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Another Approach to Generate Fuzzy Normed Spaces and Fuzzy Normed Algebras by Normed Spaces and Normed Algebras Respectively

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Abstract. In this paper, we introduce a new approach to generate a fuzzy norm using a classic norm and a continuous Archimedean t-norm (CATN). Our method involves two steps. First, we utilize a CATN to create a continuous additive generator (CAG). Then, we employ the corresponding additive generator (AG) and a classic norm to generate a fuzzy norm.

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1 Introduction

It is well-established that the fuzzy concepts hold significant importance in multiple disciplines such as engineering, medicine, management, and

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mathematics. Two topics from fuzzy theory that can be very relevant are t-norm and fuzzy norm. In this paper, we will present specific information to one of the fuzzy definitions called fuzzy norm, that is a basic notion in fuzzy functional analysis.

In 1965, Zadeh [11] introduced fuzzy sets to the world of science through a scientific paper. The fuzzy sets serve as a uncertain mathematical model in the field of analysis. First time, A .K. Katsaras, defined a concept called fuzzy norm in [4]. Also, some analytical and topological definitions using fuzzy norm are stated in [10]. In later years, T. Bag and S. K. Samanta in [1], gave a newer definition of fuzzy norm, working with that definition facilitated the procedure of proving theorems. R. Saadati and J. H. Park in [7] by using t-norms and continuous t-conorms, have reached to intuitionistic fuzzy normed spaces. O. Grigorenko, J. Minana, and O. Valero [3], introduced a method for making a fuzzy metric. T. Binzar, F. Pater, and S. Nadaban [2] investigated the relationship between fuzzy normed algebras.

The main motivation of this paper is to identify new fuzzy norms and new algebra fuzzy norms. In this paper, we state that a new CATN can be obtained from additive and multiplicative generators of a previous CATN. This method shows that countless of unknown CATN can be obtained.

The rest of the paper is organized as follows : In Section 2, we mention the definitions of t-norm, additive generator, multiplicative generator, pseudo-inverse, fuzzy norm, and etc. Some new results are stated in section 3.

2 Definitions

Definition 2.1. [3] *A triangular norm (briefly, t-norm) is a function $\star : [0, 1]^2 \longrightarrow [0, 1]$ that for each $(\alpha, \beta, \gamma) \in [0, 1]^3$ we have :*

- 1) $\alpha \star \beta = \beta \star \alpha,$
- 2) $\alpha \star (\beta \star \gamma) = (\alpha \star \beta) \star \gamma,$
- 3) $\alpha \star \beta \geq \alpha \star \gamma,$ for $\beta \geq \gamma,$
- 4) $\alpha \star 1 = \alpha.$

Definition 2.2. [5] A t -norm $\star : [0, 1]^2 \rightarrow [0, 1]$ is called Archimedean t -norm if for each $(\alpha, \beta) \in (0, 1) \times (0, 1)$ there is $k \in \mathbb{N}$ such that $\alpha^{(k)} < \beta$ where

$$\alpha^{(k)} = \overbrace{\alpha \star \alpha \star \cdots \star \alpha}^{k\text{-times}}.$$

Also, a continuous t -norm is Archimedean iff $\alpha \star \alpha < \alpha$ for all $0 < \alpha < 1$ [5].

Example 2.3. If $\star_L : [0, 1]^2 \rightarrow [0, 1]$ and $\star_p : [0, 1]^2 \rightarrow [0, 1]$ are defined by $\alpha \star_L \beta = \max\{\alpha + \beta - 1, 0\}$ and $\alpha \star_p \beta = \alpha\beta$, then \star_L and \star_p are Archimedean t -norm.

Definition 2.4. [5] Let $\Psi : [0, 1] \rightarrow [0, \infty]$ be a decreasing function. Then $\Psi^{(-1)} : [0, \infty] \rightarrow [0, 1]$ is pseudo-inverse of Ψ that is defined by

$$\Psi^{(-1)}(\gamma) = \sup\{\zeta \in [0, 1] : \Psi(\zeta) > \gamma\}$$

where we assume that $\sup \emptyset = 0$. Also if Ψ is continuous and strictly decreasing, then we have :

$$\Psi^{(-1)}(\gamma) = \begin{cases} 1, & 0 \leq \gamma < \Psi(1) \\ \Psi^{-1}(\gamma), & \Psi(1) \leq \gamma < \Psi(0) \\ 0, & \Psi(0) \leq \gamma \leq \infty \end{cases}.$$

In particular, if Ψ is continuous, strictly decreasing, and $\Psi(1) = 0$, then

$$\Psi^{(-1)}(\gamma) = \begin{cases} \Psi^{-1}(\gamma), & 0 \leq \gamma < \Psi(0) \\ 0, & \Psi(0) \leq \gamma \leq \infty \end{cases}.$$

Definition 2.5. [5] Let $\Psi : [0, 1] \rightarrow [0, 1]$ be an increasing function. Then $\Psi^{(-1)} : [0, 1] \rightarrow [0, 1]$ is pseudo-inverse of Ψ that is defined by

$$\Psi^{(-1)}(\gamma) = \sup\{\zeta \in [0, 1] : \Psi(\zeta) < \gamma\}$$

where we assume that $\sup \emptyset = 0$. Also if Ψ is continuous and strictly increasing, then we have :

$$\Psi^{(-1)}(\gamma) = \begin{cases} 0, & 0 \leq \gamma < \Psi(0) \\ \Psi^{-1}(\gamma), & \Psi(0) \leq \gamma < \Psi(1) \\ 1, & \Psi(1) \leq \gamma \leq 1 \end{cases}.$$

In particular, if Ψ is continuous, strictly increasing, and $\Psi(1) = 1$, then

$$\Psi^{(-1)}(\gamma) = \begin{cases} 0, & 0 \leq \gamma < \Psi(0) \\ \Psi^{-1}(\gamma), & \Psi(0) \leq \gamma \leq 1 \end{cases}.$$

Definition 2.6. [5] A multiplicative generator (MG) of a t -norm \star is a strictly increasing function $\varrho : [0, 1] \rightarrow [0, 1]$ which is right-continuous at $\zeta = 0$ and $\varrho(1) = 1$ such that for each $(\zeta, \gamma) \in [0, 1] \times [0, 1]$,

$$\varrho(\zeta) \varrho(\gamma) \in \text{Ran}(\varrho) \cup [0, \varrho(0)]$$

and

$$\zeta \star \gamma = \varrho^{(-1)}(\varrho(\zeta) \varrho(\gamma)).$$

Corollary 2.7. [5] Let $\varrho : [0, 1] \rightarrow [0, 1]$ be a strictly increasing function which is right-continuous at $\zeta = 0$, $\varrho(1) = 1$ and for each $(\zeta, \gamma) \in [0, 1] \times [0, 1]$,

$$\varrho(\zeta) \varrho(\gamma) \in \text{Ran}(\varrho) \cup [0, \varrho(0)].$$

If $\star : [0, 1]^2 \rightarrow [0, 1]$ is defined by

$$\zeta \star \gamma = \varrho^{(-1)}(\varrho(\zeta) \varrho(\gamma)),$$

then \star is a t -norm.

Corollary 2.8. [5] Let $\star : [0, 1]^2 \rightarrow [0, 1]$ be a t -norm. Then \star is a CATN iff there exists a continuous multiplicative generator (CMG) of \star .

Definition 2.9. [3] An additive generator (AG) of a t -norm \star is a strictly decreasing function $\xi_\star : [0, 1] \rightarrow [0, \infty]$ which is also right-continuous at $\zeta = 0$ and $\xi_\star(1) = 0$. Also for all $(\zeta, \gamma) \in [0, 1] \times [0, 1]$, The following are valid :

$$\xi_\star(\zeta) + \xi_\star(\gamma) \in \text{Ran}(\xi_\star) \cup [\xi_\star(0), \infty]$$

and

$$\zeta \star \gamma = \xi_\star^{(-1)}(\xi_\star(\zeta) + \xi_\star(\gamma)).$$

Proposition 2.10. [3] Let $\xi : [0, 1] \rightarrow [0, \infty]$ be a strictly decreasing function with $\xi(1) = 0$ and

$$\xi(\zeta) + \xi(\gamma) \in \text{Ran}(\xi) \cup [\xi(0), \infty]$$

for all $(\zeta, \gamma) \in [0, 1] \times [0, 1]$. Then $\star : [0, 1]^2 \rightarrow [0, 1]$ defined by

$$\zeta \star \gamma = \xi^{(-1)}(\xi(\zeta) + \xi(\gamma))$$

is a t -norm.

Theorem 2.11. [3] A t -norm \star is a CATN iff there exists a CAG such as ξ_\star such that for each $(\zeta, \gamma) \in [0, 1] \times [0, 1]$,

$$\zeta \star \gamma = \xi_\star^{(-1)}(\xi_\star(\zeta) + \xi_\star(\gamma)).$$

In the following, we will state that by using a CAG and a CMG of CATN \star , a new CAG and consequently a new t -norm can be obtained.

Corollary 2.12. Let \star be a CATN and ξ_\star and ϱ_\star be CAG and CMG of \star respectively. If $\xi_\star(0) = \infty$, then $h : [0, 1] \rightarrow [0, \infty]$ where $h(\zeta) = \xi_\star(\zeta) - \varrho_\star(\zeta) + 1$ is a CAG of \star' where

$$\zeta \star' \gamma = h^{(-1)}(h(\zeta) + h(\gamma))$$

for all $(\zeta, \gamma) \in [0, 1] \times [0, 1]$.

Proof. Since ξ_\star and ϱ_\star are continuous, h is continuous. As ξ_\star is strictly decreasing and ϱ_\star is strictly increasing, h is strictly decreasing. Also $h(1) = 0$ and $h(0) = \infty$. Hence the continuity of h implies that $\text{Ran}(h) = [0, \infty]$. Therefore $h(\zeta) + h(\gamma) \in \text{Ran}(h)$ for all $\zeta, \gamma \in [0, 1]$. Applying Proposition 2.10, we can conclude that h is a CAG of \star' . \square

Definition 2.13. [9] Let Z be a linear space and \star be a t -norm. A function $\aleph : Z \times \mathbb{R} \rightarrow [0, 1]$ is named a fuzzy norm on Z if for all $\zeta, \gamma \in Z$ and all $s, t \in \mathbb{R}$, we have :

- 1) $\aleph(\zeta, t) = 0$ for all $t \leq 0$.
- 2) $\aleph(\zeta, t) = 1$ for all $t > 0$ iff $\zeta = 0$.

3) $\aleph(\mu\zeta, t) = \aleph\left(\zeta, \frac{t}{|\mu|}\right)$ for all $\mu \neq 0$ and $t \in \mathbb{R}$.

4) $\aleph(\zeta + \gamma, s + t) \geq \aleph(\zeta, s) \star \aleph(\gamma, t)$ for all $s, t \in \mathbb{R}$.

5) $\aleph(\zeta, \cdot)$ is increasing on \mathbb{R} and $\lim_{t \rightarrow \infty} \aleph(\zeta, t) = 1$.

Considering the above definition, (Z, \aleph, \star) is called a fuzzy normed space.

3 Main Results

In this section, let \star be a CATN and ξ_\star be the corresponding CAG of \star . Also let $\xi_\star^{(-1)}$ be the pseudo-inverse of ξ_\star .

Theorem 3.1. *Let $(Z, \|\cdot\|)$ be a normed space, \star be a CATN and ξ_\star be a CAG of \star . Then (Z, \aleph_\star, \star) is a fuzzy normed space which for each $\zeta \in Z$ and each $t \in \mathbb{R}$ is defined as follows :*

$$\aleph_\star(\zeta, t) = \begin{cases} \xi_\star^{(-1)}\left(\frac{\|\zeta\|}{t}\right), & t > 0 \\ 0, & t \leq 0 \end{cases}.$$

Proof. If $\xi_\star(0) = \infty$, then by Definition 2.4 we have

$$\aleph_\star(\zeta, t) = \begin{cases} \xi_\star^{-1}\left(\frac{\|\zeta\|}{t}\right), & \frac{\|\zeta\|}{t} \geq 0 \\ 0, & t \leq 0 \end{cases}.$$

In this case, we only prove part 4 of Definition 2.13. Let $\zeta, \gamma \in Z$ and $s, t \in \mathbb{R}$. If $s \leq 0$ or $t \leq 0$, then $\aleph_\star(\zeta, s) = 0$ or $\aleph_\star(\gamma, t) = 0$. Hence

$$\aleph_\star(\zeta + \gamma, s + t) \geq \aleph_\star(\zeta, s) \star \aleph_\star(\gamma, t) = 0.$$

If $s > 0$ and $t > 0$, then

$$\frac{\|\zeta + \gamma\|}{s + t} \leq \frac{\|\zeta\| + \|\gamma\|}{s + t} \leq \frac{\|\zeta\|}{s} + \frac{\|\gamma\|}{t}.$$

Hence

$$\begin{aligned}
\aleph_\star(\zeta + \gamma, s + t) &= \xi_\star^{-1} \left(\frac{\|\zeta + \gamma\|}{s + t} \right) \\
&\geq \xi_\star^{-1} \left(\frac{\|\zeta\|}{s} + \frac{\|\gamma\|}{t} \right) \\
&= \xi_\star^{-1} \left(\xi_\star \left(\xi_\star^{-1} \left(\frac{\|\zeta\|}{s} \right) \right) + \xi_\star \left(\xi_\star^{-1} \left(\frac{\|\gamma\|}{t} \right) \right) \right) \\
&= \xi_\star^{(-1)} \left(\xi_\star \left(\xi_\star^{(-1)} \left(\frac{\|\zeta\|}{s} \right) \right) + \xi_\star \left(\xi_\star^{(-1)} \left(\frac{\|\gamma\|}{t} \right) \right) \right) \\
&= \xi_\star^{(-1)} \left(\frac{\|\zeta\|}{s} \right) \star \xi_\star^{(-1)} \left(\frac{\|\gamma\|}{t} \right) \\
&= \aleph_\star(\zeta, s) \star \aleph_\star(\gamma, t).
\end{aligned}$$

Now, we prove the theorem for $\xi_\star(0) < \infty$. In this case

$$\begin{aligned}
\aleph_\star(\zeta, t) &= \begin{cases} \xi_\star^{(-1)} \left(\frac{\|\zeta\|}{t} \right), & t > 0 \\ 0, & t \leq 0 \end{cases} \\
&= \begin{cases} \xi_\star^{-1} \left(\frac{\|\zeta\|}{t} \right), & 0 \leq \frac{\|\zeta\|}{t} < \xi_\star(0), t > 0 \\ 0, & \xi_\star(0) \leq \frac{\|\zeta\|}{t}, t > 0 \\ 0, & t \leq 0 \end{cases}.
\end{aligned}$$

For each $t \leq 0$, clearly $\aleph_\star(\zeta, t) = 0$. To prove the second property, if for each $t > 0$, $\aleph_\star(\zeta, t) = 1$. Then $\xi_\star^{-1} \left(\frac{\|\zeta\|}{t} \right) = 1$. Hence $\frac{\|\zeta\|}{t} = \xi_\star(1) = 0$ for all $t > 0$. Then $\zeta = 0$. Conversely, if $\zeta = 0$, then for each $t > 0$,

$$\aleph_\star(0, t) = \xi_\star^{-1}(0) = 1.$$

To prove the third part, let $\zeta \in Z$, $t \in \mathbb{R}$ and $\alpha \neq 0$. We have :

$$\begin{aligned}
\aleph_{\star}(\alpha\zeta, t) &= \begin{cases} \xi_{\star}^{-1}\left(\frac{\|\alpha\zeta\|}{t}\right), & 0 \leq \frac{\|\alpha\zeta\|}{t} < \xi_{\star}(0), t > 0 \\ 0, & \xi_{\star}(0) \leq \frac{\|\alpha\zeta\|}{t}, t > 0 \\ 0, & t \leq 0 \end{cases} \\
&= \begin{cases} \xi_{\star}^{-1}\left(\frac{\|\zeta\|}{|\alpha|}\right), & 0 \leq \frac{\|\zeta\|}{|\alpha|} < \xi_{\star}(0), t > 0 \\ 0, & \xi_{\star}(0) \leq \frac{\|\zeta\|}{|\alpha|}, t > 0 \\ 0, & t \leq 0 \end{cases} \\
&= \begin{cases} \xi_{\star}^{-1}\left(\frac{\|\zeta\|}{|\alpha|}\right), & 0 \leq \frac{\|\zeta\|}{|\alpha|} < \xi_{\star}(0), \frac{t}{|\alpha|} > 0 \\ 0, & \xi_{\star}(0) \leq \frac{\|\zeta\|}{|\alpha|}, \frac{t}{|\alpha|} > 0 \\ 0, & \frac{t}{|\alpha|} \leq 0 \end{cases} \\
&= \aleph_{\star}\left(\zeta, \frac{t}{|\alpha|}\right).
\end{aligned}$$

To prove the fourth part, let $\zeta, \gamma \in Z$ and $s, t \in \mathbb{R}$. If $s \leq 0$ or $t \leq 0$, then $\aleph_{\star}(\zeta, s) = 0$ or $\aleph_{\star}(\gamma, t) = 0$. Since $a \star 0 = 0 \star b = 0$ for all $a, b \in [0, 1]$,

$$\aleph_{\star}(\zeta + \gamma, s + t) \geq \aleph_{\star}(\zeta, s) \star \aleph_{\star}(\gamma, t) = 0.$$

Now we suppose that $s, t \in (0, \infty)$. If $\xi_{\star}(0) \leq \frac{\|\zeta\|}{s}$ or $\xi_{\star}(0) \leq \frac{\|\gamma\|}{t}$, then $\aleph_{\star}(\zeta, s) = 0$ or $\aleph_{\star}(\gamma, t) = 0$. Hence

$$\aleph_{\star}(\zeta + \gamma, s + t) \geq \aleph_{\star}(\zeta, s) \star \aleph_{\star}(\gamma, t) = 0.$$

If $0 \leq \frac{\|\zeta\|}{s} < \xi_\star(0)$ and $0 \leq \frac{\|\gamma\|}{t} < \xi_\star(0)$, then

$$\aleph_\star(\zeta, s) = \xi_\star^{(-1)}\left(\frac{\|\zeta\|}{s}\right) = \xi_\star^{-1}\left(\frac{\|\zeta\|}{s}\right)$$

and

$$\aleph_\star(\gamma, t) = \xi_\star^{(-1)}\left(\frac{\|\gamma\|}{t}\right) = \xi_\star^{-1}\left(\frac{\|\gamma\|}{t}\right).$$

Since $\xi_\star^{(-1)}$ is decreasing on $[0, \infty]$ and $\frac{\|\zeta + \gamma\|}{s+t} \leq \frac{\|\zeta\|}{s} + \frac{\|\gamma\|}{t}$,

$$\begin{aligned} \aleph_\star(\zeta + \gamma, s+t) &= \xi_\star^{(-1)}\left(\frac{\|\zeta + \gamma\|}{s+t}\right) \\ &\geq \xi_\star^{(-1)}\left(\frac{\|\zeta\|}{s} + \frac{\|\gamma\|}{t}\right) \\ &= \xi_\star^{(-1)}\left(\xi_\star\left(\xi_\star^{-1}\left(\frac{\|\zeta\|}{s}\right)\right) + \xi_\star\left(\xi_\star^{-1}\left(\frac{\|\gamma\|}{t}\right)\right)\right) \\ &= \xi_\star^{(-1)}\left(\xi_\star\left(\xi_\star^{(-1)}\left(\frac{\|\zeta\|}{s}\right)\right) + \xi_\star\left(\xi_\star^{(-1)}\left(\frac{\|\gamma\|}{t}\right)\right)\right) \\ &= \xi_\star^{(-1)}\left(\frac{\|\zeta\|}{s}\right) \star \xi_\star^{(-1)}\left(\frac{\|\gamma\|}{t}\right) \\ &= \aleph_\star(\zeta, s) \star \aleph_\star(\gamma, t). \end{aligned}$$

This shows that for each $\zeta, \gamma \in Z$ and $s, t \in \mathbb{R}$,

$$\aleph_\star(\zeta + \gamma, s+t) \geq \aleph_\star(\zeta, s) \star \aleph_\star(\gamma, t).$$

To prove the fifth part, suppose that $\zeta \in Z$, $t_1, t_2 \in \mathbb{R}$ and $t_1 < t_2$. If $t_1 \leq 0$, then

$$0 = \aleph_\star(\zeta, t_1) \leq \aleph_\star(\zeta, t_2).$$

If $t_1 > 0$, then

$$\frac{\|\zeta\|}{t_2} \leq \frac{\|\zeta\|}{t_1}.$$

In this case, if $0 \leq \frac{\|\zeta\|}{t_1} < \xi_\star(0)$, then $0 \leq \frac{\|\zeta\|}{t_2} < \xi_\star(0)$ and as a result

$$\xi_\star^{(-1)}\left(\frac{\|\zeta\|}{t_2}\right) = \xi_\star^{-1}\left(\frac{\|\zeta\|}{t_2}\right) \geq \xi_\star^{-1}\left(\frac{\|\zeta\|}{t_1}\right) = \xi_\star^{(-1)}\left(\frac{\|\zeta\|}{t_1}\right).$$

If $\xi_\star(0) \leq \frac{\|\zeta\|}{t_1}$, then

$$\xi_\star^{(-1)}\left(\frac{\|\zeta\|}{t_2}\right) \geq 0 = \xi_\star^{(-1)}\left(\frac{\|\zeta\|}{t_1}\right).$$

This means that $\aleph_\star(\zeta, t_2) \geq \aleph_\star(\zeta, t_1)$. Therefore, \aleph_\star is increasing with respect to t . Now since $\lim_{t \rightarrow \infty} \frac{\|\zeta\|}{t} = 0$, there exists an $M > 0$ such that for each $t \geq M$, $\frac{\|\zeta\|}{t} < \xi_\star(0)$. Hence,

$$\xi_\star^{(-1)}\left(\frac{\|\zeta\|}{t}\right) = \xi_\star^{-1}\left(\frac{\|\zeta\|}{t}\right).$$

So

$$\begin{aligned} \lim_{t \rightarrow \infty} \aleph_\star(\zeta, t) &= \lim_{t \rightarrow \infty} \xi_\star^{(-1)}\left(\frac{\|\zeta\|}{t}\right) \\ &= \lim_{t \rightarrow \infty} \xi_\star^{-1}\left(\frac{\|\zeta\|}{t}\right) \\ &= \xi_\star^{-1}\left(\lim_{t \rightarrow \infty} \frac{\|\zeta\|}{t}\right) \\ &= \xi_\star^{-1}(0) \\ &= 1. \end{aligned}$$

□

Example 3.2. Let $(Z, \|\cdot\|)$ be a normed space, $\alpha \star_p \beta = \alpha\beta$, $\xi_{\star_p}(\alpha) = -\text{Ln}(\alpha)$ for all $\alpha, \beta \in [0, 1]$, where $\text{Ln}(0) = -\infty$. If $\aleph_{\star_p} : Z \times \mathbb{R} \rightarrow [0, 1]$ is defined by

$$\aleph_{\star_p}(\zeta, t) = \begin{cases} e^{-\frac{\|\zeta\|}{t}}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

for all $\zeta \in Z$ and $t \in \mathbb{R}$, then $(Z, \aleph_{\star_p}, \star_p)$ is a fuzzy normed space.

Example 3.3. Let Z be a topological space and $C^b(Z)$ be the set of all complex valued, bounded and continuous functions on Z equipped with the norm

$$\|f\|_\infty = \sup \{|f(\zeta)|, \zeta \in Z\}$$

for all $f \in C^b(Z)$. If

$$\alpha \star_E \beta = \frac{\alpha\beta}{1 + (1 - \alpha)(1 - \beta)}$$

and $\xi_{\star_E}(\alpha) = \ln \frac{2 - \alpha}{\alpha}$ for each $(\alpha, \beta) \in [0, 1] \times [0, 1]$ such that $\xi_{\star_E}(0) = \infty$, then for each $t \in \mathbb{R}$ and $g \in C^b(Z)$,

$$\aleph_{\star_E}(g, t) = \begin{cases} \frac{2}{e\left(\frac{\|g\|_\infty}{t}\right) + 1}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

is a fuzzy norm on functional space $C^b(Z)$.

Theorem 3.4. [6] *Let Z be a linear space and suppose that (Z, \aleph_1, \star_1) and (Z, \aleph_2, \star_2) are fuzzy normed spaces. If there exist $c_1, c_2 > 0$ such that*

$$\aleph_2(c_1\zeta, t) \leq \aleph_1(\zeta, t) \leq \aleph_2(c_2\zeta, t)$$

for each $\zeta \in Z$ and $t \in \mathbb{R}$, then \aleph_1 and \aleph_2 are equivalent fuzzy norms on Z .

Proposition 3.5. *Let Z be a linear space and $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on Z such that $c_1\|\cdot\|_2 \leq \|\cdot\|_1 \leq c_2\|\cdot\|_2$. Also let $\star : [0, 1]^2 \rightarrow [0, 1]$ be a CATN. Define $\aleph_\star^{(1)}$ and $\aleph_\star^{(2)}$ by*

$$\aleph_\star^{(1)}(\zeta, t) = \begin{cases} \xi_\star^{(-1)}\left(\frac{\|\zeta\|_1}{t}\right), & t > 0 \\ 0, & t \leq 0 \end{cases}$$

and

$$\mathfrak{N}_\star^{(2)}(\zeta, t) = \begin{cases} \xi_\star^{(-1)}\left(\frac{\|\zeta\|_2}{t}\right), & t > 0 \\ 0, & t \leq 0 \end{cases}.$$

Then, $\mathfrak{N}_\star^{(1)}$ and $\mathfrak{N}_\star^{(2)}$ are equivalent fuzzy norms.

Proof. We show that

$$\mathfrak{N}_\star^{(2)}(c_2\zeta, t) \leq \mathfrak{N}_\star^{(1)}(\zeta, t) \leq \mathfrak{N}_\star^{(2)}(c_1\zeta, t) \quad (1)$$

for all $\zeta \in Z$ and $t \in \mathbb{R}$. Clearly

$$\mathfrak{N}_\star^{(2)}(c_1\zeta, t) = \begin{cases} \xi_\star^{(-1)}\left(\frac{c_1\|\zeta\|_2}{t}\right), & t > 0 \\ 0, & t \leq 0 \end{cases}$$

and

$$\mathfrak{N}_\star^{(2)}(c_2\zeta, t) = \begin{cases} \xi_\star^{(-1)}\left(\frac{c_2\|\zeta\|_2}{t}\right), & t > 0 \\ 0, & t \leq 0 \end{cases}$$

for all $\zeta \in Z$ and $t \in \mathbb{R}$. Since for each $\zeta \in Z$, $c_1\|\zeta\|_2 \leq \|\zeta\|_1$, $\frac{c_1\|\zeta\|_2}{t} \leq \frac{\|\zeta\|_1}{t}$ for all $t > 0$. By decreasing property of $\xi_\star^{(-1)}$ we have

$$\xi_\star^{(-1)}\left(\frac{c_1\|\zeta\|_2}{t}\right) \geq \xi_\star^{(-1)}\left(\frac{\|\zeta\|_1}{t}\right), \quad t > 0.$$

Therefore,

$$\mathfrak{N}_\star^{(2)}(c_1\zeta, t) \geq \mathfrak{N}_\star^{(1)}(\zeta, t)$$

for all $t > 0$. Similarly, it can be shown that

$$\mathfrak{N}_\star^{(1)}(\zeta, t) \geq \mathfrak{N}_\star^{(2)}(c_2\zeta, t)$$

for all $t > 0$. Hence inequality (1) hold for all $\zeta \in Z$ and $t \in \mathbb{R}$. \square

Definition 3.6. [9] Let (Z, \mathfrak{N}, \star) be a fuzzy normed space. A sequence $\{\zeta_n\}_{n \geq 1}$ in Z is said to be fuzzy convergent, if there exists $\zeta \in Z$ such that for each $0 < \alpha < 1$ and $t > 0$, there exists $K \in \mathbb{N}$ such that for all $n \geq K$,

$$\mathfrak{N}(\zeta_n - \zeta, t) > 1 - \alpha.$$

Definition 3.7. [9] Let (Z, \mathfrak{N}, \star) be a fuzzy normed space. A sequence $\{\zeta_n\}_{n \geq 1}$ in Z is said to be a fuzzy Cauchy sequence if for each $0 < \alpha < 1$ and $t > 0$, there exists $K \in \mathbb{N}$ such that for all $n > m \geq K$,

$$\mathfrak{N}(\zeta_n - \zeta_m, t) > 1 - \alpha.$$

Definition 3.8. [8] A fuzzy normed space (Z, \mathfrak{N}, \star) is said to be a fuzzy Banach space, if every fuzzy Cauchy sequence, is fuzzy convergent in Z .

Proposition 3.9. Let $(Z, \|\cdot\|)$ be a Banach space and \star be a CATN. Then $(Z, \mathfrak{N}_\star, \star)$ is a fuzzy Banach space.

Proof. Suppose that $\{\zeta_n\}_{n \geq 1}$ is a fuzzy Cauchy sequence in Z . We show that $\{\zeta_n\}_{n \geq 1}$ is a Cauchy sequence with respect to the norm $\|\cdot\|$. Let $\epsilon > 0$ be given. Since $\xi_\star : [0, 1] \rightarrow [0, \infty]$ is continuous at $\alpha_0 = 1$ and $\xi_\star(1) = 0$, there exists $0 < \alpha_1 < 1$ such that $\xi_\star(1 - \alpha_1) < \epsilon$. By Definition 3.7, for α_1 and $t = 1$, there exists $K \in \mathbb{N}$ such that for all $n > m \geq K$

$$\mathfrak{N}_\star(\zeta_n - \zeta_m, 1) = \xi_\star^{(-1)}\left(\frac{\|\zeta_n - \zeta_m\|}{1}\right) > 1 - \alpha_1 > 0.$$

So $\xi_\star^{(-1)}(\|\zeta_n - \zeta_m\|) = \xi_\star^{-1}(\|\zeta_n - \zeta_m\|)$ and consequently

$$\xi_\star^{-1}(\|\zeta_n - \zeta_m\|) > 1 - \alpha_1.$$

Because ξ_\star is strictly decreasing, we obtain

$$\|\zeta_n - \zeta_m\| < \xi_\star(1 - \alpha_1) < \epsilon$$

for all $n > m \geq K$. This shows that $\{\zeta_n\}_{n \geq 1}$ is a Cauchy sequence. The completeness of Z implies that there exists $\zeta \in Z$ such that $\zeta_n \xrightarrow{\|\cdot\|} \zeta$.

For $t > 0$, since $\lim_{n \rightarrow \infty} \frac{\|\zeta_n - \zeta\|}{t} = 0$, there exists $n_0 \in \mathbb{N}$ such that $\frac{\|\zeta_n - \zeta\|}{t} < \xi_{\star}(0)$ for all $n \geq n_0$. Hence

$$\begin{aligned} \lim_{n \rightarrow \infty} \aleph_{\star}(\zeta_n - \zeta, t) &= \lim_{n \rightarrow \infty} \xi_{\star}^{(-1)}\left(\frac{\|\zeta_n - \zeta\|}{t}\right) \\ &= \lim_{n \rightarrow \infty} \xi_{\star}^{-1}\left(\frac{\|\zeta_n - \zeta\|}{t}\right) \\ &= \xi_{\star}^{-1}\left(\lim_{n \rightarrow \infty} \frac{\|\zeta_n - \zeta\|}{t}\right) \\ &= \xi_{\star}^{-1}(0) \\ &= 1. \end{aligned}$$

So $\{\zeta_n\}_{n \geq 1}$ is fuzzy convergent to ζ . This shows that Z is a fuzzy Banach space. \square

Definition 3.10. [2] Let Z be an algebra, \star_1, \star_2 be t -norms and (Z, \aleph, \star_1) be a fuzzy normed space. If

$$\aleph(\zeta\gamma, st) \geq \aleph(\zeta, s) \star_2 \aleph(\gamma, t) \quad (2)$$

for all $\zeta, \gamma \in Z$ and $s, t \in \mathbb{R}$, then $(Z, \aleph, \star_1, \star_2)$ is called a FNA (Fuzzy normed algebra).

Corollary 3.11. Let $(Z, \|\cdot\|)$ be a normed algebra and \star_H and \star_p be defined as follows :

$$\alpha \star_H \beta = \begin{cases} 0, & \alpha = \beta = 0 \\ \frac{\alpha\beta}{\alpha + \beta - \alpha\beta}, & o.w. \end{cases}$$

and $\alpha \star_p \beta = \alpha\beta$ for all $(\alpha, \beta) \in [0, 1] \times [0, 1]$. Then $(Z, \aleph_{\star_H}, \star_H, \star_p)$ is a FNA.

Proof. To prove, we only check inequality (2) for \aleph_{\star_H} . Since $\xi_{\star_H}(\alpha) = \frac{1-\alpha}{\alpha}$ which $\xi_{\star_H}(0) = \infty$ and $\xi_{\star_H}^{(-1)}(\alpha) = \xi_{\star_H}^{-1}(\alpha) = \frac{1}{1+\alpha}$, by Theorem

3.1 we have

$$\aleph_{\star_H}(\zeta, t) = \begin{cases} \frac{t}{t + \|\zeta\|}, & t > 0 \\ 0, & t \leq 0 \end{cases}.$$

If $s \leq 0$ or $t \leq 0$, then inequality

$$\aleph_{\star_H}(\zeta\gamma, st) \geq \aleph_{\star_H}(\zeta, s) \star_P \aleph_{\star_H}(\gamma, t)$$

is obvious. If $s > 0$ and $t > 0$, we show that

$$\frac{st}{st + \|\zeta\gamma\|} \geq \left(\frac{s}{s + \|\zeta\|} \right) \left(\frac{t}{t + \|\gamma\|} \right). \quad (3)$$

But

$$\begin{aligned} \|\zeta\gamma\| &\leq \|\zeta\| \|\gamma\| \\ &\leq \|\zeta\| \|\gamma\| + s \|\gamma\| + t \|\zeta\|. \end{aligned}$$

So

$$\begin{aligned} st + \|\zeta\gamma\| &\leq st + \|\zeta\| \|\gamma\| + s \|\gamma\| + t \|\zeta\| \\ &= (s + \|\zeta\|)(t + \|\gamma\|) \end{aligned}$$

and consequently

$$\frac{st}{st + \|\zeta\gamma\|} \geq \frac{s}{s + \|\zeta\|} \cdot \frac{t}{t + \|\gamma\|}.$$

Therefore inequality (3) holds. \square

Proposition 3.12. *Let Z be a normed algebra, \star be a CATN and ξ_\star be a CAG of \star such that $\xi_\star(0) \leq 1$. Then $(Z, \aleph_\star, \star, \star)$ is a FNA.*

Proof. We only prove inequality (2) for \aleph_\star . Let $\zeta, \gamma \in Z$ and $s, t > 0$. If $\xi_\star(0) \leq \frac{\|\zeta\|}{s}$ or $\xi_\star(0) \leq \frac{\|\gamma\|}{t}$, then $\aleph_\star(\zeta, s) = 0$ or $\aleph_\star(\gamma, t) = 0$. Hence

$$\aleph_\star(\zeta\gamma, st) \geq \aleph_\star(\zeta, s) \star \aleph_\star(\gamma, t) = 0.$$

If $0 \leq \frac{\|\zeta\|}{s} < \xi_\star(0) \leq 1$ and $0 \leq \frac{\|\gamma\|}{t} < \xi_\star(0) \leq 1$, then

$$\frac{\|\zeta\gamma\|}{st} \leq \frac{\|\zeta\|}{s} \frac{\|\gamma\|}{t} \leq \frac{\|\zeta\|}{s} + \frac{\|\gamma\|}{t}.$$

Hence,

$$\begin{aligned} \aleph_\star(\zeta\gamma, st) &= \xi_\star^{(-1)}\left(\frac{\|\zeta\gamma\|}{st}\right) \\ &\geq \xi_\star^{(-1)}\left(\frac{\|\zeta\|}{s} + \frac{\|\gamma\|}{t}\right) \\ &= \xi_\star^{(-1)}\left(\xi_\star\left(\xi_\star^{-1}\left(\frac{\|\zeta\|}{s}\right)\right) + \xi_\star\left(\xi_\star^{-1}\left(\frac{\|\gamma\|}{t}\right)\right)\right) \\ &= \xi_\star^{-1}\left(\frac{\|\zeta\|}{s}\right) \star \xi_\star^{-1}\left(\frac{\|\gamma\|}{t}\right) \\ &= \xi_\star^{(-1)}\left(\frac{\|\zeta\|}{s}\right) \star \xi_\star^{(-1)}\left(\frac{\|\gamma\|}{t}\right) \\ &= \aleph_\star(\zeta, s) \star \aleph_\star(\gamma, t). \end{aligned}$$

□

We will show further that for a CAG of \star where $\xi_\star(0) > 1$, it is possible to reach an algebra fuzzy norm.

Lemma 3.13. *Suppose that \star is a CATN and ξ_\star is a CAG of \star . Then*

$$1) (k\xi_\star)^{(-1)}(\lambda) = \xi_\star^{(-1)}\left(\frac{\lambda}{k}\right), \quad 0 < k < \infty.$$

$$2) \alpha \star \beta = \xi_\star^{(-1)}(\xi_\star(\alpha) + \xi_\star(\beta)) = (k\xi_\star)^{(-1)}(k\xi_\star(\alpha) + k\xi_\star(\beta))$$

for each $0 \leq \alpha \leq 1, 0 \leq \beta \leq 1$ and $0 < k < \infty$.

Proof. 1) Suppose that $\lambda \in [0, \infty]$ and $0 < k < \infty$. Therefore

$$\begin{aligned} (k\xi_\star)^{(-1)}(\lambda) &= \sup\{0 \leq \alpha \leq 1, k\xi_\star(\alpha) > \lambda\} \\ &= \sup\left\{0 \leq \alpha \leq 1, \xi_\star(\alpha) > \frac{\lambda}{k}\right\} \\ &= \xi_\star^{(-1)}\left(\frac{\lambda}{k}\right). \end{aligned}$$

2) Let $(\alpha, \beta) \in [0, 1] \times [0, 1]$ and $0 < k < \infty$. Therefore by part 1 we have,

$$\begin{aligned} (k\xi_\star)^{(-1)}(k\xi_\star(\alpha) + k\xi_\star(\beta)) &= \xi_\star^{(-1)}\left(\frac{k\xi_\star(\alpha) + k\xi_\star(\beta)}{k}\right) \\ &= (\xi_\star)^{(-1)}(\xi_\star(\alpha) + \xi_\star(\beta)) \\ &= \alpha \star \beta. \end{aligned}$$

□

Lemma (3.13) states that t-norms produced by ξ_\star and $k\xi_\star$ are the same.

Remark 3.14. Let \star be a CATN and $0 < k < \infty$. Then the fuzzy norm \aleph_\star generated by $k\xi_\star$ as a CAG on a norm space $(Z, \|\cdot\|)$ is as follows :

$$\begin{aligned} \aleph_\star(\zeta, t) &= \begin{cases} (k\xi_\star)^{(-1)}\left(\frac{\|\zeta\|}{t}\right), & t > 0 \\ 0, & t \leq 0 \end{cases} \\ &= \begin{cases} (k\xi_\star)^{-1}\left(\frac{\|\zeta\|}{t}\right), & 0 \leq \frac{\|\zeta\|}{t} < k\xi_\star(0), t > 0 \\ 0, & k\xi_\star(0) \leq \frac{\|\zeta\|}{t}, t > 0 \\ 0, & t \leq 0 \end{cases} \end{aligned}$$

for all $\zeta \in Z$ and $t \in \mathbb{R}$.

Proposition 3.15. Let $(Z, \|\cdot\|)$ be a normed algebra, \star be a CATN and ξ_\star be a CAG of \star such that $1 < \xi_\star(0) < \infty$. If \aleph_\star is the fuzzy norm generated by $k\xi_\star$ where $k = \frac{1}{\xi_\star(0)}$, then $(Z, \aleph_\star, \star, \star)$ is a FNA.

Proof. By Lemma 3.13, $k\xi_\star$ is a CAG of \star . Since $k\xi_\star(0) = 1$, by applying Proposition 3.12 the proof is trivial. □

Proposition 3.16. Let \star be a CATN and ξ_\star be a CAG of \star such that $\xi_\star(0) = \infty$. If $h : [0, 1] \rightarrow [0, \infty]$ is defined by $h(a) = \xi_\star(e^{(a-1)})$ for each $0 \leq a \leq 1$, then

$$h^{(-1)}(b) = \begin{cases} h^{-1}(b), & 0 \leq b < \xi_\star\left(\frac{1}{e}\right) \\ 0, & b \geq \xi_\star\left(\frac{1}{e}\right) \end{cases}$$

where $h^{-1}(b) = \text{Ln}(\xi_{\star}^{-1}(b)) + 1$ for all $b \in \left[0, \xi_{\star}\left(\frac{1}{e}\right)\right]$. Moreover $\star' : [0, 1]^2 \rightarrow [0, 1]$ defined by

$$\alpha \star' \beta = h^{(-1)}(h(\alpha) + h(\beta))$$

is a CATN.

Proof. Since h is a CAG of \star' , by Theorem 2.11, \star' is a CATN. \square

Proposition 3.17. Let $(Z, \|\cdot\|)$ be a normed algebra, \star be a CATN and ξ_{\star} be a CAG of \star such that $\xi_{\star}(0) = \infty$. If $h(a) = \xi_{\star}(e^{(a-1)})$ for all $0 \leq a \leq 1$, $\alpha \star' \beta = h^{(-1)}(h(\alpha) + h(\beta))$ for all $(\alpha, \beta) \in [0, 1] \times [0, 1]$ and $\aleph_{\star'}$ is the fuzzy norm generated by kh where $k = \frac{1}{\xi_{\star}\left(\frac{1}{e}\right)}$, then

$(Z, \aleph_{\star'}, \star', \star')$ is a FNA.

Proof. Since $kh(0) = 1$, by Proposition 3.12, the proof is obvious. \square

By Propositions 3.12, 3.15 and 3.17 and using a CATN \star , a FNA can be produced. Within the next proposition, we are going appear that with each CATN, a new CATN can be made.

Proposition 3.18. Let \star be a CATN and ξ_{\star} and ϱ_{\star} be the corresponding CAG and CMG of \star respectively. If $h : [0, 1] \rightarrow [0, \infty]$ is defined by

$$h(a) = (\xi_{\star} \circ \varrho_{\star})(a)$$

for all $0 \leq a \leq 1$, then h is a CAG of \star' where

$$\alpha \star' \beta = h^{(-1)}(h(\alpha) + h(\beta))$$

for all $(\alpha, \beta) \in [0, 1] \times [0, 1]$.

Proof. Since h is a CAG of \star' , by Theorem 2.11, \star' is a CATN. \square

Corollary 3.19. Let \star be a CATN and $\xi_{\star}^{(1)}, \xi_{\star}^{(2)}$ be two CAGs of \star . If $\aleph_{\star}^{(1)}$ and $\aleph_{\star}^{(2)}$ are fuzzy norms generated by $\xi_{\star}^{(1)}$ and $\xi_{\star}^{(2)}$ on a normed space $(Z, \|\cdot\|)$ respectively, then there exists $\alpha \in (0, \infty)$ such that

$$\aleph_{\star}^{(2)}(\zeta, t) = \aleph_{\star}^{(1)}(\zeta, \alpha t)$$

for all $\zeta \in Z$ and $t \in \mathbb{R}$. In particular if $\alpha \leq 1$, then

$$\aleph_{\star}^{(2)}(\zeta, t) \leq \aleph_{\star}^{(1)}(\zeta, t)$$

for all $\zeta \in Z$ and $t \in \mathbb{R}$. If $\alpha > 1$, then

$$\aleph_{\star}^{(2)}(\zeta, t) \geq \aleph_{\star}^{(1)}(\zeta, t)$$

for all $\zeta \in Z$ and $t \in \mathbb{R}$.

Proof. Since for each CATN \star , a CAG of \star is uniquely determined up to a positive multiplicative constant [5], then we assume that

$$\xi_{\star}^{(2)} = \alpha \xi_{\star}^{(1)}$$

for some $\alpha \in (0, \infty)$. Hence

$$\begin{aligned} \aleph_{\star}^{(2)}(\zeta, t) &= \begin{cases} \left(\xi_{\star}^{(2)}\right)^{(-1)}\left(\frac{\|\zeta\|}{t}\right), & t > 0 \\ 0, & t \leq 0 \end{cases} \\ &= \begin{cases} \left(\alpha \xi_{\star}^{(1)}\right)^{(-1)}\left(\frac{\|\zeta\|}{t}\right), & t > 0 \\ 0, & t \leq 0 \end{cases} \\ &= \begin{cases} \left(\xi_{\star}^{(1)}\right)^{(-1)}\left(\frac{\|\zeta\|}{\alpha t}\right), & t > 0 \\ 0, & t \leq 0 \end{cases} \\ &= \begin{cases} \left(\xi_{\star}^{(1)}\right)^{(-1)}\left(\frac{\|\zeta\|}{\alpha t}\right), & \alpha t > 0 \\ 0, & \alpha t \leq 0 \end{cases} \\ &= \aleph_{\star}^{(1)}(\zeta, \alpha t) \end{aligned}$$

for all $\zeta \in Z$ and $t \in \mathbb{R}$. Since $\aleph_{\star}(\zeta, \cdot)$ is increasing on \mathbb{R} , the remain of the proof is obvious. \square

Conclusion

In this paper, we conclude that it is possible to reach a fuzzy norm and an algebra fuzzy norm on a normed space $(Z, \|\cdot\|)$ by applying a CATN \star using the mentioned methods.

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