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Some Remarks on Strongly Irreducible Ideals

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Abstract. A proper ideal I of a ring R is called strongly irreducible ideal (briefly, SI-ideal) whenever I contains the intersection of two ideals of R, I contains at least one of those ideals. It is clear that any prime ideal is a strongly irreducible ideal. Therefore, these ideals can be considered as generalizations of the prime ideals. From this point of view, in this paper we extend some results of prime ideals to SI-ideals. As an example, we show that the number of minimal SI-ideals in Noetherian arithmetical rings is finite and in these rings every ideal contains a finite intersection of SI-ideals. Also we give a similar result of the prime avoidance lemma for SI-ideals.

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1 Introduction

A proper ideal I of a ring R is called *strongly irreducible* if for any ideals J and K of R, the inclusion $J \cap K \subseteq I$ implies that either $J \subseteq I$ or

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$K \subseteq I$. Obviously, an ideal I is strongly irreducible if and only if for each $x, y \in R$, $Rx \cap Ry \subseteq I$ implies that $x \in I$ or $y \in I$. Prime ideals are strongly irreducible, not necessarily vice versa (note, the zero ideal in $\mathbb{Z}_8$ is strongly but not prime). Every ideal in a valuation domain is strongly irreducible. Strongly irreducible ideals were first studied by Fuchs, [6], under the name of *primitive ideals*. The term "*strongly irreducible*" was first used by Blair in [2]. The interest and the involvement of the above well-known mathematicians in the concept of strongly ideals definitely shows that these ideals play basic role in algebra. However we should also bring to the attention of the reader that there seems to be a noticable hiatus on the study of these ideals in the literature, however the reader may follow a new finding on these ideals in [7]. We refer the reader to [1], [9], and [13] for more information about strongly irreducible ideals. Throughout this paper, all rings are associative with $1 \neq 0$, not necessarily commutative rings. A ring R is called *reduced* if it has no non-zero nilpotent elements. If S is a multiplicatively closed set of the commutative ring R, then for each ideal I of R, the notation I^e is the ideal generated by $f(I)$ in $S^{-1}R$, and for each ideal J of $S^{-1}R$, the notation J^c is the ideal $f^{-1}(J)$, where $f : R \to S^{-1}R$ is the natural ring homomorphism. Everything needed about rings can be found in any of the books [10, 12].

2 Strongly Irreducible Ideal

Definition 2.1. A proper ideal I of a ring R is called strongly irreducible which we abbreviated as SI-ideal, if for any two ideals J and K of R the inclusion $J \cap K \subseteq I$ implies that $J \subseteq I$ or $K \subseteq I$.

Clearly that it follows immediately (by induction) from definition an ideal I of R is strongly irreducible if, whenever I contains the intersection of a finite list of ideals of R, I contains at least one of the ideals in the list. Obviously, in valuation rings and more generally in uniserial rings (a ring is called uniserial or chain ring if the lattice of all its ideals is linearly ordered by inclusion), any ideal is an SI-ideal. Indeed, a ring R is uniserial if and only if any proper ideal of R is an SI-ideal.

In the next proposition some basic properties about SI-ideal in commutative rings are described ([9, lemma 2.2]). we should emphasize that we have already succeded to extend these results of [9], to duo rings, see [8] (note, by a duo ring we mean a ring in which every one sided ideal is two sided). Recall that a ring R is said to be arithmetical if the lattice of all its ideals is distributive, that is, for any ideals I, J and K of R: $I + (J \cap K) = (I + J) \cap (I + K)$ or equivalently, $I \cap (J + K) = (I \cap J) + (I \cap K)$.

Proposition 2.2. *Let I be an ideal in a ring R. Then:*

(1) *If I is strongly irreducible ideal, then I is irreducible. In particular, if R is Noetherian, then every strongly irreducible ideal is primary.*

(2) *If I is a prime ideal, then I is strongly irreducible.*

(3) *If R is an arithmetical ring, then I is irreducible iff I is strongly irreducible.*

(4) *If $S \subseteq R$ is a multiplicatively closed set and if I^e is strongly irreducible of $S^{-1}R$, then I^{ec} is strongly irreducible ideal of R.*

(5) *If I is a strongly irreducible primary ideal and $S \subseteq R$ is a multiplicatively closed set such that $I \cap S = \emptyset$, then I^e is strongly irreducible ideal of $S^{-1}R$.*

(6) *Let I be a P-primary ideal, $S = R - P$, and I^e is strongly irreducible ideal of R_P, then I is strongly irreducible.*

(7) *If I strongly irreducible ideal in R and if H is an ideal contained in I, then I/H is strongly irreducible ideal in R/H. The converse holds if R is an arithmetical ring.*

(8) *A principal primary ideal of a UFD is strongly irreducible.*

The next theorem which is in [9], is also extended to duo rings in [8].

Theorem 2.3. *Let (R, M) be a local commutative ring and let I be a strongly irreducible M-primary ideal in R. If $I \subset (I :_R M)$ then:*

(1) *$(I :_R M)$ is a principal ideal.*

(2) *$I = (I :_R M)M$.*

(3) *For each ideal J in R either $J \subseteq I$ or $(I :_R M) \subseteq J$.*

The next proposition characterizes the SI-ideals of the finite product of rings and the full matrix ring.

Proposition 2.4.

(1) *Let $R_1, R_2, \cdots, R_n$ be any rings. Then any SI-ideal in $R_1 \times R_2 \times \cdots \times R_n$ is of the form $R_1 \times \cdots \times R_{i-1} \times I_i \times R_{i+1} \times ... \times R_n$ where I_i is an SI-ideal in R_i.*

(2) *J is an SI-ideal in $M_n(R)$ if and only if there exists an SI-ideal I in R such that $J = M_n(I)$.*

Proof. (1). By induction, it is enough to establish the case $n = 2$. Let $J = I_1 \times I_2$ be an SI-ideal of $R_1 \times R_2$. We show that either $I_1 = R_1$ or $I_2 = R_2$. Assume, for a contradiction, $I_1 \neq R_1$ and $I_2 \neq R_2$. Take the ideal $(R_1 \times 0) \cap (0 \times R_2) \subseteq J$. This implies that either $(R_1 \times 0) \subseteq J$ or $(0 \times R_2) \subseteq J$, a contradiction. Clearly that If I_1 and I_2 are SI-ideals in R_1 and R_2, respectively, then $I_1 \times R_2$ and $R_1 \times I_2$ are SI-ideals of $R_1 \times R_2$.
(2). By the fact that every ideal of $M_n(R)$ is of the form $M_n(I)$, where I is an ideal of R, the proof is straightforward. □

Let R and T be two rings, let J be an ideal of T and let $f : R \to T$ be a ring homomorphism. According to [3], the following ring construction called the amalgamation of R with T along J with respect to f is a subring of $R \times T$ defined by

$$R \bowtie^f J := \{(r, f(r) + j) : r \in R \ , j \in J\} .$$

This construction generalizes amalgamated duplication of a ring along an ideal that introduced and studied by D'Anna and Fontana in [4], which is the subring of $R \times R$ given by

$$R \bowtie I := \{(r, r + i) | r \in R \ , i \in I\} .$$

Our next result establish the transfer of some of strongly irreducible ideals in amalgamation of rings.

Theorem 2.5. *Let R and T be two rings and $f : R \to T$ be a ring homomorphism. For an ideal I of R and an ideal J of T, the ideal $I \bowtie^f J$ is an SI-ideal of $R \bowtie^f J$ if and only if I is an SI-ideal of R.*

Proof. Assume that $I \bowtie^f J$ is an SI-ideal of $R \bowtie^f J$. Let K and L be two ideals of R satisfy $K \cap L \subseteq I$. Thus, $(K \bowtie^f J) \cap (L \bowtie^f J) \subseteq I \bowtie^f J$. By our assumption, we deduce that either $(K \bowtie^f J) \subseteq I \bowtie^f J$ or $(L \bowtie^f J) \subseteq I \bowtie^f J$ and so either $K \subseteq I$ or $L \subseteq I$. This means that I is an SI-ideal of R. Conversely, assume that I is an SI-ideal of R. Let H be an ideal of $R \bowtie^f J$ and set $I_H = \{a \in R | (a, f(a) + j) \in H \text{ for some } j \in J\}$. Let $H_1 \cap H_2 \subseteq I \bowtie^f J$. Obviously, $I_{H_1} \cap I_{H_2} \subseteq I$. By our assumption, we infer that either $I_{H_1} \subseteq I$ or $I_{H_2} \subseteq I$. and hence we conclude that either $H_1 \subseteq I \bowtie^f J$ or $H_2 \subseteq I \bowtie^f J$, as desired. □

It is easy to see that if $\{Q_\lambda\}_{i\in\Lambda}$ be any chain of SI-ideals then $\bigcap_{\lambda\in\Lambda} Q_\lambda$ is an SI-ideal. Using Zorn's Lemma this fact shows that for every proper ideal, say I, there exists an SI-ideal over I which is minimal with respect to this property, see [1, Theorem 2.1]. Let us define this concept precisely.

Definition 2.6. Let I be an ideal of a ring R. A minimal SI-ideal over I is any SI-ideal of R, say Q, such that $I \subseteq Q$ and Q is minimal with respect to this property. A minimal SI-ideal over zero ideal is called minimal SI-ideal.

Remark 2.7. Since any proper ideal of a ring R is contained in a prime ideal of R and since every prime ideal is an SI-ideal, hence for every proper ideal there exists a minimal SI-ideal over it.

Example 2.8. In $\mathbb{Z}_{24}$, the generated ideal by 8 is a minimal SI-ideal. In $\mathbb{Z}_n$, the zero ideal is an SI-ideal if and only if n is a power of a prime number.

Recall that every prime ideal contains a minimal prime ideal. The next proposition shows that the same result is also holds for SI-ideals.

Proposition 2.9. *Any strongly irreducible ideal Q in a ring R contains a minimal strongly irreducible ideal.*

Proof. Let Σ be the set of those strongly irreducible ideals of R which are contained in Q. Since $Q \in \Sigma$ and since the intersection of any chain in Σ lies in Σ, so by Zorn's Lemma, Σ has minimal element with respect to inclusion. Clearly that such element is a minimal SI-ideal contained in Q. □

Clearly that the intersection of all minimal strongly irreducible ideals contains the intersection of all strongly irreducible ideals, so by Proposition 2.9, the inclusion is actually an equality.

It was shown that if S is a multiplicatively closed set in commutative ring R and if I^e is strongly irreducible in $S^{-1}R$; then I^{ec} is strongly irreducible in R and if I is a strongly irreducible primary ideal of R, then I^e is strongly irreducible in $S^{-1}R$.

In the next lemma, we prove two basic properties concerning minimal strongly irreducible ideals.

Lemma 2.10. *Let I be an ideal in a Noetherian commutative ring R and S is a multiplicatively closed set in R such that $I \cap S = \emptyset$. Then:*

(1) *If I is a minimal SI-ideal, then I^e is minimal SI-ideal.*

(2) *If I^e is a minimal SI-ideal, then I^{ec} is minimal SI-ideal.*

Proof. For (1) let J be an SI-ideal of $S^{-1}R$ and $J \subseteq I^e$. Then $J^c \subseteq I^{ec}$, but $I^{ec} = I$ (note, in Noetherian rings, any SI-ideal is primary) and J^c is strongly irreducible in R (by Proposision 2.2(4)). Therefore $J^c = I$ and so $J = J^{ce} = I^e$.
For (2) let J be an SI-ideal of R and $J \subseteq I^{ec}$. Then $J^e \subseteq I^{ece} = I^e$, and since J^e is an SI-ideal of R, so $J^e = I^e$ and hence $J = J^{ec} = I^{ec}$. □

It has been shown in [9, Proposition 3.5], that if I is an M-primary SI-ideal in the local Noetherian ring (R, M) with positive heitgh, then I is a regular ideal (i.e., every nonzero element of I is not a zero divisor). In [8, Proposition 3.4], a stronger result is proved that under the above conditions, R must be an integral domain (note, when R is a domain then manifestly every nonzero element of I is a nonzero divisor, indeed, a much stronger result than [9, Proposition 3.5]). Although, the proof of the following result is given for duo rings, however for the sake of the reader we present a proof for commutative ring.

Proposition 2.11. *Let (R, M) be a local commutative Noetherian ring and I be an M-primary ideal of R. If $I \neq M$ and $\mathrm{ht}(M) > 0$, then R is a domain.*

Proof. By Theorem 2.3, $(I :_R M) = Rx$ and for each ideal J of R, $J \subseteq I$ or $(I :_R M) \subseteq J$. Now, assume that Q is a minimal prime ideal of R. Then $Q \subseteq I$ or $(I :_R M) \subseteq Q$. If $(I :_R M) \subseteq Q$, then $\mathrm{ht}(I) = 0$, and this contradicts the hypothesis that $\mathrm{ht}(I) > 0$. So we may assume that $Q \subseteq I$. Hence we have $Q \subseteq I \subseteq (I :_R M) = Rx$ and notice that $x \notin Q$, for $(I :_R M) \nsubseteq Q$. To complete the proof, we show that $Q = 0$. For this purpose, assume that y is an arbitrary element of Q. Then there exists $a_1 \in R$ such that $y = a_1 x$. Since $x \notin Q$, hence $a_1 \in Q$. Similarly, for a_1 we get an element a_2, which must be in Q, such that $a_1 = a_2 x$. By continuing this process, we get elements $a_1, a_2, \cdots$ of R such that $y = a_1 x = a_2 x^2 = \cdots$ and $a_1 R \subseteq a_2 R \subseteq \cdots$. But R is Notherian, so there exists a positive integer n, such that $a_n R = a_{n+1} R$. Therefore $a_{n+1} = a_n t$ for some $t \in R$. In this case $a_n x^n = a_{n+1} x^{n+1} = a_n t x^{n+1}$. Therefore $a_n x^n (1 - tx) = \circ$ and since $1 - tx$ is an invertible element of R, we get $y = a_n x^n = \circ$. Hence $Q = \circ$ and this means that R must be an integral domain. $\square$

Corollary 2.12. *Let I be an SI-ideal ideal of a Noetherian ring R. If $\sqrt{I} = P$, $I \neq P$ and $ht(P) > 0$, then R_P is a domain.*

Recall that if Q is an SI-ideal, then it is an irreducible ideal, therefore in Noetherian ring every SI-ideal is primary. On the other hand if Q is P-primary, it does not necessarily imply that P is a minimal prime ideal. Here we raise the question that if Q is a minimal SI-ideal which is P-primary, then is P a minimal prime ideal? In the next result, we have given a partial answer to this question. We should also recall that for any ideal I in a ring R, there exists a prime ideal P which is minimal over I. However P may not be a minimal prime ideal. In the next result we suprisingly notice that when I is an SI-ideal then P is indeed a minimal prime ideal.

Corollary 2.13. *Let I be a minimal SI-ideal in a commutative Noetherian ring R and P a minimal prime ideal over I, then P is a minimal prime ideal.*

Proof. If $I = P$, the assertion holds. Now let $I \neq P$. In this case we also show that P is a minimal prime. If not, then $ht(P) > 0$. Therefore, by the previous corollary R_P must be a domain. Since I^e is a minimal SI-ideal in R_P (Corollary 2.10 (1)), we infer that $I^e = \circ$, which implies that $ht(P) = 0$, a contradiction. $\square$

The following result is the counterpart of the well-known result that every prime ideal is essential or minimal prime.

Proposition 2.14. *Let Q be an SI-ideal. Then Q is essential or minimal strongly irreducible.*

Proof. Suppose that Q is not essential, so there exists a nonzero ideal I such that $I \cap Q = 0$. Now let $Q^{'}$ be an SI-ideal and $Q^{'} \subseteq Q$. Since $I \cap Q = 0 \subseteq Q^{'}$ and $I \nsubseteq Q^{'}$, we infer that $Q \subseteq Q^{'}$. Therefore, $Q = Q^{'}$ and this completes the proof. $\square$

Definition 2.15. A subset S of a ring R is called an *i-system* of R if for any two ideals I and J of R, $(I \cap J) \cap S \neq \emptyset$ whenever $I \cap S \neq \emptyset$ and $J \cap S \neq \emptyset$.

Clearly, every m-system is an i-system. Also, an ideal Q of a ring R is an SI-ideal if and only if $R - Q$ is an i-system.

Remark 2.16. Suppose that Q is an ideal of an arithmetical ring R. If Q is not an SI-ideal, then there are ideals I, J in R such that $Q \subset I$, $Q \subset J$ and $I \cap J \subseteq Q$.

Proof. Since Q is not an SI-ideal, hence there are two ideals A, B such that $A \cap B \subseteq Q$ but $A \nsubseteq Q$ and $B \nsubseteq Q$. Now, set $I = A + Q$ and $J = B + Q$. In this case, it is clear that I and J are the desired ideals. $\square$

Lemma 2.17. *Let R be an arithmetical ring and S an i-system of R. If Q is an ideal of R which is disjoint from S and is maximal with respect to this property, then Q is an SI-ideal.*

Proof. Assume that Q is not strongly irreducible, then there exist two ideals I and J in R such that $I \cap J \subseteq Q$ but $I \nsubseteq Q$ and $J \nsubseteq Q$. By the maximality of Q, $(I + Q) \cap S \neq \emptyset$ and $(J + Q) \cap S \neq \emptyset$. Therefore, $[(I + Q) \cap (J + Q)] \cap S \neq \emptyset$. Since $[(I + Q) \cap (J + Q)] = (I \cap J) + Q = Q$, we get $Q \cap S \neq \emptyset$, a contradiction. $\square$

Theorem 2.18. *Let R be an arithmetical ring. If R has ACC on two-sided ideals, then each ideal of R contains a finite intersection of SI-ideals.*

Proof. Let Σ be the set of ideals containing no finite intersection of SI-ideals. If $\Sigma \neq \emptyset$, then by the hypothesis there exists a maximal element $Q \in \Sigma$. Hence there exist ideals I, J in R such that $Q \subset I$, $Q \subset J$ and $I \cap J \subseteq Q$. Therefore, by the maximality of Q, there exist SI-ideals $I_1, \cdots, I_n, J_1, \cdots, J_m$ with $I_1 \cap \cdots \cap I_n \subseteq I$ and $J_1 \cap \cdots \cap, J_m \subseteq J$. But in this case, $I_1 \cap \cdots \cap I_n \cap J_1 \cap \cdots \cap, J_m \subseteq Q$ which contradicts $Q \in \Sigma$. Thus $\Sigma = \emptyset$, and this completes the proof. $\square$

Corollary 2.19. *If R is the ring as the previous theorem, then there are only a finite number of minimal SI-ideals, and a finite intersection of minimal SI-ideals is zero.*

Proof. By Theorem 2.18 and Proposition 2.9, the second part holds. Now, let $Q_1, \cdots, Q_n$ be minimal SI-ideals in R with $Q_1 \cap \cdots \cap Q_n = \circ$ and Q minimal SI-ideal. Then from $Q_1 \cap \cdots \cap Q_n \subseteq Q$ we deduce that $Q_j \subseteq Q$ for some Q_j and by the minimality of Q we infer that $Q = Q_j$. Therefore, the minimal SI-ideals of R are contained in the finite set $\{Q_1, \cdots, Q_n\}$. $\square$

Definition 2.20. For each ideal I in the ring R, we define the *i-radical* of I, denoted by $\sqrt[i]{I}$, as follows:

$$\sqrt[i]{I} = \{r \in R\colon \text{ every i-system containing } r \text{ meets } I\}$$

Theorem 2.21. *(Cohen type theorem) For any arithmetical ring R and any ideal I of R, $\sqrt[i]{I}$ equals the intersection of all strongly irreducible ideals containing I. Indeed, $\sqrt[i]{I} = I$.*

Proof. Let Σ be the set of all SI-ideals containing I, $r \in \sqrt[i]{I}$ and Q be any SI-ideal with $I \subseteq Q$. If $r \notin Q$, then since $R - Q$ is an i-system and $r \in R - Q$, by definition of $\sqrt[i]{I}$, we must have $I \cap (R - Q) \neq \emptyset$ which is absord. Hence $\sqrt[i]{I} \subseteq \bigcap_{Q \in \Sigma} Q$. Conversely, let $r \in \bigcap_{Q \in \Sigma} Q$ and S be any i-system containing r. If $I \cap S = \emptyset$, then using Zorn's Lemma there exists an ideal $Q \supseteq I$ which is maximal with respect to being disjoint from S. Therefore, $r \notin Q$ and since by Lemma 2.17, Q is an SI-ideal, we have a contradiction with the choice of r. Hence $I \cap S \neq \emptyset$, i.e., $r \in \sqrt[i]{I}$, and consequently $\bigcap_{Q \in \Sigma} Q \subseteq \sqrt[i]{I}$. Thus $\sqrt[i]{I} = \bigcap_{Q \in \Sigma} Q$ and the proof is complete. The last part is evident in view of [5, Part 3]. $\square$

Definition 2.22. A nonzero element a in a duo ring R is called *strongly zero divisor* if $\langle a \rangle \nleq_e R$, i.e., there is a non zero element $b \in R$ such that $\langle a \rangle \cap \langle b \rangle = \circ$. A non strongly zero divisor is called *quasi regular* element.

In general the set of strongly zero divisors is not closed under the addition in R for example in $R = \mathbb{Z}_6$, 2 and 3 are strongly zero divisors but $3 - 2$ is not.

Example 2.23. 2 is zero divisor element in $\mathbb{Z}_4$, but $(2) \leq_e \mathbb{Z}_4$. Hence the set of all zero divisors properly contains the set of all strongly zero divisor elements.

Remark 2.24. If R is a reduce duo ring, then $a \in R$ is a strongly zero divisor if and only if a is a zero divisor, or equivalently $ann_R(a) \neq \circ$.

Proof. Let a be a zero divisor of R. Then there is a nonzero element b such that $ab = 0$. Therefore, $\langle a \rangle \langle b \rangle = \circ$ and hence $\langle a \rangle \cap \langle b \rangle = \circ$, because R is reduce. $\square$

Definition 2.25. A duo ring R is called *Goldie type ring* if every essential ideal of R contains a quasi regular element.

Proposition 2.26. *If R is a Goldie type ring then the set consist of all quasi regular elements is an i-system.*

Proof. Let S be the set of all quasi regular elements of R and I, J be two ideals in R such that $I \cap S \neq \emptyset$ and $J \cap S \neq \emptyset$. In this case, we have $I \leq_e R$ and $J \leq_e R$ and so $I \cap J \leq_e R$. Thus by definition $I \cap J$ contains a quasi regular element and consequently, $(I \cap J) \cap S \neq \emptyset$. $\square$

Theorem 2.27. *Let D be the set of all strongly zero divisors in a Goldie type arithmetical ring, then D is a union of strongly irreducible ideals.*

Proof. Let $S = R - D$ and Σ be the set of those SI-ideals which disjoint from S (note, S is an i-system by Proposition 2.26). We claim that $S = \bigcap_{Q \in \Sigma}(R - Q)$. To this end, it is clear that $S \subseteq \bigcap_{Q \in \Sigma}(R - Q)$.

Now, we show that the reverse inclusion. Suppose that $x \in \bigcap_{Q\in\Sigma}(R-Q)$ but $x \notin S$. Then we have $\langle x\rangle \cap S = \emptyset$ and so by Lemma 2.17, we can enlarge $\langle x\rangle$ to an SI-ideal, say Q, which is disjoint from S. This shows that $x \notin \bigcap_{Q\in\Sigma}(R-Q)$, a contradiction. Therefore, the equality $S = \bigcap_{Q\in\Sigma}(R-Q)$ holds and consequently $D = \bigcup_{Q\in\Sigma} Q$. □

3 Prime Avoidance Lemma extended to SI-ideals

In dealing with the topic of this section, we should admit that we are following the methods for dealing with the prime avoidance lemma in [11]. First, let us, make a definition.

Definition 3.1. If $S, Q_1, \cdots, Q_n$ are subsets of a ring R such that $S \subseteq \bigcup_{i=1}^n Q_i$ implies that S is contained in the union of a smaller number of Q_i's, we shall say that $S \subseteq \bigcup_{i=1}^n Q_i$ is reducible.

Theorem 3.2. *Let $I, Q_1, Q_2, \cdots, Q_n$, $n \geq 2$, be ideals of a ring R and $I \subseteq \bigcup_{i=1}^n Q_i$. If at most two of the Q_i's are not SI-ideal, then $I \subseteq Q_i$ for some Q_i.*

Proof. For $n = 2$, the assertion holds, even if Q_1 and Q_2 are just subgroups of R, which is a classical result in algebra. Now assume $n \geq 3$. In this case, without loss of generality we may assume that Q_1 is an SI-ideal and $Q_i \nsubseteq Q_j$ for $i \neq j$. Also by induction we may assume that $I \nsubseteq \bigcup_{i=2}^n Q_i$. Hence there is $x \in I$ such that $x \notin \bigcup_{i=2}^n Q_i$. We show that $I \subseteq Q_1$ and we are done. Let us put $J = \bigcap_{i=2}^n Q_i$ and note that for each $y \in I \cap J$ we have $x + y \notin Q_i$ for all $i \geq 2$. Therefore $x + y \in Q_1$ which means $y \in Q_1$ and so $I \cap J \subseteq Q_1$. Since Q_1 is an SI-ideal and $J \nsubseteq Q_1$, we infer that $I \subseteq Q_1$. □

Proposition 3.3. *Let I be an ideal and T a subset of a ring R. If $I + T \subseteq \bigcup_{i=1}^n Q_i$, where each Q_i is an ideal of R such that at most one of the Q_i's are not strongly irreducible, then there exists $t \in T$ such that $I \cup \{t\} \subseteq Q_i$ for some Q_i.*

Proof. For $n = 1$, we note that $I \cup \{t\} \subseteq Q_1$ for all $t \in T$, even if Q_1 is not strongly irreducible. Hence, let $n \geq 2$ and we may assume that Q_1 is strongly irreducible and $Q_i \nsubseteq Q_j$ for $i \neq j$. Now by induction we may assume that $I+T \nsubseteq \bigcup_{i=2}^{n} Q_i$. Hence there are $x \in I$ and $t \in T$ with $x+t \in Q_1 - (\bigcup_{i=2}^{n} Q_i)$. We claim that $I \cup \{t\} \subseteq Q_1$ and we are done. To this end, it suffices to show that $I \subseteq Q_1$. Let us put $J = \bigcap_{i=2}^{n} Q_i$ and note that for each $y \in I \cap J$ we have $x+t+y \notin Q_i$ for all $i \geq 2$, hence $x+t+y \in Q_1$ which means $y \in Q_1$ and therefore $I \cap J \subseteq Q_1$. Since Q_1 is an SI-ideal and $J \nsubseteq Q_1$, we infer that $I \subseteq Q_1$. $\square$

Corollary 3.4. *Let I be an ideal and T a subset of a ring R. If $I+T \subseteq \bigcup_{i=1}^{n} Q_i$, where each Q_i is an ideal of R such that at most one of the Q_i's are not strongly irreducible. If $I+T \subseteq \bigcup_{i=1}^{n} Q_i$ is irreducible, then $I \subseteq \bigcap_{i=1}^{n} Q_i$.*

Proof. For n=1 there is nothing to prove. Hence let $n \geq 2$. By Proposition 3.3, there exist $t_1 \in T$ and $1 \leq i \leq n$ such that $I \cup \{t_1\} \subseteq Q_i$. Since $I+T \subseteq \bigcup_{i=1}^{n} Q_i$ is irreducible, there exists $t_2 \in T$ such that $I+\{t_2\} \nsubseteq Q_i$. Since $I+\{t_2\} \subseteq \bigcup_{i=1}^{n} Q_i$, again by Proposition 3.3, there exists $Q_j \neq Q_i$ such that $I \cup \{t_2\} \subseteq Q_j$. Clearly that $I \subseteq Q_i \cap Q_j$ and if $n=2$, we are done. So let $n \geq 3$. In this case by the hypothesis we have $I+T \nsubseteq Q_i \cup Q_j$. Therefore there exists $t_3 \in T$ which $I+\{t_3\} \nsubseteq Q_i \cup Q_j$. Now $I+\{t_3\} \subseteq \bigcup_{i=1}^{n} Q_i$ implies that $I \cup \{t_3\} \subseteq Q_k$ for some $Q_k \neq Q_i, Q_j$, and consequently $I \subseteq Q_i \cap Q_j \cap Q_k$. By the repeat this prosses n times, the proof is complete. $\square$

Let us, before giving our final result, recall an interesting comment by Karamzadeh concerning the counterpart of the following corollary which is presented in [11] as an exercise: He invites the reader to give, if possible, a number theoritical solution to this final exercise in [11], where a variant of avoidance lemma is invoked for the solution of the exercise. We offer the same invitation to the reader, if a proof without using any variant of avoidance lemma, can be given for the next result.

Corollary 3.5. *Let $p_1, p_2, \cdots, p_{n+1}$, be distinct positive integers such that all p_i's, except possibly p_{n+1}, are prime and $(p_i, p_{n+1}) = 1$ for all $i \leq n$. Then for any positive integers $k_1, k_2, \cdots, k_{n+1}$, there exists the largest ideal G of the ring of integer numbers such that for each $g \in G$ and each p_i there exists some p_j such that $p_j^{k_j}$ divides $g + p_i^{k_i}$.*

Proof. First we note that if there exists such an ideal G we must have $G + T \subseteq \bigcup_{i=1}^{n+1} \langle p_i^{k_i} \rangle$, where $T = \{p_1^{k_1}, p_2^{k_2}, \cdots, p_{n+1}^{k_{n+1}}\}$. Since $T \subseteq$

$\bigcup_{i=1}^{n+1}\langle p_i^{k_i}\rangle$ is irreducible, we deduce that so is $G+T \subseteq \bigcup_{i=1}^{n+1}\langle p_i^{k_i}\rangle$. So by the previous corollary, we get $G \subseteq \bigcap_{i=1}^{n+1}\langle p_i^{k_i}\rangle = \langle p_1^{k_1}p_2^{k_2}\cdots p_{n+1}^{k_{n+1}}\rangle$. Since the ideal $\langle p_1^{k_1}p_2^{k_2}\cdots p_{n+1}^{k_{n+1}}\rangle$ has the property mentioned of the corollary, therefore $G = \langle p_1^{k_1}p_2^{k_2}\cdots p_{n+1}^{k_{n+1}}\rangle$ is the largest ideal with this property. □

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