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The Homological Functors of Finite P-Groups

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Abstract. In this paper, we determine the structure of various homological functors including the exterior square, Schur multiplier and symmetric square for finite two generator non-abelian prime power groups of positive type.

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1 Introduction

A group is metacyclic if there is a normal cyclic subgroup whose factor group is also cyclic. In [4], Beuerle classified the non-abelian metacyclic prime power groups into 2-generator groups of class two and class at least three. We consider $J(G)$ as the kernel of homomorphism $\kappa : G \otimes G \to G'$ with $\kappa(g \otimes h) = [g, h]$ mapping $g \otimes h$ in $G \otimes G$ to $[g, h]$ in G'. The exterior square of a group G is defined as $G \wedge G = (G \otimes G)/\nabla(G)$ in which $\nabla(G)$ denote the subgroup of $J(G)$ generated by the elements $(x \otimes x)$ for $x \in G$. Thus $\nabla(G) = \langle (x \otimes x) | x \in G \rangle \leq J(G)$.

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The structures of the non-abelian tensor square and various homological functors which include $J(G)$, $\nabla(G)$, exterior square and Schur multiplier have been investigated by some researchers. In case of an odd prime p, Bacon and Kappe in [1] studied the homological functors of finite p-groups with nilpotency class two. In [7], a similar investigation was done for infinite 2-generator groups of nilpotency class two. V. Ramachandran [8] was able to characterize the non-abelian tensor square and the homological functors of the symmetric group of order six. Furthermore, Mat Hassim *et al.* [6], were also computed the non-abelian tensor squares and some homological functors of all 2-Engel groups of order at most 16. Beurle and Kappe in [3] exclusively studied the non-abelian tensor squares and some homological functors of infinite metacyclic groups. Moreover, they computed the Schur multipliers and exterior squares of certain Bieberbach and Crystallographic groups. On the other hand, Rashid *et al.* used the Schur multiplier of a group of order $8q$ in determining whether a group of this type is capable [9].

In this paper, the structures of various homological functors, among them the exterior square, the Schur multiplier and the symmetric square, are determined for the metacyclic 2-generator p-groups of positive type. Throughout this paper, we use Beuerle's classifications [4] of all finite non-abelian metacyclic p-groups.

We first provide some preliminary definitions and results used in the subsequent sections.

2 Some Preliminaries on the Homological Functors of Groups

Metacyclic p-groups can be categorized as follows [4]:

$$G_p(\alpha, \beta, \epsilon, \delta, \pm) = \langle a, b \mid a^{p^{\alpha}} = 1, b^{p^{\beta}} = a^{p^{\alpha-\epsilon}}, bab^{-1} = a^r \rangle,$$

where $r = p^{\alpha-\delta} \pm 1$. The group G is said to be of positive or negative type if $r = p^{\alpha-\delta} + 1$ or $r = p^{\alpha-\delta} - 1$, respectively.

In the group G, each isomorphism class of metacyclic p-groups can be represented by five parameters $p, \alpha, \beta, \epsilon$ and δ. These parameters are used to measure the order, the center and abelianness of the groups as well as their nilpotency class and whether the groups are split extension or not.

The metacyclic p-groups of class two can be partitioned into two families of non- isomorphic p-groups given in the following.

Theorem 2.1. (See [4]) *Let G be a non-abelian metacyclic p-group of nilpotency class two. Then G is isomorphic to exactly one group in the following list:*

$$G \cong \langle a, b | a^{p^{\alpha}} = b^{p^{\beta}} = 1, [a, b] = a^{p^{\alpha-\delta}} \rangle, \tag{1}$$
$$\textit{where } p \textit{ is a prime}, \alpha, \beta, \delta \in \mathbb{N}, \alpha \geq 2\delta \textit{ and } \beta \geq \delta \geq 1.$$
$$\text{If } p = 2, \textit{ then in addition } \alpha + \beta > 3.$$
$$G \cong \langle a, b | a^4 = 1, b^2 = [a, b] = a^2 \rangle, \tag{2}$$
$$\textit{the group of quaternions of order } 8.$$

The following two theorems show that metacyclic p-groups (p an odd prime) of positive type of class at least three are divided into split and non-split families of non-isomorphic p-groups.

Theorem 2.2. (See [4]) *Let p be an odd prime and G a metacyclic p-group of nilpotency class at least three. Then G is isomorphic to exactly one group in the following list:*

$$G \cong \langle a, b | a^{p^{\alpha}} = b^{p^{\beta}} = 1, [b, a] = a^{p^{\alpha-\delta}} \rangle, \tag{3}$$
$$\textit{where } p \textit{ is an odd prime}, \alpha, \beta, \delta \in \mathbb{N}, \delta \leq \alpha < 2\delta, \delta \leq \beta \textit{ and}$$
$$\delta \leq \ min\ \{\alpha - 1, \beta\}.$$
$$G \cong \langle a, b | a^{p^{\alpha}} = 1, b^{p^{\beta}} = a^{p^{\alpha-\epsilon}}, [b, a] = a^{p^{\alpha-\delta}} \rangle, \tag{4}$$
$$\textit{where } p \textit{ is an odd prime}, \alpha, \beta, \delta, \epsilon \in \mathbb{N}, \delta + \epsilon \leq \alpha < 2\delta, \delta \leq \beta,$$
$$\alpha < \beta + \epsilon \textit{ and } \delta \leq \ min\ \{\alpha - 1, \beta\}.$$

Likewise, Theorem 2.3 gives a classification of metacyclic 2-groups of positive type of class at least three, which is divided into split and non-split families of non-isomorphic 2-groups.

Theorem 2.3. (See [4]) *Let G be a metacyclic 2-group of nilpotency class at least three. If G is of positive type, then G is isomorphic to* *exactly one group in the following list:*

$$G \cong \langle a,b | a^{2^\alpha} = b^{2^\beta} = 1, [b,a] = a^{2^{\alpha-\delta}} \rangle, \tag{5}$$
$$\text{where } \alpha,\beta,\delta \in \mathbb{N}, 1+\delta < \alpha < 2\delta \text{ and } \beta \geq \delta;$$
$$G \cong \langle a,b | a^{2^\alpha} = 1, b^{2^\beta} = a^{2^{\alpha-\epsilon}}, [b,a] = a^{2^{\alpha-\delta}} \rangle, \tag{6}$$
$$\text{where } \alpha,\beta,\delta,\epsilon \in \mathbb{N}, \delta+1 < \alpha < 2\delta, \delta \leq \beta, \alpha < \beta+\epsilon, \delta+\epsilon \leq \alpha$$
$$\text{and } \delta \leq \min\{\alpha-1,\beta\}.$$

Theorem 2.4. *[5] For any group G, if G^{ab} has no elements of order 2 then $\Gamma(G^{ab}) \cong \nabla(G^{ab})$ and $\nabla(G^{ab}) \cong \nabla(G)$. Moreover, if G^{ab} has no elements of order 2 then $J(G)$ is isomorphic to the direct product $\nabla(G) \times M(G)$.*

Theorem 2.5. *[2] Let G be a finite metacyclic p-group, p an odd prime. Then* $G^{ab} \cong \begin{cases} C_{p^\beta} \times C_{p^{\alpha-\delta}}, & \text{if } G \text{ is of Type (1) and } \beta \leq \alpha-\delta, \\ C_{p^{\alpha-\delta}} \times C_{p^\beta}, & \text{otherwise,} \end{cases}$

Theorem 2.6 will be used to prove some theorems in the main results for computing $J(G)$, $\nabla(G)$ and $\nabla(G/G')$ of some metacyclic p-groups with p odd.

Theorem 2.6. *[5] If G^{ab} has no elements of order 2 then, $\Gamma(G^{ab}) \cong \nabla(G^{ab})$ and $\nabla(G^{ab}) \cong \nabla(G)$. Moreover, if G^{ab} has no elements of order 2 then, $J(G)$ is isomorphic to the direct product $\nabla(G) \times M(G)$.*

Furthermore, the following theorem will be used for computing $\triangle(G)$, $G\tilde{\otimes}G$ and $\tilde{J}(G)$ of finite groups.

Theorem 2.7. *[1] Let G be a torsion group with no elements of even order. Then, $\triangle(G) = \nabla(G)$, $G\tilde{\otimes}G = G \wedge G$ and $\tilde{J}(G) = M(G)$.*

Theorem 2.8 will be used to prove some of our main results for obtaining $M(G)$ of some non-split metacyclic p-groups.

Theorem 2.8. *[6] Let $G = \langle a,b | a^m = e, b^s = a^t, bab^{-1} = a^r \rangle$ be a finite metacyclic group, where the positive integers m, r, s and t satisfy $r^s \equiv 1 (mod\ m)$, $m \big| t(r-1)$ and $t \big| m$. Then $M(G) \cong C_n$, where $n = \frac{(r-1,m)(1+r+r^2+\ldots+r^{s-1},t)}{m}$.*

3 Homological Functors of Finite Non-abelian Metacyclic p-Groups of Positive Types

We compute the homological functors for metacyclic p-groups based on their classification into types. These computations focus on groups of positive type and utilize the following preliminary theorems.

Definition 3.1. *Let $\triangle(G)$ denote the subgroup of $J(G)$ generated by the elements* $(x\otimes y)(y\otimes x)$ for $x, y \in G$. Thus $\triangle(G) = \langle (x\otimes y)(y\otimes x) | x, y \in G\rangle \leq J(G)$.

This section determines the various homological functors as outlined in the introduction for the groups described in Theorems 2.1, 2.2 and 3.5. It begins with a general result to facilitate determining these functors, but which is also of interest in its own right.

We are now in a position to compute the homological functors for the p-groups of Types 1, 3 and 4 with p an odd prime. The groups of Type 1 are divided into three cases, namely the cases where $\beta \leq \alpha - \delta$, $\alpha - \delta \leq \beta \leq \alpha$ and $\alpha \leq \beta$. The results of the computations for the case $\beta \leq \alpha - \delta$ are given in the next theorem.

Theorem 3.2. *Let G be a finite non-abelian metacyclic p-group of Type 1 ($\beta \leq \alpha - \delta$). Then,*

1. $\nabla(G) = \langle a\otimes a\rangle \times \langle (b\otimes a)(a\otimes b)\rangle \times \langle b\otimes b\rangle \cong \nabla(G^{ab}) \cong C_{p^{\alpha-\delta}} \times C^2_{p^{\beta}}$.
2. $J(G) = \langle a\otimes a\rangle \times \langle (b\otimes a)^{p^{\delta}}\rangle \times \langle (b\otimes a)(a\otimes b)\rangle \times \langle b\otimes b\rangle \cong C_{p^{\alpha-\delta}} \times C_{p^{\beta-\delta}} \times C^2_{p^{\beta}}$.
3. $G \wedge G = \langle b \wedge a\rangle \cong C_{p^{\beta}}$.
4. $M(G) = \langle (b\wedge a)^{p^{\delta}}\rangle \cong C_{p^{\beta-\delta}}$.

Proof. For (1), note that $\nabla(G) \subseteq J(G)$. From the definitions we have $\nabla(G) = \langle a\otimes a\rangle \times \langle (b\otimes a)(a\otimes b)\rangle \times \langle b\otimes b\rangle$. By Theorem 2.5, since G^{ab} has no elements of order 2 for odd p, then $\nabla(G) \cong \nabla(G^{ab}) \cong C_{p^{\alpha-\delta}} \times C^2_{p^{\beta}}$. For Case (2), observe that $\kappa : G\otimes G \to G'$ implies $J(G) = ker(k) = \langle a\otimes a\rangle \times \langle (b\otimes a)^{p^{\delta}}\rangle \times \langle (b\otimes a)(a\otimes b)\rangle \times \langle b\otimes b\rangle$. Using Theorem 2.6, $J(G) \cong C_{p^{\alpha-\delta}} \times C_{p^{\beta-\delta}} \times C^2_{p^{\beta}}$. For (3), since $G\wedge G = (G\otimes G)/\nabla(G) =$

$\langle b \wedge a \rangle$, then $G \wedge G \cong C_{p^\beta}$. For the final case, from $M(G) = J(G)/\nabla(G)$, we have $M(G) = \langle (b \wedge a)^{p^\delta} \rangle \cong C_{p^{\beta-\delta}}$. □

By using a similar method as in the proof of Theorems 3.2, the homological functors of groups of Type 1 with p an odd for the cases where $\alpha - \delta \leq \beta \leq \alpha$ and $\alpha \leq \beta$ are obtained.

For $p = 2$, the same structure applies for finding the homological functors of groups of Type 5 for the cases $\alpha - \delta \leq \beta \leq \alpha$ and $\alpha \leq \beta$.

Corollary 3.3. *Let G be a finite non-abelian metacyclic p-group of Type (5) and $\beta \leq \alpha - \delta$. Then*

1. $J(G) = C_{2^{\alpha-\delta+1}} \times C_{2^{\beta-\delta}} \times C^2_{2^\beta}$.
2. $\nabla(G) = C_{2^{\alpha-\delta+1}} \times C^2_{2^\beta}$.
3. $G \wedge G = C_{2^\beta}$.
4. $M(G) = C_{2^{\beta-\delta}}$.

Proof. These results follow directly from replacing p with 2 in the previous proofs, with adjustments for powers of 2 in the orders of the cyclic groups. □

The following theorem provides the homological functors of groups of Type 2.

Theorem 3.4. *Let G be a finite non-abelian metacyclic p-group of Type 2. Then,*

1. $G \otimes G \cong C_2^2 \times C_4^2$.
2. $M(G) \cong C_1$.
3. $G \wedge G \cong C_2$.

Proof. Let G be a group of Type (2). It is easy to see that $G' = \langle a^2 \rangle \cong C_2$. Since $G \otimes G \cong C_2 \times C_2 \times C_4 \times C_4$ then, (1) holds. To prove (2), the result follows directly from Theorem 2.8. To prove (3), observe that $M(G) \cong C_1$. From the results in [9] we have the following central extension: Therefore, we have the following central extension:

$$1 \longrightarrow M(G) \longrightarrow G \wedge G \longrightarrow G' \longrightarrow 1.$$

Hence, $G \wedge G \cong G'$ and the desired result follows. □

The second type of p-groups where p is an odd prime within our scope is of Type (3). This type splits into two cases, namely the case where $\beta \leq \alpha$ and the case where $\alpha \leq \beta$. The results of the computations for the case where $\beta \leq \alpha$ are given in the next theorem.

Theorem 3.5. *Let* G *be a finite non-abelian metacyclic p-group of Type* 3 *with* $\beta \leq \alpha$. *Then,*

1. $\nabla(G) = \langle a \otimes a \rangle \times \langle (b \otimes a)(a \otimes b) \rangle \times \langle b \otimes b \rangle \cong \nabla(G^{ab}) \cong C^2_{p^{\alpha-\delta}} \times C_{p^\beta}$,
2. $J(G) = C_{p^{\alpha-\delta}} \times C_{p^{\beta-\delta}} \times C_{p^{\alpha-\delta}} \times C_{p^\beta}$.
3. $G \wedge G = \langle b \wedge a \rangle \cong C_{p^\beta}$,
4. $M(G) = \langle (b \wedge a)^{p^\delta} \rangle \cong C_{p^{\beta-\delta}}$,
5. $\triangle(G) = \langle (a \otimes a)^2 \rangle \times \langle (b \otimes a)(a \otimes b) \rangle \times \langle (b \otimes b)^2 \rangle \cong C^2_{p^{\alpha-\delta}} \times C_{p^\beta}$,
6. $G \tilde{\otimes} G = \langle b \tilde{\otimes} a \rangle \cong C_{p^\beta}$,
7. $\tilde{J}(G) = \langle (b \tilde{\otimes} a)^{p^\delta} \rangle \cong C_{p^{\beta-\delta}}$.

Proof. Let G be a group of Type (3) with $\beta \leq \alpha$. For (1), we have $a \otimes a \in \nabla(G)$, $b \otimes b \in \nabla(G)$ and $b \otimes a \notin \nabla(G)$. We also have, $(b \otimes a)(a \otimes b) \in \nabla(G)$. Hence, $\nabla(G) = \langle a \otimes a \rangle \times \langle (b \otimes a)(a \otimes b) \rangle \times \langle b \otimes b \rangle$. Theorem 2.5 and 2.6 show that G^{ab} has no elements of order 2 and $\nabla(G) \cong \nabla(G^{ab}) \cong C^2_{p^{\alpha-\delta}} \times C_{p^\beta}$. Hence, the desired result holds. Similarly, by Definition 3.1 we get $\triangle(G) = \langle (a \otimes a)^2 \rangle \times \langle (b \otimes a)(a \otimes b) \rangle \times \langle (b \otimes b)^2 \rangle$. Since p is odd and G has no elements of even order, then $\triangle(G) = \nabla(G) \cong C^2_{p^{\alpha-\delta}} \times C_{p^\beta}$. Hence, concerning the above discussion the result holds.

To prove (2), the homomorphism $\kappa : G \otimes G \to G'$ with $\kappa(g \otimes h) = [g, h]$ where $g, h \in G$. For the generators of $G \otimes G$ we obtain $\kappa(a \otimes a) = [a, a] = 1$, $\kappa(b \otimes b) = [b, b] = 1$ and $\kappa\left((b \otimes a)(a \otimes b)\right) = [b, a][a, b] = [b, a][b, a]^{-1} = 1$. Hence, $a \otimes a$, $b \otimes b$ and $(b \otimes a)(a \otimes b) \in \ker(\kappa) = J(G)$. For $b \otimes a$, we have $\kappa(b \otimes a) = [b, a] = a^{p^{\alpha-\delta}} \neq 1$. Thus, $(b \otimes a) \notin J(G)$. However, since $|[b, a]| = |a^{p^{\alpha-\delta}}| = p^\delta$, we have $\kappa\left((b \otimes a)^{p^\delta}\right) = [b, a]^{p^\delta} = 1$. Therefore, $(b \otimes a)^{p^\delta} \in J(G)$. Now, Theorem 2.8 yields $M(G) \cong C_{p^{\beta-\delta}}$.

Using Theorem 2.5, G^{ab} has no elements of order 2. Therefore, Theorem 2.6 leads to $J(G) \cong C_{p^{\alpha-\delta}} \times C_{p^{\beta-\delta}} \times C^2_{p^\beta}$. Thus, (2) holds. Now, we prove (3). Since $\nabla(G) \lhd G \otimes G$, we have

$$\begin{aligned} G \wedge G = & (G \otimes G)/\nabla(G) \\ = & \langle a \otimes a\rangle / \langle a \otimes a\rangle \times \langle b \otimes a\rangle / \langle 1_\otimes\rangle \times \\ & \langle (b \otimes a)(a \otimes b)\rangle / \langle (b \otimes a)(a \otimes b)\rangle \times \langle b \otimes b\rangle / \langle b \otimes b\rangle \\ = & \langle b \otimes a\rangle / \langle 1_\otimes\rangle = \langle (b \otimes a)\nabla(G)\rangle = \langle b \wedge a\rangle. \end{aligned}$$

Observing the order restrictions on the generators yield $G \wedge G \cong C_{p^\beta}$, the desired result. Similarly, as $\nabla(G) \lhd J(G)$ we have,

$$\begin{aligned} M(G) = & J(G)/\nabla(G) \\ = & \Big(\langle a \otimes a\rangle \times \langle (b \otimes a)^{p^\delta}\rangle \times \langle (b \otimes a)(a \otimes b)\rangle \times \langle b \otimes b\rangle\Big) / \\ & (\langle a \otimes a\rangle \times \langle (b \otimes a)(a \otimes b)\rangle \times \langle b \otimes b\rangle) \\ = & \langle (b \otimes a)^{p^\delta}\rangle / \langle 1_\otimes\rangle = \langle (b \otimes a)^{p^\delta}\nabla(G)\rangle = \langle (b \wedge a)^{p^\delta}\rangle. \end{aligned}$$

By observing the order restrictions on the generators, we get $M(G) \cong C_{p^{\beta-\delta}}$. Thus, (3) holds. Next, since $G \otimes G = \langle a \otimes a\rangle \times \langle b \otimes a\rangle \times \langle (b \otimes a)(a \otimes b)\rangle \times \langle b \otimes b\rangle \cong C_{p^{\alpha-\delta}} \times C_{p^\beta} \times C_{p^{\alpha-\delta}} \times C_{p^\beta}$, also, using (5) and observing that $\triangle(G) \lhd G \otimes G$, we have

$$\begin{aligned} G \tilde{\otimes} G = & (G \otimes G)/\triangle(G) \\ = & (\langle a \otimes a\rangle \times \langle b \otimes a\rangle \times \langle (b \otimes a)(a \otimes b)\rangle \times \langle b \otimes b\rangle) / \\ & (\langle (a \otimes a)^2\rangle \times \langle (b \otimes a)(a \otimes b)\rangle \times \langle (b \otimes b)^2\rangle) \\ = & \langle b \otimes a\rangle / \langle 1_\otimes\rangle = \langle (b \otimes a)\triangle(G)\rangle = \langle b \tilde{\otimes} a\rangle. \end{aligned}$$

From the above considerations and using Theorem 2.7 the desired result for (6) holds. Finally, utilizing the above results and considering that

$\triangle(G) \lhd J(G)$ we have

$$\begin{aligned}\tilde{J}(G) =& J(G)/\triangle(G)\\ =& \Big(\langle a\otimes a\rangle \times \langle (b\otimes a)^{p^\delta}\rangle \times \langle (b\otimes a)(a\otimes b)\rangle \times \langle b\otimes b\rangle\Big)/\\ & \big(\langle (a\otimes a)^2\rangle \times \langle (b\otimes a)(a\otimes b)\rangle \times \langle (b\otimes b)^2\rangle\big)\\ =& \langle a\otimes a\rangle/\langle (a\otimes a)^2\rangle \times \langle (b\otimes a)^{p^\delta}\rangle/\langle 1_\otimes\rangle \times\\ & \langle (b\otimes a)(a\otimes b)\rangle/\langle (b\otimes a)(a\otimes b)\rangle \times \langle b\otimes b\rangle/\langle (b\otimes b)^2\rangle\\ =& \langle (b\otimes a)^{p^\delta}\rangle/\langle 1_\otimes\rangle = \langle (b\otimes a)^{p^\delta}\triangle(G)\rangle = \langle (b\tilde{\otimes} a)^{p^\delta}\rangle.\end{aligned}$$

Thus, by Theorem 2.7 and using the fact that p is odd and G has no elements of even order, (7) is proved. □

Using a similar method, the homological functors of groups of Type (3) for the case $\alpha \leq \beta$ can be computed.

The same calculations as in Theorems 3.5 can be applied for finding the homological functors of non-split groups of type (4). The results are stated in the following theorem.

Theorem 3.6. *Let G be a finite non-abelian metacyclic p-group of Type (4). Then*

1. $\nabla(G) = \triangle(G) \cong \nabla(G^{ab}) \cong C_{p^{\alpha-\delta}} \times C_{p^\beta} \times C_{p^{\alpha-\delta}}$,
2. $G \wedge G = G\tilde{\otimes}G \cong C_{p^{\alpha-\epsilon}}$,
3. $M(G) = \tilde{J}(G) \cong C_{p^{\alpha-\delta-\epsilon}}$,
4. $J(G) \cong C_{p^{\alpha-\delta}} \times C_{p^\beta} \times C_{p^{\alpha-\delta}} \times C_{p^{\alpha-\delta-\epsilon}}$.

Proof. Suppose that G is a group of Type (4) and p is an odd prime. For (1), using Theorem 2.5 the group G^{ab} has no elements of order 2. Thus, from 2.6 and the above consideration we have

$$\nabla(G) \cong \nabla(G^{ab}) \cong C_{p^{\alpha-\delta}} \times C_{p^\beta} \times C_{p^{\alpha-\delta}}.$$

Now, by Theorem 2.7 the desired result follows.

To prove (2), we know $\nabla(G) \lhd G \otimes G$, thus

$$\begin{aligned} G \wedge G &= (G \otimes G)\,/\nabla(G) \\ &\cong \left(C_{p^{\alpha-\delta}} \times C_{p^{\alpha-\epsilon}} \times C_{p^{\alpha-\delta}} \times C_{p^{\beta}}\right) / \left(C_{p^{\alpha-\delta}} \times C_{p^{\beta}} \times C_{p^{\alpha-\delta}}\right) \\ &\cong C_{p^{\alpha-\delta}}/C_{p^{\alpha-\delta}} \times C_{p^{\alpha-\delta}}/C_{p^{\alpha-\delta}} \times C_{p^{\alpha-\epsilon}}/C_1 \times C_{p^{\beta}}/C_{p^{\beta}} \\ &\cong C_{p^{\alpha-\epsilon}}/C_1 \cong C_{p^{\alpha-\epsilon}}. \end{aligned}$$

Hence, using Theorem 2.7 the desired result follows. The result of (3) follows from Theorem 2.8 and Theorem 2.7. Finally, for (4) since p odd, G^{ab} has no elements of order 2 and using Theorems 2.5, 2.6, (1) and (3) the result holds. □

Now, we compute the homological functors of Type (5). This type splits into two cases, namely the case where $\beta \leq \alpha$ and the case where $\alpha \leq \beta$. The results of the computations for the case where $\beta \leq \alpha$ are given in the next theorem. The remaining type of metacyclic 2-groups of positive type is Type 6. Since the generators of the non-abelian tensor square of groups of this type were not specified, some homological functors can not be computed as before. The results of the computations for $G \otimes G$, $M(G)$ and $G \wedge G$ of the case where $\beta \leq \alpha - \delta$ are given in the next theorem.

Theorem 3.7. *Let G be a finite non-abelian metacyclic 2-group of Type 6. Then*

1. $M(G) \cong C_{2^{\alpha-\delta-\epsilon}}$,
2. $G \wedge G \cong C_{2^{\delta}}$, *if* $\alpha = \delta + \epsilon$.

Proof. Let G be a group of Type 6. First, observe that (1) follows from Theorem 2.8. To prove (2), let $\alpha = \delta + \epsilon$. Then, $M(G) \cong C_1$. From commutative diagram include with exact rows and central extensions

The results in [9] give a commutative diagram, with exact rows and central extensions as columns. from which we have the following central extension:

$$1 \longrightarrow M(G) \longrightarrow G \wedge G \longrightarrow G' \longrightarrow 1.$$

Hence, concerning the above discussion, $G \wedge G \cong G'$ and the desired result follows from (i) in Theorem 2.5. □

References

[1] M. R. Bacon and L. C. Kappe, On capable p-group of nilpotency class two, *Illinois Journal of Mathematics*, 47 (2003), 49−62.

[2] A. M. A. B. Basri and K. Moradipour, On the capability non-abelian tensor square and non-commuting graph of prime power groups, *International Journal of Nonlinear Analysis and Applications*, (2024), (Accepted).

[3] J. R. Beuerle and L. C. Kappe, Infinite metacyclic groups and their non-abelian tensor squares, *Proceedings of the Edinburgh Mathematical Society*, 43 (2000), 651−662.

[4] J. R. Beuerle, An elementary classification of finite metacyclic p-groups of class at least three, *Algebra Colloquium*, 12(4) (2005), 553−562.

[5] A. Magidin and R. F. Morse, Certain homological functors of 2-generator p-groups of class 2, *Contemporary Mathematics*, 5(11) (2010), 127−166.

[6] H. I. M. Hassim, N. H. Sarmin and M. S. Mohd, On the computations of some homological functors of 2-engel groups of order at most sixteen, *Journal of Quality Measurement and Analysis*, 97 (2011), 2994−4306.

[7] N. M. Mohd Ali, *Capability and Homological Functors of Infinite Two Generator Groups of Nilpotency Class Two*, Ph.D. Thesis, UTM, (2008).

[8] V. Ramachandran and N. H. Sarmin, Computing the non-abelian tensor square and homological functors of the symmetric group of order six, *Proceedings of Regional Annual Fundamental Science Seminar*, (2008), 27−29.

[9] S. Rashid, N. H. Sarmin, A. Erfanian and N. M. Mohd Ali, On the non-abelian tensor square and capability of groups of order p^2q, *Archiv der Mathematik*, 97 (2011), 299−306.

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