. Furthermore, suppose that CCQ_F holds at $\hat{x}$, and condition A satisfies at $\hat{x}$. If $\partial_F^+ f(\hat{x})$ and $\partial_F^+ g_i(\hat{x}),\ i \in I(\widehat{x})$ are nonempty, then there exist $\lambda_i \geqslant 0,\ i \in I(\widehat{x})$, where $\lambda_i \neq 0$ for finitely many $i \in I(\hat{x})$, such that*

$$0 \in \partial_L^+ f(\hat{x}) + \sum_{i \in I(\widehat{x})} \lambda_i \partial_L^+ g_i(\hat{x}).$$

We observe that, in the above theorems The Cottle constraint qualification is defined with Fréchet subdifferential. Now, we introduce a qualification condition which is defined by Mordukhovich subdifferential.

Definition 3.11. *Let $x_0 \in M$. We say that the SIP satisfies the* `Basic Constraint Qualification` *(BCQ_L, shortly) at x_0 if*

$$N_M(x_0) \subseteq \bigcup_{\lambda \in S(\lambda)} \Big[\sum_{i \in I} \lambda_i \partial_L g_i(x_0) \Big],$$

where $S(\lambda)$ denotes the set of active constraints multipliers at x_0 defined by

$$S(\lambda) := \big\{ \lambda \in \mathbb{R}_+^{(I)} \mid \lambda_i g_i(x_0) = 0 \text{ for all } i \in I \big\}.$$

The BCQ, firstly was introduced in [4] in relation to the convex optimization problems, and it was extended in [7] to the framework of convex semi-infinite problems.

Theorem 3.12. (KKT type necessary condition for SIP using $\partial_{\mathbf{L}}$) *Let $\widehat{x}$ be an optimal solution of SIP, and BCQ_L satisfy at $\hat{x}$. Then, there exist nonnegative scalars $\lambda_i,\ i \in I(\widehat{x})$, finite number of them not vanishing, such that*

$$0 \in \partial_L f(\hat{x}) + \sum_{i \in I(\widehat{x})} \lambda_i \partial_L g_i(\hat{x}).$$

Proof. Since $\hat{x}$ is an optimal point of f on M, then the function $(f+\chi_M)$ attains its minimum at $\hat{x}$. Employing Theorem 2.1 and estimating (1), we conclude that

$$0 \in \partial_L(f+\chi_M)(\hat{x}) \subseteq \partial_L f(\hat{x}) + \partial_L \chi_M(\hat{x}) = \partial_L f(\hat{x}) + N_M(\hat{x}). \qquad (11)$$

Due to BCQ_L, there is an $\lambda \in S(\lambda)$, satisfying

$$0 \in \partial_L f(\hat{x}) + \sum_{i \in I} \lambda_i \partial_L g_i(\hat{x}).$$

With regard to the definition of $S(\lambda)$, the proof is complete. $\square$

The following theorem is immediately follows from Theorems 3.10 & 3.12, and definition of ∂_L^0.

Theorem 3.13. **(KKT type necessary condition for SIP using $\partial_{\mathbf{L}}^{\mathbf{0}}$)** *Let $\widehat{x}$ is an optimal solution of SIP. Furthermore, suppose that one of the following conditions holds:*

- *BCQ_L at $\hat{x}$.*
- *CCQ_F at $\hat{x}$, condition A at $\hat{x}$, and nonemptyness of $\partial_F^+ f(\hat{x})$ and $\partial_F^+ g_i(\hat{x})$ for $i \in I(\widehat{x})$.*

Then, there exist $\lambda_i \geqslant 0$, $i \in I(\widehat{x})$, where $\lambda_i \neq 0$ for finitely many $i \in I(\hat{x})$, such that

$$0 \in \partial_L^0 f(\hat{x}) + \sum_{i \in I(\widehat{x})} \lambda_i \partial_L^0 g_i(\hat{x}).$$

Remark 3.14. *In Theorem 3.12, ∂_L cannot be replaced by ∂_F, since the subadditive formula (1) which is used in (11) does not hold for ∂_F.*

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