

General local cohomology modules and Faltings' Local-global Principles

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Abstract. In this article, we study the Local-global Principles for the Artinianness of ordinary local cohomology modules and the finiteness of general local cohomology modules. Let R be a Noetherian ring, Φ be a system of ideals of R and N be an R -module. Assume that \mathcal{S} is a Serre subcategory of $\text{Mod}(R)$ satisfying the condition C_Φ and the *Residual Fields condition* (briefly \mathcal{RF} condition) and let \mathcal{S}_A be the class of Artinian R -modules. For $t \in \mathbb{N}_0$, we first show that the general local cohomology modules $H_\Phi^i(N) \in \mathcal{S}$ for every $i < t$ if and only if $H_{\mathfrak{b}}^i(N) \in \mathcal{S}$ for any $\mathfrak{b} \in \Phi$ and every $i < t$. Then, for a finite R -module N , we conclude that if $H_\Phi^i(N) \in \mathcal{S}_A$ for every $i < t$, thus $H_\Phi^i(N) \in \mathcal{S}$ for every $i < t$. Consequently, we show that the least non-negative integer i in which $H_\Phi^i(N)$ is not Artinian, is a lower bound for all \mathcal{S} -depth $_\Phi(N)$. Finally, we prove that if $n \in \mathbb{N}_0$ is such that N is in dimension $< n$ and $(\text{Ass}_R(H_\Phi^{h_\Phi^n(N)}(N)))_{\geq n}$ is a finite set, then $f_\Phi^n(N) = h_\Phi^n(N)$.

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1 Introduction

In this paper, R will always denote a non-trivial commutative Noetherian ring and N will denote an R -module. By a finite R -module, we mean an R -module which is finitely generated. For the set of non-negative integers and the category of all R -modules and R -homomorphisms we shall use \mathbb{N}_0 and $\text{Mod}(R)$, respectively.

It is well known that for each $i \geq 0$, for the i -th local cohomology R -module N relative to the ideal \mathfrak{b} , there is a natural R -isomorphism as follows:

$$H_{\mathfrak{b}}^i(N) \cong \varinjlim_{n \in \mathbb{N}} \text{Ext}_R^i(R/\mathfrak{b}^n, N).$$

Let $\Phi \neq \emptyset$ be a set of ideals of R . Recall that Φ is called a system of ideals if the multiplication of any two ideals of Φ , always, contains an ideal of Φ . For such a system and for any R -module N , we consider the following submodule

$$\Gamma_{\Phi}(N) := \{x \in N \mid \mathfrak{b}x = 0 \text{ for some } \mathfrak{b} \in \Phi\}.$$

Then Γ_{Φ} is a covariant, R -linear and left exact functor from $\text{Mod}(R)$ to $\text{Mod}(R)$. When Φ is taken as the power of an ideal, say \mathfrak{b} , then $H_{\Phi}^i(-)$ is naturally equivalent to the ordinary functor $H_{\mathfrak{b}}^i(-)$. As of now, we refer to $H_{\Phi}^i(N)$ as the general local cohomology module. The ordinary local cohomology and its generalization on a system of ideals, have been studied in [7, 8, 9, 12].

Also, it is well known that, Faltings' Local-global Principle for the finiteness of local cohomology modules, [11, satz1], states that for any integer $m > 0$, $H_{\mathfrak{b}}^j(N)$ is a finite R -module for every $j < m$ if and only if $H_{\mathfrak{b}R_{\mathfrak{p}}}^j(N_{\mathfrak{p}})$ is a finite $R_{\mathfrak{p}}$ -module for every $j < m$ and any $\mathfrak{p} \in \text{Spec}(R)$. Hence, another formulation of $f_{\mathfrak{b}}(N)$, the finiteness dimension of N with respect to \mathfrak{b} , (see [9, Theorem 9.6.2]), is as follows:

$$\begin{aligned} f_{\mathfrak{b}}(N) &= \inf\{j \in \mathbb{N}_0 \mid H_{\mathfrak{b}}^j(N) \text{ is not a finite } R\text{-module}\} \\ &= \inf\{f_{\mathfrak{b}R_{\mathfrak{p}}}(N_{\mathfrak{p}}) \mid \mathfrak{p} \text{ is a prime ideal of } R\}. \end{aligned}$$

Then, for any $n \in \mathbb{N}_0$, Bahmanpour et al., in [5], presented the n -th finiteness dimension $f_{\mathfrak{b}}^n(N)$ of N relative to \mathfrak{b} by

$$f_{\mathfrak{b}}^n(N) := \inf\{f_{\mathfrak{b}R_{\mathfrak{p}}}(N_{\mathfrak{p}}) \mid \mathfrak{p} \in \text{Supp}(N/\mathfrak{b}N), \dim R/\mathfrak{p} \geq n\}.$$

As some applications of this notion, it is shown that

$$\inf\{i \in \mathbb{N}_0 \mid H_{\mathfrak{b}}^i(N) \text{ is not minimax (weakly Laskerian)}\},$$

is equal to $f_{\mathfrak{b}}^1(N)$ ($f_{\mathfrak{b}}^2(N)$). An R -module N is called a minimax R -module, if N/N' is Artinian module for some finite submodule N' of N . The class of minimax modules has been studied by Zink [21], Zöschinger [22, 23] and Rudlof [18]. Moreover, an R -module N is called skinny or weakly Laskerian module, if $\text{Ass}_R N/N'$ is a finite set, for any submodule N' of N (cf. [10] or [14]).

The class of in dimension $< n$ modules is presented in [4]. The authors generalized Faltings' Local-global Principle on a complete local ring R , for any finite R -module N and ideal \mathfrak{b} of R , as:

$$f_{\mathfrak{b}}^n(N) = h_{\mathfrak{b}}^n(N) := \inf\{i \geq 0 \mid H_{\mathfrak{b}}^i(N) \text{ is not in dimension } < n\}.$$

Then Mehrvarz et al., in [15] showed that the equality $f_{\mathfrak{b}}^n(N) = h_{\mathfrak{b}}^n(N)$ is true on an arbitrary Noetherian (not necessarily complete and local) ring, too.

On the other hand, Tang in [20] proved a similar Local-global Principle for the Artinianness of local cohomology modules. He proved that, for any $m \in \mathbb{N}$, the necessary and sufficient condition for $H_{\mathfrak{b}}^i(N)$ to be Artinian for every $i < m$ is $H_{\mathfrak{b}R_{\mathfrak{p}}}^i(N_{\mathfrak{p}})$ to be Artinian for every $i < m$ and any $\mathfrak{p} \in \text{Spec}(R)$, in which R is an arbitrary Noetherian ring, \mathfrak{b} is an ideal of R and N is a finite R -module (see [20, Theorem 2.2]).

In section 2, we deal with the Serre subcategories which satisfy the conditions C_{Φ} and \mathcal{RF} (see Definitions 2.7, 2.13) and the situations that the general local cohomology modules belong to these Serre subcategories. Let \mathcal{S} be a subcategory of $\text{Mod}(R)$. Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be an exact sequence in $\text{Mod}(R)$. \mathcal{S} is said to be a Serre subcategory, whenever $M \in \mathcal{S}$ if and only if $M', M'' \in \mathcal{S}$. Assume that \mathcal{S} is a Serre subcategory of $\text{Mod}(R)$ satisfying the condition C_{Φ} and $t \in \mathbb{N}_0$. As an important achievement in this section, in Proposition 2.12, we show that $H_{\Phi}^i(N)$ belongs to \mathcal{S} for every $i < t$ if and only if $H_{\mathfrak{b}}^i(N)$ belongs to \mathcal{S} for every $\mathfrak{b} \in \Phi$ and every $i < t$.

Section 3, is about Local-global Principles. As another important result about the ordinary local cohomology modules, Theorem 3.1, shows that if N is an R -module, \mathfrak{b} an ideal of R and $t \in \mathbb{N}_0$ are such that $H_{\mathfrak{b}}^i(N)$ is \mathfrak{b} -cofinite Artinian R -module for every $i < t$, then $H_{\mathfrak{b}}^i(N) \in \mathcal{S}$ for every $i < t$ and for any Serre subcategory \mathcal{S} , which satisfies the conditions $C_{\mathfrak{b}}$ and \mathcal{RF} . An application of Theorem 3.1, we present some equivalent conditions to the Local-global Principle for Artinianness (see Proposition 3.2). In sequel, assume that \mathcal{S} is an arbitrary Serre subcategory that satisfies the conditions C_{Φ} and \mathcal{RF} , N is a finite R -module and t is a non-negative integer. Set $\mathcal{S}_{\mathcal{A}}$ as the class of Artinian R -modules. In Corollary 3.3, we show that if $H_{\Phi}^i(N)$ belongs to $\mathcal{S}_{\mathcal{A}}$ for every $i < t$, then $H_{\Phi}^i(N)$ belongs to \mathcal{S} for every $i < t$. Then, in Proposition 3.7, we show that $\inf\{i \in \mathbb{N}_0 \mid H_{\Phi}^i(N) \text{ is not Artinian}\}$ is a lower bound for all \mathcal{S} -depth $_{\Phi}(N)$ (see Definition 3.5), i.e.,

$$\text{f-depth}_{\Phi}(N) \leq \mathcal{S}\text{-depth}_{\Phi}(N).$$

As the last important result of this paper, we conclude that $f_{\Phi}^n(N) = h_{\Phi}^n(N)$, for any $n \in \mathbb{N}_0$ (see Theorem 3.10). This generalizes the main results of [4, Theorem 2.5] for an arbitrary Noetherian (not necessary complete local) ring and [15, Theorem 2.10] for an arbitrary system of ideals of R .

2 General local cohomology modules and conditions C_{Φ} and \mathcal{RF}

We begin this section with one of the results of [19] which is used in most results of this paper. Let F, T be two left exact covariant functors from $\text{Mod}(R)$ to itself. For

any $j \geq 1$, we shall use the j -th right derived functors of F, H and the composition FH by F^j, T^j and $(FT)^j$, respectively.

Proposition 2.1. (see[19, Proposition2.2]) *Assume that \mathcal{S} is a Serre subcategory of $\text{Mod}(R)$, \mathfrak{b} is an ideal of R and N is an R -module. Let T be a covariant and left exact functor of $\text{Mod}(R)$ to itself such that $(0 :_X \mathfrak{b}) = (0 :_{T(X)} \mathfrak{b})$ for any R -module X . Assume that $T(E)$ is an injective R -module for any injective R -module E . Let $n \in \mathbb{N}$ and let $\text{Ext}_R^{t-j}(R/\mathfrak{b}, T^j(N)) \in \mathcal{S}$ for $t = n, n + 1$ and every $j < n$. Then $\text{Ker}\psi \in \mathcal{S}$ and $\text{Coker}\psi \in \mathcal{S}$, in which*

$$\psi : \text{Ext}_R^n(R/\mathfrak{b}, N) \rightarrow \text{Hom}_R(R/\mathfrak{b}, T^n(N))$$

is the natural homomorphism. Thus

$$\text{Ext}_R^n(R/\mathfrak{b}, N) \in \mathcal{S} \text{ if and only if } \text{Hom}_R(R/\mathfrak{b}, T^n(N)) \in \mathcal{S}.$$

Proof. The assertion follows from [3, Proposition 3.1], by taking $F(-) = \text{Hom}_R(R/\mathfrak{b}, -)$. Because $FT(N) = F(N)$ for any R -module N . \square

Corollary 2.2. *Assume that \mathcal{S} is a Serre subcategory of $\text{Mod}(R)$, Φ is a system of ideals of R and N is an R -Module. Let $n \in \mathbb{N}_0$ be such that $\text{Ext}_R^j(R/\mathfrak{b}, H_\Phi^i(N)) \in \mathcal{S}$ for every $i, j < n$. Then $\text{Ext}_R^n(R/\mathfrak{b}, N) \in \mathcal{S}$ if and only if $\text{Hom}_R(R/\mathfrak{b}, H_\Phi^n(N)) \in \mathcal{S}$.*

Proof. Apply Proposition 2.1 for $F(N) = \text{Hom}_R(R/\mathfrak{b}, N)$ and $T(N) = \Gamma_\Phi(N)$. \square

As it is mentioned in the introduction, the authors in [4], presented the class of R -modules in dimension $< n$ for an arbitrary integer $n \in \mathbb{N}_0$. An R -module N is said to be in dimension $< n$, if $\dim \text{Supp}_R(N/N') < n$, for some finite submodule N' of N (also see the definition of the class of $\text{FD}_{\leq n}$ in [1, Definition 2.1]). It is obvious that, the class of in dimension $< n$ modules is a generalization of the class of finitely generated, Artinian, and minimax modules for some $n \in \mathbb{N}_0$ (see [15, Remark 2.2]). Moreover, it is clear that, the class of in dimension $< n$ modules are a Serre subcategory of $\text{Mod}(R)$.

Corollary 2.3. *Assume that Φ is a system of ideals of R and $s, n \in \mathbb{N}_0$ are such that the R -modules N and $H_\Phi^i(N)$ are in dimension $< n$ for every $i < s$. Thus, the R -module $\text{Hom}_R(R/\mathfrak{b}, H_\Phi^s(N)/N')$ is in dimension $< n$ for any in dimension $< n$ submodule N' of $H_\Phi^s(N)$ and all $\mathfrak{b} \in \Phi$. Consequently, $(\text{Ass}_R(H_\Phi^s(N)/N') \cap \mathbb{V}(\mathfrak{b}))_{\geq n}$ is a finite set.*

Proof. As N is in dimension $< n$, one can see that the R -module $\text{Ext}_R^i(R/\mathfrak{b}, N)$ is in dimension $< n$ for every $i \geq 0$. Now, for any in dimension $< n$ submodule N' of $H_\Phi^s(N)$, use Corollary 2.2 and the short exact sequence $0 \rightarrow N' \rightarrow H_\Phi^s(N) \rightarrow H_\Phi^s(N)/N' \rightarrow 0$. \square

We shall use Proposition 2.4 for some further results.

Proposition 2.4. *For a Serre subcategory \mathcal{S} of $\text{Mod}(R)$, an ideal \mathfrak{b} of R and an R -Module X , we have the following:*

- (i) $\mathfrak{b}X \in \mathcal{S}$ if and only if $X/(0 :_X \mathfrak{b}) \in \mathcal{S}$.
 (ii) $X \in \mathcal{S}$ if and only if there is $k \in \mathbb{N}_0$ such that $(0 :_X \mathfrak{b}^k) \in \mathcal{S}$ and $\mathfrak{b}^k X \in \mathcal{S}$.

Proof. (i) Let $\mathfrak{b} = \sum_{i=1}^n Rb_i$ where $b_i \in R$ and $n \in \mathbb{N}$. Suppose that $\mathfrak{b}X \in \mathcal{S}$. Consider the homomorphism $g : X \rightarrow (\mathfrak{b}X)^n$ by $g(x) = (\mathfrak{b}_i x)_{i=1}^n$ for every $x \in X$. Thus, the R -module $X/(0 :_X \mathfrak{b})$ is isomorphic to a submodule of $(\mathfrak{b}X)^n$. Conversely, consider the homomorphism $f : X^n \rightarrow \mathfrak{b}X$ given by $f((x_i)_{i=1}^n) = \sum_{i=1}^n b_i x_i$. Then f is surjective and $(0 :_X \mathfrak{b})^n \subseteq \text{Ker } f$, so $\mathfrak{b}X$ is a homomorphic image of $(X/(0 :_X \mathfrak{b}))^n$. Now, the assertion follows from $(X/(0 :_X \mathfrak{b}))^n \in \mathcal{S}$.

(ii) This part is immediately followed by part (i) and the short exact sequence

$$0 \rightarrow (0 :_X \mathfrak{b}^k) \rightarrow X \rightarrow X/(0 :_X \mathfrak{b}^k) \rightarrow 0.$$

□

Proposition 2.5. *Assume that \mathcal{S} is a Serre subcategory of $\text{Mod}(R)$, Φ is a system of ideals of R and N is an R -module. Suppose that $\mathfrak{b} \in \Phi$ and $k, t \in \mathbb{N}_0$ are such that $\mathfrak{b}^k H_{\Phi}^i(N) \in \mathcal{S}$ for every $i < t$. Thus $H_{\Phi}^i(N) \in \mathcal{S}$ for every $i < t$, if one of the following statements satisfies:*

- (i) $\Gamma_{\Phi}(N) \in \mathcal{S}$ and $\text{Ext}_R^i(R/\mathfrak{b}^k, N) \in \mathcal{S}$ for every $i < t$.
 (ii) $N \in \mathcal{S}$.
 (iii) $\Gamma_{\Phi}(N) \in \mathcal{S}$ and there exists $n \in \mathbb{N}_0$ such that $\mathfrak{b}^n N \in \mathcal{S}$.

Proof. First, suppose that condition (i) holds and use induction on t . For $t = 0, 1$, we do not have anything to prove. Now, assume that $t > 1$ and the assertion is valid for every $i \leq t - 2$ i.e., the R -modules $H_{\Phi}^0(N), H_{\Phi}^1(N), \dots, H_{\Phi}^{t-2}(N)$ belong to \mathcal{S} . We show that $H_{\Phi}^{t-1}(N) \in \mathcal{S}$. Since $\text{Ext}_R^{t-1}(R/\mathfrak{b}^k, N) \in \mathcal{S}$ and $H_{\Phi}^i(N) \in \mathcal{S}$, for every $i < t - 1$, we have $(0 :_{H_{\Phi}^{t-1}(N)} \mathfrak{b}^k) \in \mathcal{S}$, by Corollary 2.2. On the other hand, by the assumption, $\mathfrak{b}^k H_{\Phi}^{t-1}(N) \in \mathcal{S}$. Now, the result holds by Proposition 2.4 (ii). Under each condition (ii) and (iii), use part (i) and Proposition 2.4 (ii). □

As a corollary of Proposition 2.5, we achieve the following result that generalizes [9, Proposition 9.1.2].

Corollary 2.6. *Assume that \mathcal{S} is a Serre subcategory of $\text{Mod}(R)$, N is an R -module such that $N \in \mathcal{S}$ and $t \in \mathbb{N}_0$. Let \mathfrak{b} be an ideal of R such that $\mathfrak{b} \subseteq \sqrt{0 : H_{\mathfrak{b}}^i(N)}$ for every $i < t$. Then $H_{\mathfrak{b}}^i(N)$ belongs to \mathcal{S} for every $i < t$.*

Assume that \mathfrak{b} is an ideal of R and \mathcal{S} is a Serre subcategory of $\text{Mod}(R)$. The authors in [2], studied the Serre subcategories which satisfy the condition $C_{\mathfrak{b}}$. It is said that \mathcal{S} satisfies the condition $C_{\mathfrak{b}}$, if for any \mathfrak{b} -torsion R -module X , the condition $(0 :_X \mathfrak{b}) \in \mathcal{S}$ implies that $X \in \mathcal{S}$ (see [2, Definition 2.1]). The Examples 2.4 and 2.5 of [2], show that all the classes of zero R -modules, Artinian modules, modules

with finite support, \mathfrak{b} -cofinite Artinian modules and the class of all R -modules M with $\dim M \leq s$, where $s \in \mathbb{N}_0$, satisfy the condition $C_{\mathfrak{b}}$. Recall that an R -module X is said to be \mathfrak{b} -cofinite whenever $\text{Supp}_R(X) \subseteq V(\mathfrak{b})$ and $\text{Ext}_R^i(R/\mathfrak{b}, X)$ is a finite R -module for every $i \in \mathbb{N}_0$ (see [13]).

Now, in this position, for an arbitrary system of ideals Φ of R , we introduce the condition C_{Φ} and then we shall conclude some results on the general local cohomology modules. An R -module X is said to be a Φ -torsion module if $\Gamma_{\Phi}(X) = X$ and it is said to be a Φ -torsion-free module if $\Gamma_{\Phi}(X) = 0$.

Definition 2.7. Assume that \mathcal{S} is a Serre subcategory of $\text{Mod}(R)$ and Φ is a system of ideals of R . We say that \mathcal{S} satisfies the condition C_{Φ} , precisely when for every Φ -torsion R -module X , the condition $(0 :_X \mathfrak{b}) \in \mathcal{S}$ for some $\mathfrak{b} \in \Phi$ implies that $X \in \mathcal{S}$.

Remark 2.8. By Proposition 2.4 (ii), it can be said that, \mathcal{S} satisfies the condition C_{Φ} , whenever for any Φ -torsion R -module N , the condition $(0 :_N \mathfrak{a}^k) \in \mathcal{S}$ for some $\mathfrak{a} \in \Phi$ and $k \in \mathbb{N}_0$, implies that $\mathfrak{a}^k N \in \mathcal{S}$. In addition, it is easy to see that if \mathcal{S} satisfies the condition C_{Φ} , then \mathcal{S} satisfies the condition $C_{\mathfrak{b}}$, for any $\mathfrak{b} \in \Phi$.

Example 2.9. Let R be a Noetherian ring of finite dimension d . Let Φ be an arbitrary system of ideals of R and $\mathcal{S} = \{M \in \text{Mod}(R) \mid \dim M \leq d\}$. Then, it is clear that \mathcal{S} satisfies the condition C_{Φ} .

The next examples show that the class of Artinian R -modules does not satisfy the condition C_{Φ} , for some system of ideals Φ .

Example 2.10. ([6, Example 3.7]) Let R be a Gorenstein ring of finite dimension d such that has infinite maximal ideal \mathfrak{m} with $\text{ht } \mathfrak{m} = d$. Let

$$\Psi = \{\mathfrak{m} \in \text{Max}(R) \mid \text{ht } \mathfrak{m} = d\} \text{ and } \Phi = \{\mathfrak{a} \mid \mathfrak{a} \text{ is an ideal of } R \text{ and } \dim R/\mathfrak{a} \leq 0\}.$$

Then, it is easy to see that Φ is a system of ideals of R and $\Psi \subseteq \Phi$. By [6, Example 3.7], we have

$$H_{\Phi}^i(R) = \begin{cases} \bigoplus_{\mathfrak{m} \in \Psi} E(R/\mathfrak{m}) & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}$$

Thus, $H_{\Phi}^d(R)$ is not Artinian R -module. On the other hand, since $\text{Ext}_R^d(R/\mathfrak{a}, R)$ is an Artinian R -module for all $\mathfrak{a} \in \Phi$, so $(0 :_{H_{\Phi}^d(R)} \mathfrak{a}) = \text{Hom}_R(R/\mathfrak{a}, H_{\Phi}^d(R))$ is Artinian R -module for all $\mathfrak{a} \in \Phi$, by Corollary 2.2. Hence, the class of Artinian R -modules does not satisfy the condition C_{Φ} .

Example 2.11. Let \mathbb{Z} be the ring of integers and let p be a prime number of \mathbb{Z} . Set $\Phi = \{(0), p\mathbb{Z}\}$. Then, Φ is a system of ideals of \mathbb{Z} . It is clear that $\Gamma_{\Phi}(\mathbb{Z}) = \mathbb{Z}$ and so $H_{\Phi}^i(\mathbb{Z}) = 0$ for all $i \geq 1$, by [12, Lemma 2.4]. Since $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z})$ is Artinian \mathbb{Z} -module and \mathbb{Z} is not Artinian, thus the class of Artinian \mathbb{Z} -modules does not satisfy the condition C_{Φ} .

As a generalization of [2, Theorem 2.9] and [6, Corollary 2.14], we present the following proposition which is one of the useful results in this article.

Proposition 2.12. *Assume that Φ is a system of ideals of R and $t \in \mathbb{N}_0$. For any Serre subcategory \mathcal{S} of $\text{Mod}(R)$ which satisfies the condition C_Φ and any R -module N the following conditions are equivalent:*

- (i) $H_\Phi^i(N) \in \mathcal{S}$ for every $i < t$;
- (ii) $\text{Ext}_R^j(R/\mathfrak{b}, H_\Phi^i(N)) \in \mathcal{S}$ for any $\mathfrak{b} \in \Phi$ and every $i, j < t$.
- (iii) $\text{Ext}_R^i(R/\mathfrak{b}, N) \in \mathcal{S}$ for any $\mathfrak{b} \in \Phi$ and every $i < t$;
- (iv) $H_\mathfrak{b}^i(N) \in \mathcal{S}$ for any $\mathfrak{b} \in \Phi$ and every $i < t$.

Proof. (i) \Rightarrow (ii) Let $\mathfrak{b} \in \Phi$ and consider the following resolution of R/\mathfrak{b}

$$\mathcal{A}_\bullet : \cdots \longrightarrow A_s \longrightarrow A_{s-1} \longrightarrow \cdots \longrightarrow A_1 \longrightarrow A_0 \longrightarrow 0,$$

in which A_i are free R -modules with finite ranks for every $i \geq 0$.

Then $\text{Ext}_R^j(R/\mathfrak{b}, H_\Phi^i(N)) = H^j(\text{Hom}_R(\mathcal{A}_\bullet, H_\Phi^i(N)))$ is a subquotient of direct sum of finite copies of $H_\Phi^i(N)$ for every i, j . Now, the assertion holds by the fact that \mathcal{S} is a Serre subcategory.

(ii) \Rightarrow (iii) Use Corollary 2.2.

(iii) \Rightarrow (iv) As \mathcal{S} satisfies the condition $C_\mathfrak{b}$ for every $\mathfrak{b} \in \Phi$, thus this implication is true, by Remark 2.8 and [2, Theorem 2.9].

(iv) \Rightarrow (i) We show that $H_\Phi^i(N) \in \mathcal{S}$ by induction on t . For $t = 0$, we do not have anything to show. Suppose that $t = 1$. By hypothesis, $\Gamma_\mathfrak{b}(N) = H_\mathfrak{b}^0(N) \in \mathcal{S}$, and so $(0 :_{\Gamma_\Phi(N)} \mathfrak{b}) = (0 :_N \mathfrak{b}) = (0 :_{\Gamma_\mathfrak{b}(N)} \mathfrak{b}) \in \mathcal{S}$. Therefore, the assertion follows as \mathcal{S} satisfies the condition C_Φ . Now, let $t > 1$ and assume that the assertion is settled for every $i \leq t - 2$. We prove that $H_\Phi^{t-1}(N) \in \mathcal{S}$. Since, by hypothesis, $H_\mathfrak{b}^i(N) \in \mathcal{S}$ for any ideal \mathfrak{b} in Φ and every $i \leq t - 1$, $\text{Ext}_R^i(R/\mathfrak{b}, N) \in \mathcal{S}$ for every $i \leq t - 1$, by [2, Theorem 2.9]. By using inductive hypothesis, $H_\Phi^i(N) \in \mathcal{S}$ for every $i \leq t - 2$. Therefore $(0 :_{H_\Phi^{t-1}(N)} \mathfrak{b}) \in \mathcal{S}$, by Corollary 2.2. Now, the assertion holds as \mathcal{S} satisfies the condition C_Φ . \square

Definition 2.13. Let R be a Noetherian ring. A Serre subcategory \mathcal{S} of $\text{Mod}(R)$ is said to satisfy the *Residual Fields condition* (briefly \mathcal{RF} condition) if $R/\mathfrak{n} \in \mathcal{S}$ for any $\mathfrak{n} \in \text{Max}(R)$.

Example 2.14. (i) It is obvious that all of the classes of Noetherian R -modules, Artinian R -modules, R -modules with finite support and the class of all R -modules X with $\dim_R X \leq n$, where $n \in \mathbb{N}_0$, satisfy the \mathcal{RF} condition.

(ii) Assume that R is a non-local Noetherian ring and \mathfrak{b} is an ideal of R . Let $\mathfrak{n} \in \text{Max}(R)$ be such that $\mathfrak{n} \notin V(\mathfrak{b})$. Set

$$\mathcal{S} := \{M \in \text{Mod}(R) \mid M \text{ is an } \mathfrak{b}\text{-torsion } R\text{-module}\}.$$

Clearly, \mathcal{S} does not satisfy the \mathcal{RF} condition. However, the next lemma, (part iii), shows that when (R, \mathfrak{n}) is a local ring, any non-zero Serre subcategory \mathcal{S} of $\text{Mod}(R)$ satisfies the \mathcal{RF} condition.

Lemma 2.15. *Assume that \mathcal{S} is an arbitrary Serre subcategory and that $\mathcal{S}_{\mathcal{FL}}$ is the class of the finite length R -modules. Then*

- (i) \mathcal{S} is non-zero if and only if there is $\mathfrak{n} \in \text{Max}(R)$ such that $R/\mathfrak{n} \in \mathcal{S}$.
- (ii) $\mathcal{S}_{\mathcal{FL}} \subseteq \mathcal{S}$ if and only if \mathcal{S} satisfies the condition \mathcal{RF} .
- (iii) If R is a local ring and \mathcal{S} is non-zero, then \mathcal{S} satisfies the condition \mathcal{RF} .

Proof. (i)(\Rightarrow) Let $0 \neq M \in \mathcal{S}$ and $0 \neq x \in M$. Thus, there exists $\mathfrak{n} \in \text{Max}(R)$ such that $(0 :_R Rx) \subseteq \mathfrak{n}$. Now, as $Rx \in \mathcal{S}$, considering the natural epimorphism

$$Rx \cong R/(0 :_R Rx) \rightarrow R/\mathfrak{n},$$

we get $R/\mathfrak{n} \in \mathcal{S}$.

(\Leftarrow) It is obvious.

(ii) Assume that $\mathcal{S}_{\mathcal{FL}} \subseteq \mathcal{S}$ and \mathfrak{n} is an arbitrary maximal ideal of R . Since R/\mathfrak{n} has finite length, R/\mathfrak{n} belongs to \mathcal{S} . For inverse, let $L \in \mathcal{S}_{\mathcal{FL}}$ have the length l . Thus, there exists a chain

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_l = L$$

of R -submodules of N in which L_j/L_{j-1} is isomorphic to R/\mathfrak{n} for some $\mathfrak{n} \in \text{Max}(R)$ and all $1 \leq j \leq l$. Therefore, the proof is completed by induction on l .

(iii) The assertion follows from (i) and (ii). \square

3 Faltings' Local-global Principles

We begin this section with the following theorem, as an important result of the article. This is applied in the proof of Proposition 3.2.

Theorem 3.1. *Assume that \mathfrak{b} is an ideal of R , N is an R -module and $t \in \mathbb{N}_0$. Thus a necessary and sufficient condition for $H_{\mathfrak{b}}^i(N)$ to be a \mathfrak{b} -cofinite Artinian R -module is that $H_{\mathfrak{b}}^i(N) \in \mathcal{S}$ for every $i < t$ and any Serre subcategory \mathcal{S} satisfies the conditions $C_{\mathfrak{b}}$ and \mathcal{RF} .*

Proof. (\Rightarrow) Suppose that $H_{\mathfrak{b}}^i(N)$ is \mathfrak{b} -cofinite Artinian R -module for every $i < t$ and \mathcal{S} is an arbitrary Serre subcategory such that satisfies the conditions $C_{\mathfrak{b}}$ and \mathcal{RF} . Using induction on t , we prove that $H_{\mathfrak{b}}^i(N) \in \mathcal{S}$ for every $i < t$. When t is zero, we do not have anything to prove. Let $t = 1$. Since $\ell_R(0 :_{\Gamma_{\mathfrak{b}}(N)} \mathfrak{b}) < \infty$, $(0 :_{\Gamma_{\mathfrak{b}}(N)} \mathfrak{b}) \in \mathcal{S}$, by Lemma 2.15 (ii). Thus, $\Gamma_{\mathfrak{b}}(N) \in \mathcal{S}$ as \mathcal{S} satisfies the condition $C_{\mathfrak{b}}$.

Now, assume that $t > 1$ and the result is true for $i = 0, \dots, t - 2$. Using Proposition 2.12 and Corollary 2.2 for the category of \mathfrak{b} -cofinite Artinian R -modules

and $\Phi = \{\mathfrak{b}^j \mid j \geq 0\}$, since $H_{\mathfrak{b}}^i(N)$ is \mathfrak{b} -cofinite Artinian R -module for every $i \leq t-2$, $\ell_R(0 :_{H_{\mathfrak{b}}^{t-1}(N)} \mathfrak{b}) < \infty$ and so similar to the argument of the case $t = 1$, we get $(0 :_{H_{\mathfrak{b}}^{t-1}(N)} \mathfrak{b}) \in \mathcal{S}$.

(\Leftarrow) Choose \mathcal{S} as the category of \mathfrak{b} -cofinite Artinian R -modules. \square

The following result presents more equivalent conditions to the Local-global Principle for the Artinianness of ordinary local cohomology modules than those have been proven in [20, Theorem 2.2].

Proposition 3.2. *Assume that \mathfrak{b} is an ideal, N is an R -module and $t \in \mathbb{N}_0$ is such that the R -module $\text{Ext}_R^i(R/\mathfrak{b}, N)$ is finite for every $i < t$. Then the following conditions are equivalent:*

- (i) $H_{\mathfrak{b}}^i(N)$ is Artinian for every $i < t$;
- (ii) $H_{\mathfrak{b}}^i(N)$ is \mathfrak{b} -cofinite Artinian for every $i < t$;
- (iii) $(H_{\mathfrak{b}}^i(N))_{\mathfrak{p}}$ is Artinian for every $i < t$ and any prime ideal \mathfrak{p} of R ;
- (iv) $\text{Supp}(H_{\mathfrak{b}}^i(N)) \subseteq \text{Max}(R)$ for every $i < t$;
- (v) $H_{\mathfrak{b}}^i(N)$ belongs to \mathcal{S} for every $i < t$ and any Serre subcategory \mathcal{S} , which satisfies the conditions $C_{\mathfrak{b}}$ and \mathcal{RF} ;
- (vi) $\text{Ext}_R^i(R/\mathfrak{b}, N)$ belongs to \mathcal{S} for every $i < t$ and any Serre subcategory \mathcal{S} which satisfies the conditions $C_{\mathfrak{b}}$ and \mathcal{RF} .

Proof. The conditions (i) to (iv) are equivalent by [6, Theorem 3.2]. Also, conditions (ii) and (v) are equivalent by Theorem 3.1. Finally, (v) and (vi) are equivalent by Proposition 2.12, considering $\Phi = \{\mathfrak{b}^i \mid i \geq 0\}$.

\square

Corollary 3.3. *Assume that Φ is a system of ideals of R , N is a finite R -module and $\mathcal{S}_{\mathcal{A}}$ is the subcategory of Artinian R -modules of $\text{Mod}(R)$. Let $t \in \mathbb{N}_0$ be such that $H_{\Phi}^i(N) \in \mathcal{S}_{\mathcal{A}}$ for every $i < t$. Then $H_{\Phi}^i(N) \in \mathcal{S}$ for every $i < t$ and any Serre subcategory \mathcal{S} of $\text{Mod}(R)$, which satisfies the conditions C_{Φ} and \mathcal{RF} . Specially, for $\Phi = \{\mathfrak{b}^i \mid i \geq 0\}$, where \mathfrak{b} is an arbitrary ideal of R .*

Proof. Since $H_{\Phi}^i(N) \in \mathcal{S}_{\mathcal{A}}$ for every $i < t$, $H_{\mathfrak{b}}^i(N) \in \mathcal{S}_{\mathcal{A}}$ for every $i < t$ and any $\mathfrak{b} \in \Phi$, by applying [6, Corollary 2.7] for the class of Artinian R -modules. Therefore, Propositions 3.2 (part i \Rightarrow v) implies that $H_{\mathfrak{b}}^i(N) \in \mathcal{S}$ for every $i < t$ and any $\mathfrak{b} \in \Phi$. Now, the assertion follows from Propositions 2.12 (part iv \Rightarrow i). \square

Corollary 3.4. *Assume that \mathfrak{b} is an ideal of R such that $\dim R/\mathfrak{b} = 0$ and N is a finite R -module. Then $H_{\mathfrak{b}}^i(N) \in \mathcal{S}$ for any Serre subcategory \mathcal{S} satisfies the conditions $C_{\mathfrak{b}}$ and \mathcal{RF} and every $i \in \mathbb{N}_0$.*

Proof. As $\text{Supp}_R(H_{\mathfrak{b}}^i(N)) \subseteq \text{Max}(R)$ for every $i \geq 0$, the result holds easily by Proposition 3.2. \square

Let M be an R -module. As an interesting generalization of various regular sequences on M to a Serre subcategory \mathcal{S} of $\text{Mod}(R)$, the authors in [2, Definition 2.6], introduced the concept of \mathcal{S} -sequences on M . Then, in [2, Lemma 2.14 and Definition 2.15], for an ideal \mathfrak{a} of R and a Serre subcategory \mathcal{S} satisfying the condition $C_{\mathfrak{a}}$ with $M/\mathfrak{a}M \notin \mathcal{S}$, they showed that the lengths of all maximal \mathcal{S} -sequences on M in \mathfrak{a} , are equal. They denoted this number by $\mathcal{S}\text{-depth}_{\mathfrak{a}}(M)$ and in [2, Theorem 2.18], they proved that

$$\mathcal{S}\text{-depth}_{\mathfrak{a}}(M) = \min\{i \geq 0 \mid H_{\mathfrak{a}}^i(M) \notin \mathcal{S}\}.$$

In this stage, we first introduce the concept of $\mathcal{S}\text{-depth}_{\Phi}(N)$, in which N is an R -module, Φ is a system of ideals of R and \mathcal{S} is a Serre subcategory of $\text{Mod}(R)$ satisfies the condition C_{Φ} . Then, in Proposition 3.7, we show that for any Serre subcategory \mathcal{S} satisfying the conditions C_{Φ} and \mathcal{RF} , the following inequality holds:

$$\inf\{j \in \mathbb{N}_0 \mid H_{\Phi}^j(N) \text{ is not Artinian}\} \leq \mathcal{S}\text{-depth}_{\Phi}(N).$$

Definition 3.5. Assume that Φ is a system of ideals of R , N is an R -module and \mathcal{S} is a Serre subcategory of $\text{Mod}(R)$ satisfying the condition C_{Φ} . We define $\mathcal{S}\text{-depth}_{\Phi}(N)$ as:

$$\mathcal{S}\text{-depth}_{\Phi}(N) = \inf\{j \geq 0 \mid H_{\Phi}^j(N) \notin \mathcal{S}\}$$

if the infimum exists, ∞ otherwise.

Remark 3.6. Let \mathcal{S} be mentioned in Definition 3.5. According to Proposition 2.12 and the paragraph before Definition 3.5, it is obvious that if N is a finite R -module and $\mathfrak{b} \in \Phi$ is such that $N/\mathfrak{b}N$ does not belong to \mathcal{S} , then $\mathcal{S}\text{-depth}_{\Phi}(N) \in \mathbb{N}_0$ and

$$\mathcal{S}\text{-depth}_{\Phi}(N) = \min\{\mathcal{S}\text{-depth}_{\mathfrak{b}}(N) \mid \mathfrak{b} \in \Phi\}.$$

In addition, for an arbitrary ideal \mathfrak{b} of R , with choosing suitable Serre subcategories \mathcal{S} which satisfy the condition $C_{\mathfrak{b}}$ and $\Phi = \{\mathfrak{b}^i \mid i \geq 0\}$ in Definition 3.5, we can obtain the concepts of $\text{depth}_{\mathfrak{b}}(N)$, $\text{f-depth}_{\mathfrak{b}}(N)$ and $\text{g-depth}_{\mathfrak{b}}(N)$. Recall that, by [20, Theorem 3.4], [16, Theorem 3.1] and [17, Proposition 5.2], we have

$$\text{f-depth}_{\mathfrak{b}}(N) = \inf\{i \geq 0 \mid H_{\mathfrak{b}}^i(N) \text{ is not Artinian}\},$$

and

$$\text{g-depth}_{\mathfrak{b}}(N) = \inf\{i \geq 0 \mid \text{Supp}_R H_{\mathfrak{b}}^i(N) \text{ is not a finite set}\}.$$

The next result indicates that $\text{f-depth}_{\Phi}(N)$, i.e., the least integer $j \in \mathbb{N}_0$ that $H_{\Phi}^j(N)$ is not Artinian, is a lower bound for $\mathcal{S}\text{-depth}_{\Phi}(N)$, for all Serre subcategory \mathcal{S} of $\text{Mod}(R)$ satisfying the conditions C_{Φ} and \mathcal{RF} .

Proposition 3.7. Assume that Φ is a system of ideals of R and N is a finite R -module. Let $\mathcal{S}_{\mathcal{A}}$ be the category of Artinian R -modules and \mathcal{S} be a Serre subcategory of $\text{Mod}(R)$ satisfying the conditions C_{Φ} and \mathcal{RF} . Then

$$\text{f-depth}_{\Phi}(N) = \inf\{j \geq 0 \mid H_{\Phi}^j(N) \notin \mathcal{S}_{\mathcal{A}}\} \leq \mathcal{S}\text{-depth}_{\Phi}(N).$$

Specially, for any ideal \mathfrak{b} of R that $N/\mathfrak{b}N \notin \mathcal{S}$,

$$\text{f-depth}_{\mathfrak{b}}(N) = \inf\{j \in \mathbb{N}_0 \mid \text{Ext}_R^j(R/\mathfrak{b}, N) \notin \mathcal{S}_{\mathcal{A}}\} \leq \mathcal{S}\text{-depth}_{\mathfrak{b}}(N).$$

Proof. The assertions are obtained from Corollary 3.3 and Proposition 3.2. \square

Corollary 3.8. *Suppose that (R, \mathfrak{n}) is a local ring. Let N be a finite R -module and \mathfrak{b} be an ideal of R . Then for any non-zero Serre subcategory \mathcal{S} satisfying the condition $C_{\mathfrak{b}}$ and $N/\mathfrak{b}N \notin \mathcal{S}$, we get*

$$\text{f-depth}_{\mathfrak{b}}(N) \leq \mathcal{S}\text{-depth}_{\mathfrak{b}}(N).$$

Specially,

$$\text{depth}_{\mathfrak{b}}(N) \leq \text{f-depth}_{\mathfrak{b}}(N) \leq \text{g-depth}_{\mathfrak{b}}(N).$$

Proof. The assertions will be obtained from Proposition 3.7 and Lemma 2.15 (i). \square

As it is mentioned in the introduction, the most important result of [15] is its Theorem 2.10, which shows that the equality $f_{\mathfrak{b}}^n(N) = h_{\mathfrak{b}}^n(N)$ holds for any finite R -module N and any ideal \mathfrak{b} on an arbitrary Noetherian ring R . Following, Theorem 3.10, as the last important result of this article, generalizes the main results of [4, Theorem 2.5], [15, Theorem 2.10] and [19, Theorem 2.17]. To do this, the following definition is needed.

Definition 3.9. Assume that Φ is a system of ideals of R and N is an R -module. Let $n \in \mathbb{N}_0$ and $\Phi_{\mathfrak{p}} := \{\mathfrak{b}R_{\mathfrak{p}} \mid \mathfrak{b} \in \Phi\}$ for any $\mathfrak{p} \in \text{Spec}(R)$. We define

$$h_{\Phi}^n(N) := \inf\{j \geq 0 \mid H_{\Phi}^j(N) \text{ is not in dimension } < n\}$$

and

$$f_{\Phi}^n(N) := \inf\{f_{\Phi_{\mathfrak{p}}}(N_{\mathfrak{p}}) \mid \mathfrak{p} \in \text{Supp}_R(N) \text{ and } \dim R/\mathfrak{p} \geq n\}.$$

Theorem 3.10. *Assume that R is a Noetherian ring and that Φ is a system of ideals of R . Let $n \in \mathbb{N}_0$ be such that the R -module N is in dimension $< n$. Suppose that the set $(\text{Ass}_R H_{\Phi}^{h_{\Phi}^n(N)}(N))_{\geq n}$ is finite. Then $h_{\Phi}^n(N) = f_{\Phi}^n(N)$.*

Proof. Put $s := h_{\Phi}^n(N)$. For every $i < s$, $\dim \text{Supp}(H_{\Phi}^i(N)/N') < n$ for some finite submodule N' of $H_{\Phi}^i(N)$. So for any $\mathfrak{q} \in (\text{Spec}(R))_{\geq n}$, we obtain $(H_{\Phi}^i(N)/N')_{\mathfrak{q}}$ is zero. Consequently, $H_{\Phi_{\mathfrak{q}}}^i(N_{\mathfrak{q}})$ is a finite $R_{\mathfrak{q}}$ -module. Hence $s \leq f_{\Phi}^n(N)$. Now, we prove that $s = f_{\Phi}^n(N)$. On the contrary, suppose that $s < f_{\Phi}^n(N)$. Assume that

$$(\text{Ass}_R H_{\Phi}^s(N))_{\geq n} = \{\mathfrak{q}_1, \mathfrak{q}_2, \dots, \mathfrak{q}_r\}.$$

For any $1 \leq j \leq r$, $\mathfrak{q}_j \in (\text{Spec}(R))_{\geq n}$. Thus, as $s < f_{\Phi}^n(N)$, $(H_{\Phi}^s(N))_{\mathfrak{q}_j}$ is a finite $R_{\mathfrak{q}_j}$ -module. So that for every $1 \leq j \leq r$, there exists a finite R -submodule M_j of $H_{\Phi}^s(N)$ such that $(H_{\Phi}^s(N))_{\mathfrak{q}_j} \cong (M_j)_{\mathfrak{q}_j}$. Put $L_1 := M_1 + M_2 + \dots + M_r$. Then L_1 is a finite R -submodule (and Φ -torsion) of $H_{\Phi}^s(N)$ and we get

$$(\text{Ass}_R H_{\Phi}^s(N)/L_1)_{\geq n} \cap (\text{Ass}_R H_{\Phi}^s(N))_{\geq n} = \emptyset.$$

Next, we prove that the set $(\text{Ass}_R H_{\Phi}^s(N)/L_1)_{\geq n}$ is finite, too. According to [7, Lemma 2.1] and since the set $(\text{Ass}_R H_{\Phi}^s(N))_{\geq n}$ is finite, there are ideals $\mathfrak{b}_1, \mathfrak{b}_2, \dots, \mathfrak{b}_r \in \Phi$ such that

$$(\text{Ass}_R H_{\Phi}^s(N))_{\geq n} \subseteq \bigcup_{j=1}^r (\text{Ass}_R H_{\mathfrak{b}_j}^s(N))_{\geq n} \subseteq \bigcup_{j=1}^r (\mathbb{V}(\mathfrak{b}_j))_{\geq n}.$$

As Φ is a system of ideals of R , there is $\mathfrak{b} \in \Phi$ such that $\mathfrak{b} \subseteq \prod_{j=1}^r \mathfrak{b}_j$ and thus

$$(\text{Ass}_R H_{\Phi}^s(N))_{\geq n} \subseteq \bigcup_{j=1}^r (\mathbb{V}(\mathfrak{b}_j))_{\geq n} \subseteq (\mathbb{V}(\mathfrak{b}))_{\geq n}.$$

We claim that $(\text{Supp}_R H_{\Phi}^s(N))_{\geq n} \subseteq (\mathbb{V}(\mathfrak{b}))_{\geq n}$. For this purpose, let \mathfrak{q} be an arbitrary element of $(\text{Supp}_R H_{\Phi}^s(N))_{\geq n}$. Then, there is $\mathfrak{p} \in (\text{Ass}_R H_{\Phi}^s(N))_{\geq n}$ such that $\mathfrak{p} \subseteq \mathfrak{q}$. So $\mathfrak{p} \in (\mathbb{V}(\mathfrak{b}))_{\geq n}$. Since $\mathfrak{b} \subseteq \mathfrak{p} \subseteq \mathfrak{q}$ and $\dim R/\mathfrak{q} \geq n$, we get $\mathfrak{q} \in (\mathbb{V}(\mathfrak{b}))_{\geq n}$ as required. Therefore

$$(\text{Ass}_R H_{\Phi}^s(N)/L_1)_{\geq n} \subseteq (\text{Supp}_R H_{\Phi}^s(N)/L_1)_{\geq n} \subseteq (\text{Supp}_R H_{\Phi}^s(N))_{\geq n} \subseteq (\mathbb{V}(\mathfrak{b}))_{\geq n}.$$

Also, as for every $i < s$, $H_{\Phi}^i(N)$ is an R -module in dimension $< n$, by Corollary 2.3, $(\text{Ass}_R (H_{\Phi}^s(N)/L_1) \cap \mathbb{V}(\mathfrak{b}))_{\geq n}$ is a finite set and so, $(\text{Ass}_R H_{\Phi}^s(N)/L_1)_{\geq n}$ is a finite set as well. Now, as

$$(\text{Ass}_R H_{\Phi}^s(N)/L_1)_{\geq n} \cap (\text{Ass}_R H_{\Phi}^s(N))_{\geq n} = \emptyset,$$

from [15, Lemma 2.5], we conclude that

$$\bigcap_{\mathfrak{q} \in (\text{Ass}_R H_{\Phi}^s(N))_{\geq n}} \mathfrak{q} \subsetneq \bigcap_{\mathfrak{q} \in (\text{Ass}_R H_{\Phi}^s(N)/L_1)_{\geq n}} \mathfrak{q}.$$

Using a similar argument, there exists a submodule L_2/L_1 of $H_{\Phi}^s(N)/L_1$ such that

$$(\text{Ass}_R H_{\Phi}^s(N)/L_2)_{\geq n} \cap (\text{Ass}_R H_{\Phi}^s(N)/L_1)_{\geq n} = \emptyset,$$

and so

$$\bigcap_{\mathfrak{q} \in (\text{Ass}_R H_{\Phi}^s(N)/L_1)_{\geq n}} \mathfrak{q} \subsetneq \bigcap_{\mathfrak{q} \in (\text{Ass}_R H_{\Phi}^s(N)/L_2)_{\geq n}} \mathfrak{q}.$$

Proceeding in the same way, we can find a strictly chain of ideals of R as follows:

$$\bigcap_{\mathfrak{q} \in (\text{Ass}_R H_{\Phi}^s(N))_{\geq n}} \mathfrak{q} \subsetneq \bigcap_{\mathfrak{q} \in (\text{Ass}_R H_{\Phi}^s(N)/L_1)_{\geq n}} \mathfrak{q} \subsetneq \bigcap_{\mathfrak{q} \in (\text{Ass}_R H_{\Phi}^s(N)/L_2)_{\geq n}} \mathfrak{q} \subsetneq \dots$$

which is not stable and this is a contradiction. Therefore $s = f_{\Phi}^n(N)$. \square

Corollary 3.11. *Assume that R is a Noetherian ring and that \mathfrak{b} is an ideal of R . Let N be a finite R -module. Then for every $n \in \mathbb{N}_0$, $f_{\mathfrak{a}}^n(N) = h_{\mathfrak{a}}^n(N)$.*

Proof. Apply $\Phi = \{\mathfrak{b}^j \mid j > 0\}$ and $s = h_{\mathfrak{b}}^n(M)$ in Theorem 3.10 and Corollary 2.3. \square

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References

- [1] M. Aghapournahr and K. Bahmanpour, Cofiniteness of weakly Laskerian local cohomology modules, *Bull. Math. Soc. Sci. Math. Roumanie*, **105** (2014), no. 4, 347-356.
- [2] M. Aghapournahr and L. Melkersson, Local cohomology and Serre subcategories, *J. Algebra*, **320** (2008), no. 3, 1275-1287.
- [3] M. Aghapournahr and L. Melkersson, A natural map in local cohomology, *Ark. Mat.*, **48** (2010), no. 2, 243-251.
- [4] D. Asadollahi and R. Naghipour, Faltings' Local-global Principle for the finiteness of local cohomology modules, *Comm. Algebra*, **43** (2015), no. 3, 953-958.
- [5] K. Bahmanpour, R. Naghipour and M. Sedghi, Minimaxness and cofiniteness properties of local cohomology modules, *Comm. Algebra*, **41** (2013), no. 8, 2799-2814.
- [6] M. Behrouzian and M. Aghapournahr, Lower bounds of certain general local cohomology modules, *Comm. Algebra*, **48** (2020), no. 6, 2406-2417.
- [7] M. H. Bijan-Zadeh, Torsion theories and local cohomology over Commutative Noetherian ring, *J. London Math. Soc. (2)*, **19** (1979), no. 3, 402-410.
- [8] M. H. Bijan-Zadeh, A common generalization of Local cohomology theories, *Glasg. Math. J.*, **21** (1980), no. 2, 173-181.
- [9] M. P. Brodmann and R. Y. Sharp, *Local Cohomology: an Algebraic Introduction with Geometric Applications*, Cambridge Univ. Press, Cambridge (1998).
- [10] K. Divaani-Azar and A. Mafi, Associated primes of local cohomology modules, *Proc. Amer. Math. Soc.*, **133** (2005), no. 3, 655-660.
- [11] G. Faltings, Der Endlichkeitssatz in der lokalen kohomologie, *Math. Ann.*, **255** (1981), no. 1, 45-56.
- [12] A. Hajikarimi, Cofiniteness with respect to a Serre subcategory, *Math. Notes*, **89** (2011), 121-130.
- [13] R. Hartshorn, Affine duality and cofiniteness, *Invent. Math.*, **9** (1970), 145-164.

- [14] K. Khashyarmansh, On the finiteness properties of extension and torsion functors of local cohomology modules, *Proc. Amer. Math. Soc.*, **135** (2007), 1319-1327.
- [15] A. A. Mehrvarz, R. Naghipour and M. Sedghi, Faltings' Local-global Principle for the finiteness of local cohomology modules over Noetherian rings, *Comm. Algebra*, **43** (2015), no. 11, 4860-4872.
- [16] L. Melkersson, Some applications of a criterion for Artinianness of a module, *J. Pure Appl. Algebra*, **101** (1995), no. 3, 291-303.
- [17] L.T. Nhan, On generalized regular sequences and the finiteness for associated primes of local cohomology modules, *Comm. Algebra*, **33** (2005), no. 3, 793-806.
- [18] P. Rudlof, On minimax and related modules, *Can. J. Math.*, **44** (1992), 154-166.
- [19] M.Y. Sadeghi, Kh. Ahmadi Amoli and M. Chaghamirza, General local cohomology modules in view of low points and high points, *Ukrainian Math. J.*, **75** (2023), 698-711.
- [20] Z. Tang, Local-global Principle for the Artinianness of local cohomology modules, *Comm. Algebra*, **40** (2012), no. 1, 58-63.
- [21] T. Zink, Endlichkeitsbedingungen für modulnüber einem Noetherschen ring, *Math. Nachr.*, **64** (1974), 239-252, (in German).
- [22] H. Zöschinger, Minimax-moduln, *J. Algebra*, **102** (1986), 1-32.
- [23] H. Zöschinger, Über die maximalbedingung für radikalvolle untermoduln, *Hokkaido Math. J.*, **17** (1988), 101-116.

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